H-measures and variants

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Averaging of strongly inhomogeneious materials

First works at the end of 19th century (mathematical physics)
Poisson, Maxwell, Rayleigh

Examples of strongly inhomogeneous media:
fibre reinforced materials (reinforced glass, concrete)
layered materials (sperr holz)
gas concrete
porous media (interesting e.g. for oil extraction)
leaf

Inhomogeneous structures are usually better than homogeneous (they better optimise in order to achieve some property).
Shape optimisation

The notion of *effective property:*
the simplest structure of inhomogeneities — periodic structure (true for crystals, man-made materials, . . .)
apply the averaging procedure, which will produce the effective coefficients (constants in this case!)
we replace a strongly inhomogeneious material by a homogeneous one — thus the name *homogenisation* (Ivo Babuška, 1974)

Mathematical approaches to homogenisation

There are three different frameworks when approaching the homogenisation (in the mathematical sense):

Periodic assumption: assume that the coefficients are periodic

This assumption is realistic for crystals and man-made materials. However, it turns out that it gives reasonably good results even when it is not satisfied.

Weak convergences: general approach introduced by L. Tartar and F. Murat.

The passage from microscale to the macroscale is modelled by various weak convergences.

Probabilistic approach: the behaviour is assumed to follow some probabilistic distribution

In fact, we know that our laws are deterministic (continuum mechanics; nothing to do with quantum mechanical uncertainty).

We shall follow the second approach.

Asymptotic expansions using multiple scales

When working with periodic functions on $[0,1]$ it is natural to denote the period by $\varepsilon = 1/n$. Thus we have (Dirichlet b.c. on both sides):

$$\begin{cases} -\left(a_\varepsilon(x)u'_\varepsilon(x)\right)' = f(x) \\ \quad u_\varepsilon(0) = u_\varepsilon(1) = 0 \, . \end{cases}$$

Assume the asymptotic expansion

$$u_\varepsilon(x) = u^0(x,y) + \varepsilon u^1(x,y) + \varepsilon^2 u^2(x,y) + \cdots \, ,$$

where $y = x/\varepsilon = nx$ is the *fast variable*. ($\frac{d}{dx} = \partial_x + \frac{1}{\varepsilon}\partial_y$), thus:

$$\begin{aligned} u'_\varepsilon &= \partial_x u^0 + \frac{1}{\varepsilon}\partial_y u^0 + \varepsilon\left(\partial_x u^1 + \frac{1}{\varepsilon}\partial_y u^1\right) + \varepsilon^2\left(\partial_x u^2 + \frac{1}{\varepsilon}\partial_y u^2\right) + \cdots \\ &= \frac{1}{\varepsilon}\partial_y u^0 + \left(\partial_x u^0 + \partial_y u^1\right) + \varepsilon\left(\partial_x u^1 + \partial_x u^2\right) + \varepsilon^2(\cdots) + \cdots \, . \end{aligned}$$

Note that u_ε depends only on x, while the functions on the right in general depend both on x and y.
Additionally assume that the above functions depend 1–periodically on y; more precisely consider the mapping: $y \mapsto \check{y}$; $\mathbf{R} \longrightarrow S^1$, e.g.
$y \mapsto \check{y} = (\cos 2\pi y, \sin 2\pi y)$, while a is defined on S^1.

The heat flux

If we assumed that $a_\varepsilon(x) = a((x/\varepsilon)^\vee)$, after inserting the expansion into the equation we obtain:

$$p_\varepsilon(x) := a_\varepsilon(x) u'_\varepsilon(x) \; .$$

Rewrite the equation in the form:

$$-f = p'_\varepsilon(x) = \frac{1}{\varepsilon^2}\partial_y(a_\varepsilon \partial_y u^0) + \frac{1}{\varepsilon}\left(\partial_x(a_\varepsilon \partial_y u^0) + \partial_y(a_\varepsilon \partial_x u^0 + a_\varepsilon \partial_y u^1)\right)$$
$$+ \partial_x(a_\varepsilon \partial_x u^0 + a_\varepsilon \partial_y u^1) + \partial_y(a_\varepsilon \partial_x u^1 + a_\varepsilon \partial_y u^2) + \varepsilon(\cdots) + \cdots$$

The expansions are equal if the terms multiplying the same powers are equal, so

$$\partial_y(a_\varepsilon \partial_y u^0) = 0 \; ,$$

which is an ODE on the circle for $u^0(x, \cdot)$, with a solution unique up to a constant. Therefore, $u^0(x, \cdot)$ does not depend on y; this fact simplifies the asymptotic expansion.

Considering the form of u^ε, p^ε can also be written in a form of an asymptotic expansion:

$$p_\varepsilon(x) = p^0(x, y) + \varepsilon p^1(x, y) + \varepsilon^2 p^2(x, y) + \cdots \; ,$$

which allows us to further rewrite the equation as

$$-\left(\partial_x + \frac{1}{\varepsilon}\partial_y\right) p_\varepsilon = f \; .$$

After equating the corresponding terms . . .

$$\begin{aligned}
\frac{1}{\varepsilon} &\quad : \quad -\partial_y p^0 = 0 \\
1 &\quad : \quad -\partial_x p^0 - \partial_y p^1 = f \\
\varepsilon &\quad : \quad -\partial_x p^1 - \partial_y p^2 = 0
\end{aligned}$$

By the first equality, neither p^0 depends on y.
After defining the mean-value operator (in y): $\tilde{\varphi} := \fint_{S^1} \varphi$, and applying it to the second equation, we get:

$$-\partial_x p^0 = f \ .$$

This is the macroscopic equation.
What is the relationship between p^0 and $(u^0)'$?

Study of the local problem

We assumed that $a_\varepsilon(x) = a((x/\varepsilon)^\vee)$, so after inserting the expansion into the equation we also obtain:

$$\partial_y(a\partial_x u^0) + \partial_y(a\partial_y u^1) = 0 \ ,$$

thus after integrating

$$a(y)[\partial_x u^0(x) + \partial_y u^1(x,y)] = C(x) \ ,$$

and

$$\partial_y u^1(x,y) = \frac{C(x)}{a(y)} - (u^0)'(x) \ .$$

After we take the mean value in y (i.e. integrate once more), and denote $A^{-1} := (1/a)^\sim$, we finally get:

$$\partial_y u^1(x,y) = (A/a(y) - 1)(u^0)'(x) \ .$$

Returning to the equation we started with, if we find a function w such that

$$-\partial_y(a(y)w'(y)) = a' \ ,$$

then $u^1 = (u^0)'w$.

Back to the macroscopic equation

In terms of u-s it reads:

$$a(u^0)'' + \partial_y(aw)(u^0)'' + aw'(u^0)'' + \partial_y(a\partial_y u^2) = -f \ ,$$

which after applying the averaging operator becomes

$$(u^0)''[\tilde{a} + 0 + \widetilde{aw'}] + 0 = -f \ .$$

Returning to the equation for a: $w' = C/a - 1$, thus after integrating over the period we get $C = A$.
Finally we get:

$$A(u^0)'' = -f \ ,$$

which is the same equation as the one for u_ε, except that the effective coefficients are the harmonic mean of a.

The precise mathematical result in the periodic setting

Theorem. *Let* $f \in \mathrm{L}^2(\langle 0,1\rangle)$, $u_\varepsilon \in \mathrm{H}^1(\langle 0,1\rangle)$ *the solution of problem*

$$\begin{cases} -\big(a_\varepsilon(x)u_\varepsilon'(x)\big)' = f(x) \\ u_\varepsilon(0) = u_\varepsilon(1) = 0 \ , \end{cases}$$

then $u_\varepsilon \rightharpoonup u^0$, *where* u^0 *solves the same equation, but with*

$$\widetilde{\left(\frac{1}{a_1}\right)}$$

instead of $\frac{1}{a_\varepsilon}$. ∎

We shall see the proof of this theorem in a more general setting then periodic.

The general result

The problem

$$\begin{cases} -\big(a_n(x)u_n'(x)\big)' = f(x) \\ u_n(0) = u_n(1) = 0 \ . \end{cases}$$

Assume that $\alpha < a_n < \beta$, and $f \in \mathrm{L}^2(\langle 0,1\rangle)$.

The equation can be written in the variational form:

$$(\forall\, v \in \mathrm{H}_0^1(\langle 0,1\rangle)) \qquad \int_0^1 a_n u_n' v' = \int_0^1 f v \ .$$

As the expression on the left is equivalent to the scalar product on $\mathrm{H}_0^1(\langle 0,1\rangle)$, while on the right we have a bounded linear functional, so by the Riesz representation theorem there is a unique $u_n \in \mathrm{H}_0^1(\langle 0,1\rangle)$ representing the right hand side.

The Riesz representation theorem gives us also the bound on the norm of u_n; in particular, the bound on the $\|u_n'\|_{\mathrm{L}^2(\langle 0,1\rangle)}$. Thus, u_n' is bounded, and therefore has a weak limit (say, u_∞).
On the other hand, $p_n' := (a_n u_n')' = -f$, so $p_n = a_n u_n' \in \mathrm{L}^2(\langle 0,1\rangle)$, $p_n' = f \in \mathrm{L}^2(\langle 0,1\rangle)$ and it is bounded in $\mathrm{H}_0^1(\langle 0,1\rangle)$. Therefore we can pass once more to a subsequence $p_n \rightharpoonup p$ in $\mathrm{H}^1(\langle 0,1\rangle)$, which by the Rellich compact embedding gives $p_n \longrightarrow p$ in $\mathrm{L}^2(\langle 0,1\rangle)$.

The general result (cont.)

Rewritting the equation as:

$$u_n' = \frac{1}{a_n} p_n \ ,$$

we can pass to the limit in the product (as $\frac{1}{\alpha} \geqslant \frac{1}{a_n} \geqslant \frac{1}{\beta}$ and there is a further subsequence such that $\frac{1}{a_n} \overset{*}{\rightharpoonup} \frac{1}{a_\infty}$), or after taking the derivative:

$$-\left(\frac{u_\infty'}{\frac{1}{a_\infty}} \right)' = -p' = f \ .$$

Thus the effective coefficients are a_∞.

In the periodic case, the limit of $1/a_n$ is the mean value, and the effective coefficients are constant. Any limit u_∞ has to satisfy the equation, which has the unique solution—thus the whole sequence converges.

In the general case we only know the result for an accummulation point (note that above, for simplicity, we did not explicitly write down each passage to a subsequence).

Classical example—in variational formulation

Stationary diffusion:

$$\begin{cases} -\text{div}\,(\mathbf{A}\nabla u) = f \\ \;+ \text{ boundary conditions} \end{cases}$$

Consider it on open and bounded $\Omega \subseteq \mathbf{R}^d$. We search for an $u \in \mathrm{H}^1(\Omega)$, satisfying the boundary conditions and the equation in the sense of distributions.
For simplicity take homogeneous Dirichlet boundary conditions, i.e. $u \in \mathrm{H}^1_0(\Omega)$.
For $\mathbf{A}$ we assume it is from $\mathrm{L}^\infty(\Omega)$, so the bilinear form

$$a(u,v) := \int_\Omega \mathbf{A}\nabla u \cdot \nabla v$$

is well defined. If we additionally set $L(v) := \langle f, v\rangle$ (in the real case), it is a bounded linear functional for $f \in \mathrm{H}^{-1}(\Omega) = (\mathrm{H}^1_0(\Omega))'$.
The variational formulation reads: find $u \in \mathrm{H}^1_0(\Omega)$ such that

$$(\forall\, v \in \mathrm{H}^1_0(\Omega)) \quad a(u,v) = L(v)$$

(because of density it is enough to take $v \in \mathrm{C}^\infty_c(\Omega)$).

Classical example (cont.)

Additionally assume the ellipticity:

$$(\exists\, \alpha \in \mathbf{R}^+)(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d)\ \mathbf{A}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \alpha|\boldsymbol{\xi}|^2\ (\text{a.e. } \mathbf{x} \in \Omega)\ ,$$

so that $a(u,u) \geqslant \alpha\|\nabla u\|^2_{\mathrm{L}^2(\Omega)}$.
On the bounded Ω one also has the Poincaré inequality:

$$(\exists\, C > 0)(\forall\, u \in \mathrm{C}^\infty_c(\Omega)) \quad \|u\|_{\mathrm{L}^2(\Omega)} \leqslant C\|\nabla u\|_{\mathrm{L}^2(\Omega)}\ ,$$

which by density implies that $\|\nabla u\|_{\mathrm{L}^2(\Omega)}$ is a norm equivalent to the standard one on $\mathrm{H}^1_0(\Omega)$. Finally, this gives us an inequality

$$a(u,u) \geqslant \frac{\alpha}{1+C}\|u\|^2_{\mathrm{H}^1_0(\Omega)}\ ,$$

and we can apply the Lax-Milgram lemma.

For a Hilbert space V, a bilinear form a which is V-elliptic, and L a bounded functional, the above problem has the unique solution.

$$\mathrm{H}^{-1}(\Omega) = \{f = g_0 + \operatorname{div} \mathsf{g}_1 \in \mathcal{D}'(\Omega) : g_0 \in \mathrm{L}^2(\Omega), \mathsf{g}_1 \in \mathrm{L}^2(\Omega; \mathbf{R}^d)\}\ ,$$

and we have the Gelfand triplet:

$$\mathrm{H}^1_0(\Omega) \subseteq \mathrm{L}^2(\Omega) \equiv (\mathrm{L}^2(\Omega))' \subseteq \mathrm{H}^{-1}(\Omega)\ .$$

If we take $f \in \mathrm{L}^2(\Omega)$, then $\mathsf{E} := \nabla u$ has good tangential components (as $\operatorname{rot} \mathsf{E} = 0$), while $\mathsf{D} := \mathbf{A}\mathsf{E}$ has good normal components (as $\operatorname{div} \mathsf{D} = f$).

H-convergence

Define

$$\mathcal{M}(\alpha,\beta;\Omega) := \{\mathbf{A} \in \mathrm{L}^{\infty}(\Omega;\mathbf{R}^{d\times d}) : \mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \alpha|\boldsymbol{\xi}|^2 \;\&\; \mathbf{A}^{-1}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \frac{1}{\beta}|\boldsymbol{\xi}|^2\} \,.$$

Theorem. *For any sequence* $(\mathbf{A}_n)$ *in* $\mathcal{M}(\alpha,\beta;\Omega)$ *there is an accummulation point (i.e. a subsequence and its limit* $\mathbf{A}_{\mathit{eff}} \in \mathcal{M}(\alpha,\beta;\Omega)$*) for the H-convergence.*

∎

A sequence $\mathbf{A}_n$ H-converges to $\mathbf{A}_\infty$ if for any $f \in \mathrm{H}^{-1}(\Omega)$ the sequence of solutions (u_n) of the problems

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}_n \nabla u_n) = f \\ \qquad\qquad u_{|_\Gamma} = 0 \,, \end{cases}$$

weakly converges to a limit u_∞ in $\mathrm{H}^1_0(\Omega)$, while the sequence $\mathbf{A}_n\nabla u_n$ converges weakly in $\mathrm{L}^2(\Omega)$ to $\mathbf{A}_\infty\nabla u_\infty$.
u_∞ is the unique solution of the above problem with $\mathbf{A}_\infty$

The H-convergence indeed comes from a topology.

G-convergence

Earlier, another weak convergence was introduced by Spagnolo (and De Giorgi). When we have that our problem is well-posed, even that there is a bijection between f and u, we can define the operator: $A_n(u) = f$ and its inverse $u = A_n^{-1} f$.
Take V to be a reflexive separable Banach space, V' being its dual. A continuous linear operator $A : V \longrightarrow V'$ is coercive if

$$(\exists\, \alpha > 0)(\forall\, u \in V) \qquad \langle Au, u \rangle \geqslant \alpha \|u\|_V^2 \; .$$

Theorem. *If A is coercive, and $f \in V'$, then the equation $Au = f$ has a unique solution in V, and it holds:*

$$\|u\|_V = \|A^{-1} f\|_V \leqslant \frac{1}{\alpha} \|f\|_{V'} \; .$$

∎

A sequence (A_n) of coercive operators in $\mathcal{L}(V; V')$ G-converges to an invertible operator A_∞ if one has

$$(\forall\, f, g \in V') \quad \lim_n \langle g, A_n^{-1} f \rangle = \langle g, A_\infty^{-1} f \rangle \; .$$

This is exactly the weak convergence of operators A_n^{-1}.

G-convergence (cont.)

By $E(\alpha, M)$ denote the class of all coercive operators $A \in \mathcal{L}(V; V')$ such that

$$\langle Au, u \rangle \geqslant \alpha \|u\|_V^2 \qquad \text{and} \qquad \|A\| \leqslant M \ .$$

Theorem. *Any sequence in $E(\alpha, M)$ has an accummulation point in the G-convergence sense, which is in $E(\alpha, \frac{M^2}{\alpha})$.* ∎

The difficulty with G-converdence is the identification of limit coefficients.

The heat conducting equation

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad u(0) = u_0, \end{cases}$$

in an abstract form, suitable to the application of variational techniques (cf. Dautray & Lions, Ch. XVIII, §3).
Recall that for stationary diffusion we had V, H real separable Hilbert spaces, such that

$$V \hookrightarrow H \hookrightarrow V'\ ,$$

where $H \equiv H'$, while V' is the dual of V, with continuous and dense imbeddings (i.e. V, H, V' form a Gel'fand triple).

Lemma. *Let X, Y be Hilbert spaces, $X \hookrightarrow Y$, and $a, b \in \overline{\mathbf{R}}$. Then*

$$W(a, b; X, Y) := \{u \in \mathrm{L}^2([a, b]; X) : u' \in \mathrm{L}^2([a, b]; Y)\}$$

is a Hilbert space with the norm $\|u\|_W^2 = \|u\|^2_{\mathrm{L}^2([a,b];X)} + \|u'\|^2_{\mathrm{L}^2([a,b];Y)}$.
Furthermore, $\mathrm{C}_c^\infty([a, b]; X)$ (the restrictions of functions from $\mathrm{C}_c^\infty(\mathbf{R}; X)$ to $[a, b] \cap \mathbf{R}$) is dense in $W(a, b; X, Y)$.
In particular, for $X = V$, $Y = V'$, and H a Hilbert space such that V, H, V' form a Gel'fand triple,

$$W(a, b; V) := W(a, b; V, V') \hookrightarrow \mathrm{C}([a, b]; H).$$

(for the proof cf. Dautray & Lions) ∎

Variational formulation

Let $a(t;\cdot,\cdot)$ denote a continuous bilinear form on $V \times V$ such that $a(\cdot;u,v)$ is measurable on $[0,T]$ for $u,v \in V$, $T \in \mathbf{R}^+$, and that there are $M, \alpha \in \mathbf{R}^+$ such that

$$\begin{cases} a(t;u,v) \leqslant M\|u\|_V \|v\|_V, & (\text{a.e. } t \in [0,T]),\ u,v \in V \\ a(t;u,u) \geqslant \alpha \|u\|_V^2, & (\text{a.e. } t \in [0,T]),\ u \in V. \end{cases}$$

Then for $(\text{a.e. } t \in [0,T])$ the form a defines $A(t) \in \mathcal{L}(V;V')$ by

$${}_{V'}\langle\, A(t)u, v \,\rangle_V := a(t;u,v),$$

where

$$\sup_{t\in[0,T]} \|A(t)\|_{\mathcal{L}(V;V')} \leqslant M\ .$$

The existence of solutions of

$$\begin{cases} \dfrac{d}{dt}\langle\, u(\cdot) \mid v \,\rangle_H + a(\cdot;u(\cdot),v) = \langle f(\cdot), v\rangle, \quad v \in V \\ \qquad\qquad\qquad u(0) = u_0, \end{cases}$$

is guaranteed by:

Theorem. *If $u_0 \in H$, $f \in \mathrm{L}^2([0,T];V')$, and a on $V \times V$ satisfies the above bound, there exists a unique solution $u \in W(0,T;V)$.* ∎

Relaxed assumptions

Note that the initial condition makes sense, because we look for the solution in $W(0,T;V) \hookrightarrow C([0,T];H)$.

The Theorem remains valid if we relax the assumptions on a; if there are $\lambda \in \mathbf{R}$ and $\alpha \in \mathbf{R}^+$ such that

$$a(t;u,u) + \lambda \|u\|_H^2 \geqslant \alpha \|u\|_V^2, \qquad (\text{a.e. } t \in [0,T]),\ u \in V.$$

Indeed, for $\tilde{u} = ue^{-kt}, k \in \mathbf{R}$ it is true that

$$\begin{cases} \dfrac{d}{dt}\langle\, \tilde{u}(\cdot) \mid v \,\rangle_H + k\langle\, \tilde{u}(\cdot) \mid v \,\rangle_H + a(\cdot;\tilde{u}(\cdot),v) = \langle e^{-kt} f(\cdot), v\rangle \\ \hfill \tilde{u}(0) = u_0; , \end{cases}$$

so $\tilde{a}(t;u,v) = k\langle\, u \mid v \,\rangle_H + a(t;u,v)$, with $k = \lambda$, satisfies the theorem.

The solution is obtained by the Gal'erkin method as a strong limit in $L^2(0,T;V) \cap L^\infty(0,T;H)$ of a sequence of approximative solutions u_m satisfying

$$\|u_m(t)\|_H^2 + \alpha \int_0^t \|u_m(s)\|_V^2\, ds \leqslant C \left(\|u_0\|_H^2 + \int_0^T \|f(s)\|_{V'}^2\, ds \right), \quad t \in [0,T]\ .$$

Thus u has to satisfy the same bounds; furthermore, for $t \in [0,T]$ it satisfies the *energy equality*

$$\frac{1}{2}\|u(t)\|_H^2 + \int_0^t a(s;u(s),u(s))ds = \frac{1}{2}\|u(0)\|_H^2 + \int_0^t \langle f(s), u(s)\rangle ds,$$

where the rhs represents the energy.

Relaxed assumptions (cont.)

From the equation: $\partial_t u = f - A(t)u$, and $\sup_{t\in[0,T]} \|A(t)\|_{\mathcal{L}(V;V')} \leqslant M$ we get

$$\begin{aligned} \|\partial_t u\|_{L^2(0,T;V')} &\leqslant \|f\|_{L^2(0,T;V')} + M\|u\|_{L^2(0,T;V)} \\ &\leqslant C\left(\|u_0\|_H + \|f\|_{L^2(0,T;V')}\right). \end{aligned}$$

These bounds were obtained under the assumption $A(t;u,u) \geqslant \alpha\|u\|_V^2$. However, usually we shall have only a weaker condition; what bounds can we prove?

For $\tilde{u} = ue^{-\lambda t}$ the above holds, while it is a solution of

$$\begin{cases} \dfrac{d}{dt}\langle\, \tilde{u}(\cdot) \mid v \,\rangle_H + \tilde{a}(\cdot;\tilde{u}(\cdot),v) = \langle e^{-\lambda t} f(\cdot), v\rangle \\ \qquad\qquad\qquad\qquad\qquad \tilde{u}(0) = u_0, \end{cases}$$

where $\tilde{a}(t;u,v) = \lambda\langle\, u \mid v \,\rangle_H + a(t;u,v)$. Thus

$$\begin{aligned} \|\tilde{u}(t)\|_H^2 + \alpha\int_0^t \|\tilde{u}(s)\|_V^2\,ds &\leqslant \tilde{C}\left(\|u_0\|_H^2 + \int_0^T \|e^{-\lambda s}f(s)\|_{V'}^2\,ds\right) \\ &\leqslant \tilde{C}\left(\|u_0\|_H^2 + \int_0^T \|f(s)\|_{V'}^2\,ds\right), \quad t\in[0,T]. \end{aligned}$$

Variational formulation (cont.)

On the other hand, for $t \in [0,T]$

$$\|\tilde{u}(t)\|_H^2 + \alpha \int_0^t \|\tilde{u}(s)\|_V^2 \, ds = e^{-2\lambda t}\|u(t)\|_H^2 + \alpha \int_0^t e^{-2\lambda s}\|u(s)\|_V^2 \, ds$$
$$\geqslant e^{-2\lambda T}\left(\|u(t)\|_H^2 + \alpha \int_0^t \|u(s)\|_V^2 \, ds\right).$$

Comparing them we get

$$\|u(t)\|_H^2 + \alpha \int_0^t \|u(s)\|_V^2 \, ds \leqslant C\left(\|u_0\|_H^2 + \int_0^T \|f(s)\|_{V'}^2 \, ds\right), \quad t \in [0,T].$$

Do we still have the bound?

By inserting $\tilde{u} = ue^{-\lambda t}$ and $\tilde{a}(t;u,v) = \lambda\langle\, u \mid v \,\rangle_H + a(t;u,v)$ we get

$$\frac{1}{2}e^{-2\lambda t}\|u(t)\|_H^2 + \int_0^t e^{-2\lambda s}\Big(\lambda\|u(s)\|_H^2 + a(s;u(s),u(s))\Big)ds$$
$$= \frac{1}{2}\|u(0)\|_H^2 + \int_0^t e^{-2\lambda s}\langle f(s),u(s)\rangle ds\ ,$$

or

$$0 = \int_0^t \frac{1}{2}\frac{d}{ds}\left(e^{-2\lambda s}\|u(s)\|_H^2\right) + e^{-2\lambda s}\Big(\lambda\|u(s)\|_H^2 + a(s;u(s),u(s)) - \langle f(s),u(s)\rangle\Big)ds$$
$$= \int_0^t e^{-2\lambda s}\Big(\frac{1}{2}\frac{d}{ds}\|u(s)\|_H^2 + a(s;u(s),u(s)) - \langle f(s),u(s)\rangle\Big)ds.$$

The result

Taking the derivative we get that the energy inequality remains valid

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 + a(t;u(t),u(t)) = \langle f(t), u(t)\rangle \ .$$

In our case $V = \mathrm{H}^1(\mathbf{R}^d), H = \mathrm{L}^2(\mathbf{R}^d)$, and

$$a(t;u,v) = {}_{\mathrm{H}^{-1}(\Omega)}\langle\, -\mathsf{div}\,(\mathbf{A}\nabla u), v\,\rangle_{\mathrm{H}^1(\Omega)} = \langle\, \mathbf{A}\nabla u \mid \nabla v\,\rangle_{\mathrm{L}^2(\Omega)},$$

where $\mathbf{A} \in \mathrm{L}^\infty(\mathbf{R}_0^+ \times \mathbf{R}^d; \mathrm{M}_{d\times d}(\mathbf{R}))$. Additionally assuming the ellipticity: there is an $\alpha > 0$ such that $\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \alpha|\boldsymbol{\xi}|^2$ for $\boldsymbol{\xi} \in \mathbf{R}^d$,

$$a(t;u,u) = \langle\, \mathbf{A}\nabla u \mid \nabla u\,\rangle_{\mathrm{L}} \geqslant \alpha\|\nabla u\|_{\mathrm{L}}^2 = \alpha(\|u\|_{\mathrm{H}^1(\omega)}^2 - \|u\|_{\mathrm{L}}^2),$$

or a defined above satisfies boundedness from above and below.

Corollary. *There is a unique solution* u *of*

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbf{R}^d)) \\ \qquad\qquad u(0) = u_0 \in \mathrm{L}^2(\mathbf{R}^d), \end{cases}$$

$u \in W(0,T;\mathrm{H}^1(\mathbf{R}^d)) \hookrightarrow \mathrm{C}(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d))$, *where* $\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbf{R}^d))}$ *and* $\|u\|_{W(0,T;\mathrm{H}^1(\mathbf{R}^d))}$ *are bounded from above by the norms of* u_0 *and* f*.* ∎

Sequence of problems

If we consider a sequence of problems:

$$\begin{cases} \partial_t u_n - \mathrm{div}\,(\mathbf{A}\nabla u_n) = f_n \rightharpoonup f \quad \text{in} \quad \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega)) \\ u_n(0) = u_n^0 \rightharpoonup u^0 \quad \text{in} \quad \mathrm{L}^2(\Omega). \end{cases}$$

As $(f_n), (u_n^0)$ are bounded, the solutions are as well: (u_n) in $W(0,T;\mathrm{H}^1(\Omega))$, and on a subsequence we have its convergence. The strong convergence of (u_n) we can obtain by the following

Lemma. (the Aubin compactness) *Let B_0, B_1 and B_2 be Banach spaces with $B_1 \hookrightarrow B_2$ continuously and $B_0 \hookrightarrow B_1$ compactly. If (u_n) is bounded in $\mathrm{L}^p([0,T];B_0)$ and (u_n') in $\mathrm{L}^p([0,T];B_2)$ for a $T < \infty$ and $p \in \langle 1, \infty \rangle$, then (u_n) is contained in a compact in $\mathrm{L}^p([0,T];B_1)$.* ∎

We take $B_0 = \mathrm{H}_0^1(\Omega), B_1 = \mathrm{L}^2(\Omega), B_2 = \mathrm{H}^{-1}(\Omega)$, Ω bounded. From the uniform bounds in n, the sequence (u_n) is relatively compact in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$, thus we have that $u_n \to u$ (up to a subsequence) in $\mathrm{L}^2(\mathbf{R}^+ \times \Omega)$.

Non-stationary heat conduction

Consider a domain $Q = \langle 0, T \rangle \times \Omega$, where $\Omega \subseteq \mathbf{R}^d$ is open:

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad\quad u(0,\cdot) = u_0 \ . \end{cases}$$

More precisely: $V := \mathrm{H}^1_0(\Omega)$, $V' := \mathrm{H}^{-1}(\Omega)$ and $H := \mathrm{L}^2(\Omega)$,
Gel'fand triple: $V \hookrightarrow H \hookrightarrow V'$.
For time dependent functions: $\mathcal{V} := \mathrm{L}^2(0,T;V)$, $\mathcal{V}' := \mathrm{L}^2(0,T;V')$ (which is indeed the dual of $\mathcal{V}$) and $\mathcal{H} := \mathrm{L}^2(0,T;H)$, we again have a Gel'fand triple: $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$.
Additionally assume that $\mathbf{A} \in \mathrm{L}^\infty(Q;\mathrm{M}_{d\times d})$ satisfies:

$$\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \alpha|\boldsymbol{\xi}|^2$$
$$\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \frac{1}{\beta}|\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}|^2 \ ,$$

i.e. it belongs to $\mathcal{M}(\alpha,\beta;Q)$.
With such coefficients the problem is well posed, i.e.:
$\|u\|_{\mathcal{W}} \leqslant c_1\|u_0\|_H + c_2\|f\|_{\mathcal{V}'}$, where $\mathcal{W} := \{u \in \mathcal{V} : u' \in \mathcal{V}'\}$.

Parabolic operators

Parabolic operator $\mathcal{P} \in \mathcal{L}(\mathcal{W}; \mathcal{V}')$

$$\mathcal{P}u := \partial_t u - \operatorname{div}(\mathbf{A}\nabla u)$$

is an isomorphisms of $\mathcal{W}_0 := \{u \in \mathcal{W} : u(0,\cdot) = 0\}$ onto $\mathcal{V}'$.
Spagnolo introduced *G-convergence* for more general parabolic operators:

$$\mathcal{P}_{\mathcal{A}} := \partial_t + \mathcal{A} : \mathcal{W} \longrightarrow \mathcal{V}' \ ,$$

where $(\mathcal{A}u)(t) := A(t)u(t)$, with $A(t) \in \mathcal{L}(V; V')$ such that for $\varphi, \psi \in V$

$$t \mapsto \langle A(t)\varphi, \psi\rangle \quad \text{is measurable}$$

$$\lambda_0 \|\varphi\|_V^2 \leqslant \langle A(t)\varphi, \varphi\rangle \leqslant \Lambda_0 \|\varphi\|_V^2$$

$$|\langle A(t)\varphi, \psi\rangle| \leqslant M\sqrt{\langle A(t)\varphi, \varphi\rangle}\sqrt{\langle A(t)\psi, \psi\rangle} \ ,$$

where λ_0, Λ_0 and M are some positive constants.
The set of all such operators $\mathcal{P}_{\mathcal{A}}$ is denoted by $\mathcal{P}(\lambda_0, \Lambda_0, M)$.
For $A(t) = -\operatorname{div}(\mathbf{A}(t,\cdot),\cdot)$ we write $\mathcal{P}_{\mathbf{A}}$ instead of $\mathcal{P}_{\mathcal{A}}$.

G-convergence

The adjoint operator to $\mathcal{A}$ is $\mathcal{A}^* : \mathcal{V}' \longrightarrow \mathcal{V}$:

$$(\forall\, u, v \in \mathcal{V}) \quad {}_{\mathcal{V}'}\langle\, \mathcal{A}u, v \,\rangle_{\mathcal{V}} = {}_{\mathcal{V}}\langle\, u, \mathcal{A}^* v \,\rangle_{\mathcal{V}'} \; .$$

The *formal* adjoint of $\mathcal{P}_{\mathcal{A}}$ is then

$$\mathcal{P}^*_{\mathcal{A}} u = -\partial_t u + \mathcal{A}^* u \; .$$

Note ($\mathcal{W}_T := \{u \in \mathcal{W} : u(T, \cdot) = 0\}$)

$$(\forall\, u \in \mathcal{W}_0)(\forall\, v \in \mathcal{W}_T) \quad {}_{\mathcal{V}'}\langle\, \mathcal{P}_{\mathcal{A}} u, v \,\rangle_{\mathcal{V}} = {}_{\mathcal{V}}\langle\, u, \mathcal{P}^*_{\mathcal{A}} v \,\rangle_{\mathcal{V}'} \; .$$

A sequence $\mathcal{P}_{\mathcal{A}_n} \in \mathcal{P}(\lambda_0, \Lambda_0, M)$ *G-converges* to $\mathcal{P}_{\mathcal{A}}$ (and we write $\mathcal{P}_{\mathcal{A}_n} \xrightharpoonup{G} \mathcal{P}_{\mathcal{A}}$) if for any $f \in \mathcal{V}'$

$$\mathcal{P}^{-1}_{\mathcal{A}_n} f \rightharpoonup \mathcal{P}^{-1}_{\mathcal{A}} f \quad \text{in } \mathcal{W} \; .$$

If $V \hookrightarrow H$ is compact, then $\mathcal{P}^{-1}_{\mathcal{A}_n} f \longrightarrow \mathcal{P}^{-1}_{\mathcal{A}} f$ (strongly!) in $\mathcal{H}$.

Compactness

If $V \hookrightarrow H \hookrightarrow V'$ (continuous inclusions) are also compact, which is the case for our equation ($\mathrm{H}_0^1(\Omega)$ is by the Rellich theorem compactly embedded in $\mathrm{L}^2(\Omega)$), Spagnolo proved the compactness of G-convergence:

For any $\mathcal{P}_{\mathcal{A}_n} \in \mathcal{P}(\lambda_0, \Lambda_0, M)$ there is a subsequence $\mathcal{P}_{\mathcal{A}_{n'}}$ and a $\mathcal{P}_{\mathcal{A}} \in \mathcal{P}(\lambda_0, M^2\Lambda_0, \sqrt{\Lambda_0/\lambda_0}M)$, such that $\mathcal{P}_{\mathcal{A}_{n'}} \xrightarrow{G} \mathcal{P}_{\mathcal{A}}$.

Here we again see that the choice of bounds is not as good as we would wish. Of course, in the particular case we are interested in, the above inequalities are satisfied.

Let us prove this theorem.

Proof of compactness

As $\mathcal{V}$ is reflexive and separable, by the Cantor diagonal procedure one can find a subsequence (for which we shall keep the original notation) and a bounded linear operator $\mathcal{B} : \mathcal{V}' \longrightarrow \mathcal{W}_0$ such that

$$(\forall\, f \in \mathcal{V}') \qquad \mathcal{P}_n^{-1} f \rightharpoonup \mathcal{B} f \ .$$

Indeed, let $\mathcal{F} = \{f_1, f_2, \ldots\}$ be a countable dense subset of $\mathcal{V}'$. The sequence $\mathcal{P}_{\mathcal{A}_n}^{-1} f_1$ is bounded in $\mathcal{W}_0$, so it has a weakly convergent subsequence; denote it by $\mathcal{P}_{\mathcal{A}_{n_k^1}}^{-1} f_1$.

Next we apply the same argument to the subsequence $\mathcal{P}_{\mathcal{A}_{n_k^1}}^{-1} f_2$, which has a weakly convergent subsequence $\mathcal{P}_{\mathcal{A}_{n_k^2}}^{-1} f_2$.

Continuing in the same way, for each $m \in \mathbf{N}$ we have a weakly convergent subsequence $\mathcal{P}_{\mathcal{A}_{n_k^m}}^{-1} f_l$, for $l \in 1..m$.

Now we can construct a diagonal subsequence by taking $\mathcal{P}_{\mathcal{A}_{n_k^k}}^{-1}$.

Proof of compactness (cont.)

We have to show that $\mathcal{B}$ is a parabolic operator.
To this end define: $\mathcal{A}_n v := A_n(t)v$, $u_n := \mathcal{P}_n^{-1} f$ and $\mathcal{K}f := f - (\mathcal{B}f)'$.
Thus we get:

$$u_n' + \mathcal{A}_n u_n = f$$
$$(\mathcal{B}f)' + \mathcal{K}f = f$$

After multiplying the first by u_n and the second by $\mathcal{B}f$, and taking into account the initial condition, we get

$$\frac{1}{2}\|u_n(T)\|_H^2 + {}_{\mathcal{V}'}\langle\, \mathcal{A}_n u_n, u_n \,\rangle_{\mathcal{V}} = {}_{\mathcal{V}'}\langle\, f, u_n \,\rangle_{\mathcal{V}}$$
$$\frac{1}{2}\|(\mathcal{B}f)(T)\|_H^2 + {}_{\mathcal{V}'}\langle\, \mathcal{K}f, \mathcal{B}f \,\rangle_{\mathcal{V}} = {}_{\mathcal{V}'}\langle\, f, \mathcal{B}f \,\rangle_{\mathcal{V}} \;.$$

As $u_n = \mathcal{P}_n^{-1} f \rightharpoonup \mathcal{B}f$ u $\mathcal{W}$, so

$${}_{\mathcal{V}'}\langle\, \mathcal{A}_n u_n, u_n \,\rangle_{\mathcal{V}} \rightharpoonup {}_{\mathcal{V}'}\langle\, \mathcal{K}f, \mathcal{B}f \,\rangle_{\mathcal{V}} \;,$$

thus $u_n' \rightharpoonup (\mathcal{B}f)'$ and $\mathcal{A}_n u_n \rightharpoonup f - (\mathcal{B}f) = \mathcal{K}f$ in $\mathcal{V}'$.

$\mathcal{B}$ *is injective.*

Indeed, take $\mathcal{B}f = 0$, thus ${}_{\mathcal{V}'}\langle\, \mathcal{A}_n u_n, u_n \,\rangle_{\mathcal{V}} \rightharpoonup 0$. However, from the coercitivity then $\|u_n\|_{\mathcal{V}} \longrightarrow 0$, so $\mathcal{K}f = 0$, giving $f - (\mathcal{B}f)' = 0$.

Proof of compactness (cont.)

$\mathcal{B}(\mathcal{V}')$ is dense in $\mathcal{V}$.
Indeed, if $(\forall\, g \in \mathcal{V}')\ {}_{\mathcal{V}'}\langle\, \mathcal{K}f, \mathcal{B}g\,\rangle_{\mathcal{V}} = 0$, then in particular ${}_{\mathcal{V}'}\langle\, \mathcal{K}f, \mathcal{B}f\,\rangle_{\mathcal{V}} = 0$, so ${}_{\mathcal{V}'}\langle\, \mathcal{K}f, \mathcal{B}f\,\rangle_{\mathcal{V}} \leqslant 0$, and by coercitivity $\|u_n\|_{\mathcal{V}} \longrightarrow 0$, finally giving $\mathcal{B}f = 0$ and $f = 0$.
We next define $\mathcal{A} : \mathcal{B}(\mathcal{V}') \longrightarrow \mathcal{V}'$ by

$$\mathcal{A}(\mathcal{B}f) := \mathcal{K}f\ ,$$

such that for $u := \mathcal{B}f$ we have

$$u' + \mathcal{A}u = f\ ,$$

$\mathcal{A}_n u_n \rightharpoonup \mathcal{A}u$ and ${}_{\mathcal{V}'}\langle\, \mathcal{A}_n u_n, u_n\,\rangle_{\mathcal{V}} \longrightarrow {}_{\mathcal{V}'}\langle\, \mathcal{A}u, u\,\rangle_{\mathcal{V}}$.
Now we can use the uniform estimates on $\mathcal{A}_n$:

$$|{}_{\mathcal{V}'}\langle\, \mathcal{A}_n u_n, v\,\rangle_{\mathcal{V}}| \leqslant M\sqrt{{}_{\mathcal{V}'}\langle\, \mathcal{A}_n u_n, u_n\,\rangle_{\mathcal{V}}}\sqrt{\Lambda_0}\|v\|_{\mathcal{V}}$$
$$ {}_{\mathcal{V}'}\langle\, \mathcal{A}_n u_n, u_n\,\rangle_{\mathcal{V}} \geqslant \lambda_0\|u_n\|^2_{\mathcal{V}}\ .$$

Proof of compactness (cont.)

After passing to the limit ($\|u\| \leqslant \liminf \|u_n\|$):

$$|_{\mathcal{V}'}\langle \mathcal{A}_n u, v \rangle_{\mathcal{V}}| \leqslant M\sqrt{_{\mathcal{V}'}\langle \mathcal{A}u, u \rangle_{\mathcal{V}}}\sqrt{\Lambda_0}\|v\|_{\mathcal{V}}$$

$$_{\mathcal{V}'}\langle \mathcal{A}u, u \rangle_{\mathcal{V}} \geqslant \lambda_0\|u\|_{\mathcal{V}}^2 \, .$$

Taking $v = u$ in the first inequality:

$$|_{\mathcal{V}'}\langle \mathcal{A}u, u \rangle_{\mathcal{V}}| \leqslant M\sqrt{\Lambda_0}\sqrt{_{\mathcal{V}'}\langle \mathcal{A}u, u \rangle_{\mathcal{V}}}\|u\|_{\mathcal{V}} \, ,$$

so $_{\mathcal{V}'}\langle \mathcal{A}u, u \rangle_{\mathcal{V}} \leqslant M^2\Lambda_0\|u\|_{\mathcal{V}}^2$, and then again

$$|_{\mathcal{V}'}\langle \mathcal{A}_n u, v \rangle_{\mathcal{V}}| \leqslant M^2\Lambda_0\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}} \, .$$

Therefore $\mathcal{A} : \mathcal{B}(\mathcal{V}') \longrightarrow \mathcal{V}'$ is bounded in the topology of $\mathcal{V}$, so it can be extended by continuity to all of $\mathcal{V}$, and the extension $\tilde{\mathcal{A}}$ defines an isomorphism $d/dt + \tilde{\mathcal{A}}$ of $\mathcal{W}_0$ onto $\mathcal{V}'$.

In particular:

$$(\frac{d}{dt} + \tilde{\mathcal{A}})\mathcal{B}f = f \, , \qquad f \in \mathcal{V}$$

implies $\mathcal{B} = (d/dt + \tilde{\mathcal{A}})^{-1}$.

Finally, the form of operator: $(\mathcal{A}u)(t) := A(t)u(t)$ is a consequence of the fact that it commutes with multiplication by bounded functions in t (as a weak limit of such operators).

And the Theorem is proven.

H-convergence

If each A_n is of the form: $A_n(t)u = -\text{div}\,(\mathbf{A}_n(t,\cdot)\nabla u) \quad , u \in V$, the limit is of the same form, where the matrix coefficients $\mathbf{A}$ satisfy the same type of bounds, but with different constants. Also, in such a case, on the subsequence we have the convergence

$$\mathbf{A}_{n'}\nabla u_{n'} \rightharpoonup \mathbf{A}\nabla u \quad \text{in } \mathrm{L}^2(Q;\mathbf{R}^d)\,.$$

The above motivates the following definition [DM, ŽKO]:

A sequence of matrix functions $\mathbf{A}_n \in \mathcal{M}(\alpha,\beta;Q)$ H-converges to $\mathbf{A} \in \mathcal{M}(\alpha',\beta';Q)$ if for any $f \in \mathcal{V}'$ and $u_0 \in H$ the solutions of parabolic problems

$$\begin{cases} \partial_t u_n - \text{div}\,(\mathbf{A}_n\nabla u_n) = f \\ \qquad\qquad\qquad u_n(0,\cdot) = u_0\,. \end{cases}$$

satisfy

$$u_n \rightharpoonup u \quad \textit{in } \mathcal{V}$$

$$\mathbf{A}_n\nabla u_n \rightharpoonup \mathbf{A}\nabla u \quad \textit{in } \mathrm{L}^2(Q;\mathbf{R}^d)\,.$$

Moreover, $\mathbf{A} \in \mathcal{M}(\alpha,\beta;Q)$.

H-convergence still has the advantage of the proper choice of constants (the limit stays in the chosen set).

Remark 1

In the definition of H-convergence it is enough to consider $u_0 = 0$.

Indeed, assume it is valid only for the homogeneous initial condition. For $u_0 \in H$ and $f \in \mathcal{V}'$, let u_n be the solution.
The sequence of solutions (u_n) is bounded in $\mathcal{W}$ and, due to the reflexivity, has a weakly converging subsequence. Its limit we denote by u.
Applying the locality of G-convergence we get that $u_t - \operatorname{div}(\mathbf{A}\nabla u) = f$ on Q.
As the imbedding $\mathcal{W} \hookrightarrow \mathrm{C}([0,T]; H)$ is compact, we have the strong convergence of the subsequence in $\mathrm{C}([0,T]; H)$. This means that the initial condition is preserved on the limit, i.e. $u(0, \cdot) = u_0$.
Thus, any weak accumulation point of (u_n) satisfies the equation with the initial condition u_0, and therefore the accumulation point is unique, which means that the whole sequence converges weakly to that solution u in $\mathcal{W}$.
Now we obtain $\mathbf{A}_n\nabla u_n \rightharpoonup \mathbf{A}\nabla u$ in $\mathrm{L}^2(Q; \mathbf{R}^d)$ for a subsequence. An argument as above, based on the uniqueness of the accumulation point, gives us finally that the whole sequence converges, i.e. that $\mathbf{A}_n \xrightarrow{H} \mathbf{A}$.

Remark 2

The parabolic H-convergence is generated by a topology.

$$X := \bigcup_{n\in\mathbf{N}} \mathcal{M}(\frac{1}{n}, n; Q) ,$$

for $f \in \mathcal{V}'$, define $R_f : X \longrightarrow \mathcal{W}_0$ and $Q_f : X \longrightarrow \mathrm{L}^2(Q; \mathbf{R}^d)$:

$$R_f(\mathbf{A}) := u , \quad \text{where } u \text{ solves} \quad \begin{cases} u_t - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ u(0,\cdot) = 0 \end{cases} ,$$

with the weak topology assumed on $\mathcal{W}_0$;
and $Q_f(\mathbf{A}) := \mathbf{A}\nabla u$, with the weak topology on $\mathrm{L}^2(Q; \mathbf{R}^d)$.
On X, define the weakest topology such that R_f and Q_f are continuous. It is not metrisable.

However, the relative topology on $\mathcal{M}(\alpha, \beta; Q)$ is metrisable.

H-convergent sequence depending on a parameter

Theorem. *Let $P \subseteq \mathbf{R}$ be an open set and the sequence $\mathbf{A}_n : Q \times P \to \mathrm{M}_{d\times d}(\mathbf{R})$ such that $\mathbf{A}_n(\cdot, p) \in \mathcal{M}(\alpha, \beta; Q)$ for $p \in P$. Moreover, suppose that $p \mapsto \mathbf{A}_n(\cdot, p)$ is a C^k mapping from P to $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d}(\mathbf{R}))$ with derivatives up to order k equicontinuous on every compact set $K \subseteq P$:*

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)(\forall\, p, q \in K)(\forall\, n \in \mathbf{N})(\forall\, i \in \{0, \dots, k\})$$
$$|p - q| < \delta \quad \Longrightarrow \quad \|\mathbf{A}_n^{(i)}(\cdot, p) - \mathbf{A}_n^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d}(\mathbf{R}))} < \varepsilon\,.$$

Then, there exists a subsequence $(\mathbf{A}_{n_k})$ such that for every $p \in P$

$$\mathbf{A}_{n_k}(\cdot, p) \xrightarrow{H} \mathbf{A}(\cdot, p) \ \text{ in } \mathcal{M}(\alpha, \beta; Q)\,,$$

and $p \mapsto \mathbf{A}(\cdot, p)$ is a C^k mapping from P to $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d}(\mathbf{R}))$. ∎

Analytical dependence

Theorem. *Let* $P \subseteq \mathbf{R}$ *be open set and the sequence* $\mathbf{A}_n : Q \times P \to \mathrm{M}_{d\times d}(\mathbf{R})$ *such that* $\mathbf{A}_n(\cdot, p) \in \mathcal{M}(\alpha, \beta; Q)$ *for* $p \in P$*. Moreover, suppose that* $p \mapsto \mathbf{A}_n(\cdot, p)$ *is analytic mapping from* P *to* $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d}(\mathbf{R}))$*.*
Then, there exists a subsequence $(\mathbf{A}_{n_k})$ *such that for every* $p \in P$

$$\mathbf{A}_{n_k}(\cdot, p) \xrightarrow{H} \mathbf{A}(\cdot, p) \ \text{ in } \mathcal{M}(\alpha, \beta; Q)\,,$$

and $p \mapsto \mathbf{A}(\cdot, p)$ *is analytic mapping from* P *to* $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d}(\mathbf{R}))$*.* ∎

Homogenisation: parabolic H-convergence
A brief review of elliptic theory
Existence and uniqueness for parabolic equation
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Model based on Stokes equation
Model based on time-dependent Stokes

Setting of the problem

A sequence of parabolic problems

$$(*) \qquad \begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}^n \nabla u_n) = f \\ \qquad\qquad u_n(0,\cdot) = u_0 \ . \end{cases}$$

where $\mathbf{A}^n$ is a perturbation of $\mathbf{A}_0 \in \mathrm{C}(Q; \mathrm{M}_{d\times d})$, which is bounded from below; for small γ function $\mathbf{A}^n$ is analytic in γ:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma \mathbf{B}^n(t,\mathbf{x}) + \gamma^2 \mathbf{C}^n(t,\mathbf{x}) + o(\gamma^2) \ ,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{0}$ in $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d})$).
Then (after passing to a subsequence if needed)

$$\mathbf{A}^n_\gamma \xrightarrow{H} \mathbf{A}^\infty_\gamma = \mathbf{A}_0 + \gamma \mathbf{B}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2) \ ;$$

the limit being measurable in $t, \mathbf{x}$, and analytic in γ.

No first-order term on the limit

Theorem. *The effective conductivity matrix $\mathbf{A}_\gamma^\infty$ admits the expansion*

$$\mathbf{A}_\gamma^\infty(t,\mathbf{x}) = \mathbf{A}_0(t,\mathbf{x}) + \gamma^2 \mathbf{C}_0(t,\mathbf{x}) + o(\gamma^2) \;.$$

∎

Indeed, take $u \in \mathrm{L}^2([0,T];\mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle;\mathrm{H}^{-1}(\Omega))$, and define $f_\gamma := \partial_t u - \operatorname{div}(\mathbf{A}_\gamma^\infty \nabla u)$, and $u_0 := u(0,\cdot) \in \mathrm{L}^2(\Omega)$.

Next, solve $(*)$ with $\mathbf{A}_\gamma^n$, f_γ and u_0, the solution u_γ^n.
Of course, f_γ and u_γ^n analytically depend on γ.

Because of H-convergence, we have the weak convergences in $\mathrm{L}^2(Q)$:

$$(\dagger) \qquad \begin{aligned} \mathsf{E}_\gamma^n &:= \nabla u_\gamma^n \rightharpoonup \nabla u \\ \mathsf{D}_\gamma^n &:= \mathbf{A}_\gamma^n \mathsf{E}_\gamma^n \rightharpoonup \mathbf{A}_\gamma^\infty \nabla u \;. \end{aligned}$$

Expansions in Taylor serieses (similarly for f_γ and u_γ^n):

$$\begin{aligned} \mathsf{E}_\gamma^n &= \mathsf{E}_0^n + \gamma \mathsf{E}_1^n + \gamma^2 \mathsf{E}_2^n + o(\gamma^2) \\ \mathsf{D}_\gamma^n &= \mathsf{D}_0^n + \gamma \mathsf{D}_1^n + \gamma^2 \mathsf{D}_2^n + o(\gamma^2) \;. \end{aligned}$$

No first-order term on the limit (cont.)

Inserting (†) and equating the terms with equal powers of γ:

$$\mathsf{E}_0^n = \nabla u \,, \qquad \mathsf{D}_0^n = \mathbf{A}_0 \nabla u$$
$$\mathsf{D}_1^n = \mathbf{A}_0 \mathsf{E}_1^n + \mathbf{B}^n \nabla u \rightharpoonup 0 \quad \text{in } \mathrm{L}^2(Q) \,.$$

Also, D_1^n converges to $\mathbf{B}_0 \nabla u$ (the term in expansion with γ^1)

$$\mathsf{D}_\gamma^n \rightharpoonup \mathbf{A}_\gamma^\infty \nabla u = \mathbf{A}_0 \nabla u + \gamma \mathbf{B}_0 \nabla u + \gamma^2 \mathbf{C}_0 \nabla u + o(\gamma^2) \,.$$

Thus $\mathbf{B}_0 \nabla u = 0$, and as $u \in \mathrm{L}^2([0,T]; \mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle; \mathrm{H}^{-1}(\Omega))$ was arbitrary, we conclude that $\mathbf{B}_0 = \mathbf{0}$.
For the quadratic term we have:

$$\mathsf{D}_2^n = \mathbf{A}_0 \mathsf{E}_2^n + \mathbf{B}^n \mathsf{E}_1^n + \mathbf{C}^n \nabla u \rightharpoonup \lim \mathbf{B}^n \mathsf{E}_1^n = \mathbf{C}_0 \nabla u \,,$$

and this is the limit we still have to compute.

Periodic homogenisation — an example

In the periodic case the explicit formulae for the homogenisation limit are known [BLP].
Together with Fourier analysis:
leading terms in expansion for the small amplitude homogenisation limit.

Periodic functions—functions defined on $T := S^1 = \mathbf{R}/\mathbf{Z}$, $Y := \mathbf{R}^d/\mathbf{Z}^d$ and $Z := \mathbf{R}^{1+d}/\mathbf{Z}^{1+d}$
We implicitly assume projections of $\mathbf{x} \mapsto \mathbf{y} \in Y$, etc.
For given $\rho \in \langle 0, \infty \rangle$ we define the sequence $\mathbf{A}_n$ by

$$\mathbf{A}_n(t, \mathbf{x}) = \mathbf{A}(n^\rho t, n\mathbf{x}) \,.$$

Then $\mathbf{A}_n$ H-converges to a constant $\mathbf{A}_\infty$ defined by

$$\mathbf{A}_\infty \mathsf{h} = \int_Z \mathbf{A}(\tau, \mathbf{y})(\mathsf{h} + \nabla w(\tau, \mathbf{y}))\, d\tau d\mathbf{y} \,.$$

For given h, w is a solution of some BVP, depending on ρ.

Three different cases depending on ρ

$\rho \in \langle 0, 2\rangle$: $w(\tau, \cdot)$ is the unique solution of

$$-\mathsf{div}\,(\mathbf{A}(\tau,\cdot)(\mathsf{h} + \nabla w(\tau,\cdot))) = 0$$
$$w(\tau,\cdot) \in \mathrm{H}^1(Y)\,,\ \int_Y w(\tau,\mathbf{y})\,d\mathbf{y} = 0\,,$$

$\rho = 2$: w is defined by

$$\partial_t w - \mathsf{div}\,(\mathbf{A}(\mathsf{h} + \nabla w)) = 0$$
$$w \in \mathrm{L}^2(T;\mathrm{H}^1(Y))\,,\ \partial_t w \in \mathrm{L}^2(T;\mathrm{H}^{-1}(Y))\,,\ \int_Z w\,d\tau d\mathbf{y} = 0\,.$$

$\rho \in \langle 2, \infty\rangle$: define $\widetilde{\mathbf{A}}(y) = \int_0^1 \mathbf{A}(\tau,\mathbf{y})\,d\tau$ and w as the solution of

$$-\mathsf{div}\,(\widetilde{\mathbf{A}}(\mathsf{h} + \nabla w)) = 0$$
$$w \in \mathrm{H}^1(Y)\,,\ \int_Y w\,d\mathbf{y} = 0\,.$$

Periodic small-amplitude homogenisation

A sequence of small perturbations of a constant coercive matrix $\mathbf{A}_0 \in \mathrm{M}_{d\times d}$:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma\mathbf{B}^n(t,\mathbf{x})\,,$$

where $\mathbf{B}^n(t,\mathbf{x}) = \mathbf{B}(n^\rho t, n\mathbf{x})$, $\mathbf{B}$ is Z-periodic L^∞ matrix function satisfying $\int_Z \mathbf{B}\,d\tau d\mathbf{y} = 0$.
For γ small enough, (eventually passing to a subsequence) we have the H-convergence to a limit depending analytically on γ:

$$\mathbf{A}^n_\gamma \xrightarrow{H} \mathbf{A}^\infty_\gamma = \mathbf{A}_0 + \gamma\mathbf{B}_0 + \gamma^2\mathbf{C}_0 + o(\gamma^2)$$

and a formula for $\mathbf{A}^\infty_\gamma$:

$$\begin{aligned}\mathbf{A}^\infty_\gamma \mathsf{h} &= \int_Z (\mathbf{A}_0 + \gamma\mathbf{B}))(\mathsf{h} + \nabla w_\gamma)\,d\tau d\mathbf{y}\\ &= \mathbf{A}_0\mathsf{h} + \int_Z \mathbf{A}_0\nabla w_\gamma + \gamma\int_Z \mathbf{B}\mathsf{h} + \gamma\int_Z \mathbf{B}\nabla w_\gamma = \mathbf{A}_0\mathsf{h} + \gamma\int_Z \mathbf{B}\nabla w_\gamma\,.\end{aligned}$$

Periodic small-amplitude homogenisation (cont.)

In the last equality the second term equals zero by Gauss' theorem, as w_γ is a periodic function. Similarly for the third term.

Since w_γ is a solution of some (initial–)boundary value problem, depending on ρ, it also depends analytically on γ:

$$w_\gamma = w_0 + \gamma w_1 + o(\gamma)\,.$$

The first order term vanishes, as $\mathbf{A}_0$ is constant.

$$\mathbf{A}_\gamma^\infty \mathsf{h} = \mathbf{A}_0 \mathsf{h} + \gamma^2 \int_Z \mathbf{B} \nabla w_1 + o(\gamma^2)\,,$$

so we conclude that $\mathbf{B}_0 = \mathbf{0}$ and $\mathbf{C}_0 \mathsf{h} = \int_Z \mathbf{B} \nabla w_1$.
From this formula, using the Fourier series, one can calculate the second-term approximation $\mathbf{C}_0$. Off course, we must treat separately each one of the above three cases for ρ.

The case $\rho \in \langle 0, 2\rangle$ on the limit

Fix $\tau \in [0,1]$; the BVP with coefficient $\mathbf{A}_0 + \gamma\mathbf{B}$ instead of $\mathbf{A}$ and the above expression for w, we see that w_1 solves

$$(\ddagger)\quad -\mathsf{div}\left(\mathbf{A}_0 \nabla w_1(\tau,\cdot)\right) = \mathsf{div}\left(\mathbf{B}\mathsf{h}\right),\ w_1(\tau,\cdot) \in \mathrm{H}^1(Y),\ \int_Y w_1(\tau,\mathbf{y})\,d\mathbf{y} = 0$$

Expanding w_1 in the Fourier series gives us $(J = \mathbf{Z} \times (\mathbf{Z}^d \setminus \{\mathbf{0}\}))$

$$w_1 = \sum_{(l,\mathsf{k})\in J} a_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})},$$

because of $\int_Y w_1(\tau,\mathbf{y})\,d\mathbf{y} = 0$.
Straightforward calculation gives us

$$\nabla w_1 = \sum_J 2\pi i \mathsf{k}\, a_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})},$$

$$\mathsf{div}\, \mathbf{A}_0 \nabla w_1 = \sum_J (2\pi i)^2 \mathbf{A}_0 \mathsf{k}\cdot\mathsf{k}\, a_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}.$$

The case $\rho \in \langle 0, 2\rangle$ on the limit (cont.)

For $\mathbf{B}$ denote $I := \mathbf{Z}^{d+1} \setminus \{\mathbf{0}\}$

$$\mathbf{B} = \sum_I \mathbf{B}_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}\,,$$

$$\operatorname{div} \mathbf{B}\mathsf{h} = \sum_I 2\pi i\, \mathbf{B}_{l\mathsf{k}}\mathsf{h}\cdot\mathsf{k}\, e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}\,.$$

(‡) leads to a relation among corresponding Fourier coefficients

$$2\pi i \mathbf{A}_0 \mathsf{k}\cdot\mathsf{k}\, a_{l\mathsf{k}} = -\mathbf{B}_{l\mathsf{k}}\mathsf{h}\cdot\mathsf{k}\,, \quad (l,\mathsf{k}) \in \mathbf{Z}^{d+1}\,,$$

$$\text{i.e.} \quad a_{l\mathsf{k}} = \begin{cases} -\dfrac{\mathbf{B}_{l\mathsf{k}}\mathsf{h}\cdot\mathsf{k}}{2\pi i \mathbf{A}_0\mathsf{k}\cdot\mathsf{k}}\,, & (l,\mathsf{k}) \in J \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Finally, we obtain

$$\begin{aligned} \mathbf{C}_0\mathsf{h} &= \int_Z \mathbf{B}\nabla w_1\, d\tau d\mathbf{y} \\ &= \int_Z \left(\sum_I \mathbf{B}_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}\right)\left(\sum_J (2\pi i \mathsf{k}') a_{l'\mathsf{k}'} e^{2\pi i (l'\tau + \mathsf{k}'\cdot\mathbf{y})}\right) d\tau d\mathbf{y} \end{aligned}$$

The case $\rho \in \langle 0, 2 \rangle$ on the limit (cont.)

Due to orthogonality, for the non-vanishing terms in the above product of two series we have $l' = -l$ and $\mathsf{k}' = -\mathsf{k}$. Therefore,

$$\begin{aligned} \mathbf{C}_0 \mathsf{h} &= -2\pi i \sum_J \mathbf{B}_{l\mathsf{k}} \mathsf{k} a_{-l,-\mathsf{k}} \\ &= -\sum_J \mathbf{B}_{l\mathsf{k}} \mathsf{k} \frac{\mathbf{B}_{-l,-\mathsf{k}} \mathsf{h} \cdot \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}} \mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}} \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} \mathsf{h}\,, \end{aligned}$$

where the last equality holds since $\mathbf{B}$ is a real matrix function i.e. $\overline{\mathbf{B}_{l\mathsf{k}}} = \mathbf{B}_{-l,-\mathsf{k}}$. We conclude

$$\mathbf{C}_0 = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}} \mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}} \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}}\,.$$

The case $\rho = 2$ on the limit

The calculation is similar to the first case. The only difference appears in the equation for $w_1 = \sum_{(l,\mathsf{k})\in I} a_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}$:

$$\partial_\tau w_1 - \mathsf{div}\,(\mathbf{A}_0 \nabla w_1(\tau,\cdot)) = \mathsf{div}\,(\mathbf{B}\mathsf{h})\,,$$

implying the following relation for the Fourier coefficients

$$(l - 2\pi i \mathbf{A}_0 \mathsf{k}\cdot\mathsf{k} a_{l\mathsf{k}}) = \mathbf{B}_{l\mathsf{k}}\mathsf{h}\cdot\mathsf{k}\,,\quad (l,\mathsf{k}) \in I\,,$$

and the formula for the second order approximation of the H-limit:

$$\mathbf{C}_0 = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}}\mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}}\mathsf{k}}{\frac{l}{2\pi i} + \mathbf{A}_0\mathsf{k}\cdot\mathsf{k}}\,.$$

The case $\rho \in \langle 2, \infty \rangle$ on the limit

In this case w_1 does not depend on τ. Introducing

$$\widetilde{\mathbf{B}}(\mathbf{y}) := \int_0^1 \mathbf{B}(\tau, \mathbf{y})\, d\tau$$

this case actually has the same behaviour as the one in elliptic setting, giving the formula

$$\mathbf{C}_0 = - \sum_{\mathbf{Z}^d \setminus \{\mathbf{0}\}} \frac{\widetilde{\mathbf{B}}_{\mathsf{k}} \mathsf{k} \otimes \widetilde{\mathbf{B}}_{\mathsf{k}} \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} \,.$$

Parabolic small-amplitude homogenisation—general case

Let us continue what we were doing before ...
For the quadratic term we have:

$$\mathsf{D}_2^n = \mathbf{A}_0\mathsf{E}_2^n + \mathbf{B}^n\mathsf{E}_1^n + \mathbf{C}^n\nabla u \longrightarrow \lim \mathbf{B}^n\mathsf{E}_1^n = \mathbf{C}_0\nabla u \;,$$

and this is the limit we shall express using only the parabolic variant H-measure $\boldsymbol{\mu}$.

u_1^n satisfies the equation $(*)$ with coefficients $\mathbf{A}_0$, div $(\mathbf{B}^n\nabla u)$ on the right hand side and the homogeneous innitial condition.

By applying the Fourier transform (as if the equation were valid in the whole space), and multiplying by $2\pi i\boldsymbol{\xi}$, for $(\tau,\boldsymbol{\xi}) \neq (0,0)$ we get

$$\widehat{\nabla u_1^n}(\tau,\boldsymbol{\xi}) = -\frac{(2\pi)^2\,(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\,(\widehat{\mathbf{B}^n\nabla u})(\tau,\boldsymbol{\xi})}{2\pi i\tau + (2\pi)^2\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}}.$$

Expression for the quadratic correction

As $(\boldsymbol{\xi}\otimes\boldsymbol{\xi})/(2\pi i\tau+(2\pi)^2\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi})$ is constant along branches of paraboloids $\tau=c\boldsymbol{\xi}^2, c\in\overline{\mathbf{R}}$, we have ($\varphi\in \mathrm{C}_c^\infty(Q)$)

$$\lim_n\Big\langle\, \varphi\mathbf{B}^n \mid \nabla u_1^n \,\Big\rangle = -\lim_n\Big\langle\, \widehat{\varphi\mathbf{B}^n} \mid \frac{(2\pi)^2\,(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\,(\widehat{\mathbf{B}^n\nabla u})}{2\pi i\tau+(2\pi)^2\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}} \,\Big\rangle$$

$$= -\Big\langle \boldsymbol{\mu}, \varphi\frac{(2\pi)^2\boldsymbol{\xi}\otimes\boldsymbol{\xi}\otimes\nabla u}{-2\pi i\tau+(2\pi)^2\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}}\Big\rangle,$$

where $\boldsymbol{\mu}$ is the parabolic variant H-measure associated to $(\mathbf{B}^n)$, a measure with four indices (the first two of them not being contracted above).

Expression for the quadratic correction (cont.)

By varying function $u \in \mathrm{C}^1(Q)$ (e.g. choosing ∇u constant on $\langle 0, T \rangle \times \omega$, where $\omega \subseteq \Omega$) we get

$$\int_{\langle 0,T \rangle \times \omega} C_0^{ij}(t, \mathbf{x}) \phi(t, \mathbf{x}) dt d\mathbf{x} = -\Big\langle \boldsymbol{\mu}^{ij}, \phi \frac{(2\pi)^2 \boldsymbol{\xi} \otimes \boldsymbol{\xi}}{-2\pi i \tau + (2\pi)^2 \mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} \Big\rangle,$$

where $\boldsymbol{\mu}^{ij}$ denotes the matrix measure with components $\left(\boldsymbol{\mu}^{ij}\right)_{kl} = \mu_{iklj}$.

Examples

For the periodic example of small-amplitude homogenisation, we get the same results by applying the variant H-measures, as with direct calculations performed above.

Homogenisation: parabolic H-convergence
- A brief review of elliptic theory
- Existence and uniqueness for parabolic equation
- More realistic assumptions
- Homogenisation of non-stationary heat conduction
- G-convergence
- Proof of compactness
- H-convergence

Small-amplitude homogenisation of heat equation
- Setting of the problem (parabolic case)
- Periodic small-amplitude homogenisation
- Application in small-amplitude homogenisation

Homogenisation of a model based on the Stokes equation
- Model based on Stokes equation
- Model based on time-dependent Stokes

Stationary case

(Tartar, 1976 and 1984)
$\Omega \subseteq \mathbf{R}^3$ open set, $\mathsf{u}_n \rightharpoonup \mathsf{u}_0$ in $\mathrm{H}^1_{\mathrm{loc}}(\Omega;\mathbf{R}^3)$

$$\begin{cases} -\nu\triangle \mathsf{u}_n + \mathsf{u}_n \times \operatorname{rot}(\mathsf{v}_0 + \lambda \mathsf{v}_n) + \nabla p_n = \mathsf{f}_n \\ \operatorname{div} \mathsf{u}_n = 0\ . \end{cases}$$

Not a realistic model, but contains the terms: $\mathsf{u} \times \operatorname{rot} \mathsf{A}$ resulting from the Lorentz force $q(\mathsf{u} \times \mathsf{B})$ in electrostatics, or in fluids $(\nabla \mathsf{u})\mathsf{u} = \mathsf{u} \times \operatorname{rot}(-\mathsf{u}) + \nabla \frac{|\mathsf{u}|^2}{2}$.

Theorem. *There is a subsequence and* $\mathbf{M} \geqslant 0$, *depending on the choice of the subsequence, such that the limit* u_0 *satisfies:*

$$\begin{cases} -\nu\triangle \mathsf{u}_0 + \mathsf{u}_0 \times \operatorname{rot} \mathsf{v}_0 + \lambda^2 \mathbf{M}\mathsf{u}_0 + \nabla p_0 = \mathsf{f}_0 \\ \operatorname{div} \mathsf{u}_0 = 0\ , \end{cases}$$

and it holds:

$$\nu|\nabla \mathsf{u}_n|^2 \rightharpoonup \nu|\nabla \mathsf{u}_0|^2 + \lambda^2 \mathbf{M}\mathsf{u}_0 \cdot \mathsf{u}_0 \qquad \text{in } \mathcal{D}'(\Omega)\ .$$

∎

Explicit formula via H-measures

Can $\mathbf{M}$ be computed directly from $\mathsf{v}_n \longrightarrow 0$ in $\mathrm{L}^2(\Omega;\mathbf{R}^3)$ (also bounded in $\mathrm{L}^3(\Omega;\mathbf{R}^3)$)? Yes! (Tartar, 1990)

$$\mathbf{M} = \frac{1}{\nu}\left\langle\!\left\langle \boldsymbol{\mu}, (\mathsf{v}^2 - (\mathsf{v}\cdot\boldsymbol{\xi})^2)\boldsymbol{\xi}\otimes\boldsymbol{\xi}\right\rangle\!\right\rangle \;.$$

Note. The meaning of the formula: $(\forall\,\varphi \in \mathrm{C}_c^\infty(\Omega))$

$$\int_\Omega \mathbf{M}(\mathbf{x})\varphi(\mathbf{x})\,d\mathbf{x} = \frac{1}{\nu}\left[\langle \mathrm{tr}\boldsymbol{\mu}, \varphi \boxtimes (\boldsymbol{\xi}\otimes\boldsymbol{\xi})\rangle - \langle \boldsymbol{\mu}, \varphi \boxtimes (\boldsymbol{\xi}\otimes\boldsymbol{\xi})\otimes(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\rangle\right] \;.$$

$\mathbf{M}$ is not only a measure, but a function.

What in the time-dependent case?

Stationary model motivated the introduction of H-measures.
Time-dependent led to a variant.
Tartar with Chun Liu and Konstantina Trevisa some twenty years ago; only written record in Multiscales 2000.
M. Lazar and myself — wrote it down (technical difference in the scaling).

Time dependent case

On $\mathbf{R}^3$ (we need good estimates for the pressure).

Tartar's model from 1985:

$$\begin{cases} \partial_t \mathsf{u}_n - \nu \triangle \mathsf{u}_n + \mathsf{u}_n \times \operatorname{rot}(\mathsf{v}_0 + \lambda \mathsf{v}_n) + \nabla p_n = \mathsf{f}_n \\ \operatorname{div} \mathsf{u}_n = 0 \,. \end{cases}$$

Assume that

$$\mathsf{u}_n \longrightarrow \mathsf{u}_0 \quad \text{in } \mathrm{L}^2([0,T];\mathrm{H}^1(\mathbf{R}^3;\mathbf{R}^3)) \,,$$

$$\mathsf{u}_n \overset{*}{\longrightarrow} \mathsf{u}_0 \quad \text{in } \mathrm{L}^\infty([0,T];\mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3)) \,.$$

and (p_n) is bounded in $\mathrm{L}^2([0,T] \times \mathbf{R}^3)$.

Oscillation in (v_n) generates oscillation in $(\nabla \mathsf{u}_n)$, which dissipates energy via viscosity.

This should be visible from the macroscopic equation (the equation satisfied by u_0).

Sufficient assumptions on v_n and f_n

$\mathsf{f}_n = \operatorname{div} \mathbf{G}_n$, with $\mathbf{G}_n \longrightarrow \mathbf{G}_0$ in $\mathrm{L}^2([0,T] \times \mathbf{R}^3; \mathrm{M}_{3\times 3})$

$\mathsf{v}_0 \in \mathrm{L}^2([0,T]; \mathrm{L}^\infty(\mathbf{R}^3; \mathbf{R}^3)) + \mathrm{L}^\infty([0,T]; \mathrm{L}^3(\mathbf{R}^3; \mathbf{R}^3))$

$\mathsf{v}_n = \mathsf{a}_n + \mathsf{b}_n$, where

$\mathsf{a}_n \overset{*}{\longrightarrow} 0$ in $\mathrm{L}^q([0,T]; \mathrm{L}^\infty(\mathbf{R}^3; \mathbf{R}^3))$, for some $q > 2$,

$\mathsf{b}_n \overset{*}{\longrightarrow} 0$ in $\mathrm{L}^\infty([0,T]; \mathrm{L}^r(\mathbf{R}^3; \mathbf{R}^3))$, for some $r > 3$.

Homogenised equation

Theorem. There is a subsequence and a function $\mathbf{M} \geqslant \mathbf{0}$ such that the limit u_0 satisfies:

$$\begin{cases} \partial_t \mathsf{u}_0 - \nu \triangle \mathsf{u}_0 + \mathsf{u}_0 \times \operatorname{rot} \mathsf{v}_0 + \lambda^2 \mathbf{M} \mathsf{u}_0 + \nabla p_0 = \mathsf{f}_0 \\ \operatorname{div} \mathsf{u}_0 = 0 \, , \end{cases}$$

and that we have the convergence

$$\nu |\nabla \mathsf{u}_n|^2 \longrightarrow \nu |\nabla \mathsf{u}_0|^2 + \lambda^2 \mathbf{M} \mathsf{u}_0 \cdot \mathsf{u}_0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^{1+3}) \, .$$

There is a new term, $\mathbf{M}$, in the macroscopic equation.
How can it be computed?

Oscillating test functions

$$\begin{cases} -\partial_t \mathsf{w}_n - \nu \triangle \mathsf{w}_n + \mathsf{k} \times \operatorname{rot} \mathsf{v}_n + \nabla r_n = 0 \\ \operatorname{div} \mathsf{w}_n = 0 \ , \end{cases}$$

supplemented by requirements:

$\mathsf{w}_n \longrightarrow 0$ in $\mathrm{L}^2([0,T];\mathrm{H}^1(\mathbf{R}^3;\mathbf{R}^3))$, and
$\mathsf{w}_n \overset{*}{\longrightarrow} 0$ in $\mathrm{L}^\infty([0,T];\mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3))$.

Sufficient to take homogeneous condition at $t = T$,
and (additional assumption) v_n bounded in $\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3))$.
This in particular gives r_n bounded in $\mathrm{L}^2([0,T] \times \mathbf{R}^3)$.

$$\nu \int_{\mathbf{R}^{1+3}} \varphi |\nabla \mathsf{w}_n|^2 \, d\mathbf{y} \longrightarrow \int_{\mathbf{R}^{1+3}} \varphi \mathbf{M}\mathsf{k} \cdot \mathsf{k} \, d\mathbf{y} \ ,$$

$\mathbf{M} \in \mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^3;\mathrm{M}_{3\times 3 \times)})$ and $\langle \mathbf{M}\mathsf{k} \mid \mathsf{k} \rangle \geqslant 0, \quad \mathsf{k} \in \mathbf{R}^3$.

Can we avoid w_n?

Theorem. Let $\boldsymbol{\mu}$ be a variant H-measure associated to a subsequence of (v_n).

$$\int_{\mathbf{R}^{1+3}} \mathbf{M}(t,\mathbf{x})\phi(t,\mathbf{x})dtd\mathbf{x} =$$

$$= 4\pi^2\nu\Big\langle\Big(\mathrm{tr}\boldsymbol{\mu}|\boldsymbol{\xi}|^2 - \boldsymbol{\mu}\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\Big)\frac{(\boldsymbol{\xi}\otimes\boldsymbol{\xi})}{\tau^2+\nu^2 4\pi^2|\boldsymbol{\xi}|^4}, \phi\boxtimes 1\Big\rangle,$$

with $\phi \in \mathrm{C}_c^\infty(\langle 0,T\rangle\times\mathbf{R}^3)$.

Proof.

For w_n we have (with $0 \leqslant \mathbf{M} \in \mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^3;\mathrm{M}_{3\times 3})))$:

$$\nu \int_{\mathbf{R}^{1+3}} \varphi |\nabla \mathsf{w}_n|^2 \, d\mathbf{y} \longrightarrow \int_{\mathbf{R}^{1+3}} \varphi \mathbf{M}\mathsf{k} \cdot \mathsf{k} \, d\mathbf{y} \ .$$

From estimates on r_n and v_n we get $\mathsf{w}_n' \longrightarrow 0$ in $\mathrm{L}^2(0,T;\mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^3))$, and compactness lemma gives us $\mathsf{w}_n \to 0$ in $\mathrm{L}^2_{\mathrm{loc}}([0,T] \times \mathbf{R}^3)$.

Therefore:

$$\lim_n \int_{\mathbf{R}^{1+3}} |\varphi \nabla \mathsf{w}_n|^2 \, d\mathbf{y} = \lim_n \int_{\mathbf{R}^{1+3}} |\nabla(\varphi \mathsf{w}_n)|^2 \, d\mathbf{y} \ .$$

Localise ...

Localise by multiplying the auxilliary problem by $\varphi \in \mathrm{C}_c^\infty(\langle 0, T\rangle \times \mathbf{R}^3)$

$$-\partial_t(\varphi \mathsf{w}_n) - \nu \triangle(\varphi \mathsf{w}_n) + \mathsf{k} \times \mathsf{rot}\,(\varphi \mathsf{v}_n) = -\nabla(\varphi r_n) + \mathsf{q}_n \ ,$$

$$\mathsf{q}_n = -(\partial_t \varphi)\mathsf{w}_n - \nu(\triangle\varphi)\mathsf{w}_n - 2\nu(\nabla \mathsf{w}_n)\nabla\varphi + \mathsf{k} \times (\nabla\varphi \times \mathsf{v}_n) + r_n \nabla\varphi \ ,$$

$\mathsf{q}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^{1+3})$ (and also strongly in $\mathrm{H}^{-\frac{1}{2},-1}(\mathbf{R}^{1+3})$).

As $\mathsf{w}_n \rightharpoonup 0$ in $\mathrm{L}^2([0,T]; \mathrm{H}^1(\mathbf{R}^3))$, so localised w_n and $\nabla \mathsf{w}_n$ converge weakly in L^2.

Of course, localised v_n and r_n converge weakly in L^2 as well.
From boundedness of the support of φ, we have strong convergence in $\mathrm{H}^{-\frac{1}{2},-1}$.

The Fourier transform

$$(-2\pi i\tau + \nu 4\pi^2 \boldsymbol{\xi}^2)\widehat{\varphi \mathsf{w}_n} = -\mathsf{k} \times \left((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n} \right) - 2\pi i \widehat{\varphi r_n} \boldsymbol{\xi} + \hat{\mathsf{q}}_n \ ,$$

and dividing by $(-2\pi i\tau + \nu 4\pi^2 \boldsymbol{\xi}^2)$ we get

$$\widehat{\varphi \mathsf{w}_n} = \frac{-\mathsf{k} \times \left((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n} \right) - 2\pi i \widehat{\varphi r_n} \boldsymbol{\xi} + \hat{\mathsf{q}}_n}{-2\pi i\tau + \nu 4\pi^2 \boldsymbol{\xi}^2} \ .$$

The penultimate term disappears if we project it to the plane $\perp \boldsymbol{\xi}$ (projection $P_{\boldsymbol{\xi}}$).

div $\mathsf{w}_n = 0$, so $\boldsymbol{\xi} \cdot \hat{\mathsf{w}}_n = 0$; which does not hold for div $(\varphi \mathsf{w}_n) = \nabla \varphi \cdot \mathsf{w}_n$. However, the RHS converges strongly in L^2 to 0, so in the Fourier space:

$$2\pi \boldsymbol{\xi} \cdot \widehat{\varphi \mathsf{w}_n} \longrightarrow 0 \ .$$

Projection by $P_{\boldsymbol{\xi}}$

After projection

$$\widehat{\varphi \mathsf{w}_n} = \frac{-P_{\boldsymbol{\xi}}\Big(\mathsf{k} \times \Big((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\Big)\Big) + P_{\boldsymbol{\xi}} \hat{\mathsf{q}}_n}{-2\pi i \tau + \nu 4\pi^2 \boldsymbol{\xi}^2} + \mathsf{d}_n \ ,$$

with $\mathsf{d}_n \longrightarrow 0$ in L^2.
By Plancherel

$$\begin{aligned}
\lim_n \int_\Omega \nu |\nabla(\varphi \mathsf{w}_n)|^2 \, d\mathbf{x} &= \lim_n \int_{\mathbf{R}}^{1+d} \nu 4\pi^2 |\widehat{(\varphi \mathsf{w}_n)}|^2 d\tau d\boldsymbol{\xi} \\
&= \lim_n \int_{\mathbf{R}}^{1+d} \nu 4\pi^2 \boldsymbol{\xi}^2 \left| \frac{P_{\boldsymbol{\xi}}\Big(\mathsf{k} \times \Big((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\Big) + \hat{\mathsf{q}}_n\Big)}{-2\pi i \tau + \nu 4\pi^2 \boldsymbol{\xi}^2} \right|^2 d\tau d\boldsymbol{\xi} \\
&= \lim_n \int_{\mathbf{R}}^{1+d} \nu \boldsymbol{\xi}^2 \frac{\left| P_{\boldsymbol{\xi}}\Big(\mathsf{k} \times \Big((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\Big) + \hat{\mathsf{q}}_n\Big) \right|^2}{\tau^2 + \nu 4\pi^2 \boldsymbol{\xi}^4} d\tau d\boldsymbol{\xi}
\end{aligned}$$

Applying the Lemma (analysis)

$$\frac{|\boldsymbol{\xi}|\hat{\mathsf{q}}_n}{\sqrt{\tau^2 + \nu 4\pi^2 \boldsymbol{\xi}^4}} \to 0 \quad \text{in} \qquad \mathrm{L}^2(\mathbf{R}^{1+3}) \ .$$

By $P_{\boldsymbol{\eta}}$

$$\left| P_{\boldsymbol{\eta}}\Big(\mathsf{k} \times (\boldsymbol{\eta} \times \mathbf{a})\Big) \right|^2 = (\mathsf{k} \cdot \boldsymbol{\eta})^2 \Big(|\mathsf{a}|^2 - |\mathsf{a} \cdot \boldsymbol{\eta}_0|^2 \Big)$$

where $\boldsymbol{\eta}_0$ is the unit vector in the direction of $\boldsymbol{\eta}$.
Note that k and $\boldsymbol{\eta}$ are real, while only a is complex. Therefore:

$$\lim_n \int_\Omega \nu |\nabla(\varphi \mathsf{w}_n)|^2 \, d\mathbf{x}$$

$$= \lim_n \int_{\mathbf{R}^3} \boldsymbol{\xi}^2 \frac{\Big(\mathsf{k} \cdot 2\pi i \boldsymbol{\xi}\Big)^2 \Big(|\widehat{\varphi \mathsf{v}_n}|^2 - \left| \widehat{\varphi \mathsf{v}_n} \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right|^2 \Big)}{\tau^2 + \nu 4\pi^2 \boldsymbol{\xi}^4} \, d\boldsymbol{\xi} \ .$$

Finally (after some algebra)

$$\lim_n \int_{\mathbf{R}^3} \boldsymbol{\xi}_0^2 \frac{\Big(\mathsf{k}\cdot 2\pi i\boldsymbol{\xi}_0\Big)^2\Big(|\widehat{\varphi\mathsf{v}_n}|^2 - \Big|\widehat{\varphi\mathsf{v}_n}\cdot\frac{\boldsymbol{\xi}_0}{|\boldsymbol{\xi}_0|}\Big|^2\Big)}{\tau_0^2+\nu 4\pi^2\boldsymbol{\xi}_0^4}\, d\boldsymbol{\xi} =$$

$$= \frac{1}{\nu}\langle \mathsf{tr}\boldsymbol{\mu}, (\frac{\boldsymbol{\xi}_0\cdot\mathsf{k}}{\tau_0^2+\nu 4\pi^2\boldsymbol{\xi}_0^4})^2\varphi\overline{\varphi}\rangle$$

$$- \frac{1}{\nu}\langle \boldsymbol{\mu}, (\frac{\boldsymbol{\xi}_0\cdot\mathsf{k}}{\tau_0^2+\nu 4\pi^2\boldsymbol{\xi}_0^4})^2\varphi\overline{\varphi}\boldsymbol{\xi}\otimes\boldsymbol{\xi}\rangle\ .$$