H-measures and variants

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Why a parabolic variant?

Parabolic pde-s are:

- well studied, and we have good theory for them
- in some cases we even have explicit solutions (by formulae)

$1:2$ is certainly a good ratio to start with

Besides the immediate applications (which motivated this research), related to the properties of parabolic equations, applications are possible to other equations and problems involving the scaling $1:2$.
Naturally, after understanding this ratio $1:2$, other ratios should be considered as well, as required by intended applications.

Terminology: *classical* as opposed to *parabolic or variant* H-measures.
The sphere we replace by:

$$\sigma^4(\tau,\boldsymbol{\xi}) := (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4 = 1\ , \text{ or}$$

$$\sigma_1^2(\tau,\boldsymbol{\xi}) := |\tau| + (2\pi|\boldsymbol{\xi}|)^2 = 1\ .$$

finally we chose the ellipse

$$\rho^2(\tau,\boldsymbol{\xi}) := |\boldsymbol{\xi}/2|^2 + \sqrt{(\boldsymbol{\xi}/2)^4 + \tau^2} = 1\ .$$

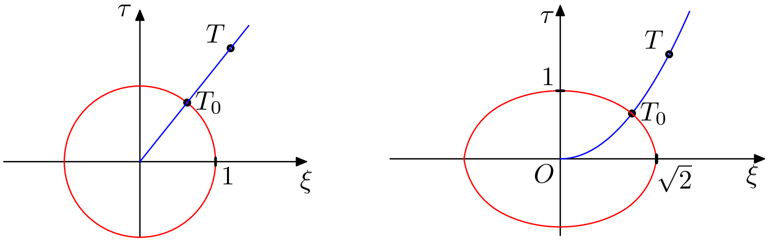
Notation.
For simplicity (2D): $(t,x) = (x^0,x^1) = \mathbf{x}$ and $(\tau,\xi) = (\xi_0,\xi_1) = \boldsymbol{\xi}$.

We use the Fourier transform in both space and time variables.

Rough geometric idea

Take a sequence $u_n \rightharpoonup 0$ in $L^2(\mathbf{R}^2)$, and integrate $|\widehat{\varphi u_n}|^2$ along

rays and project onto S^1 parabolas and project onto P^1



In $\mathbf{R}^2$ we have a compact curve (a surface in higher dimensions):

$$S^1 \ldots\ r^2(\tau,\xi) := \tau^2 + \xi^2 = 1 \qquad P^1 \ldots\ \rho^2(\tau,\xi) := (\xi/2)^2 + \sqrt{(\xi/2)^4 + \tau^2} = 1$$

and projection $\mathbf{R}^2_* = \mathbf{R}^2 \setminus \{0\}$ onto the curve (surface):

$$p(\tau,\xi) := \left(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\right) \qquad \pi(\tau,\xi) := \left(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\right)$$

Analytic picture

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier P_a, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{P_a u} = a\hat{u}$.
The norm is again equal to $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Delicate part: a is given only on S^1 or P^1.
We extend it by the projections, p or π: if α is a function defined on a compact surface, we take $a := \alpha \circ p$ or $a := \alpha \circ \pi$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau, \xi)}, \frac{\xi}{r(\tau, \xi)}\Big) \qquad a(\tau, \xi) := \alpha\Big(\frac{\tau}{\rho^2(\tau, \xi)}, \frac{\xi}{\rho(\tau, \xi)}\Big)$$

The precise scaling is contained in the projections, not the surface.

Now we can state the main theorem.

Existence of H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, then there exists its subsequence and a complex matrix Radon measure $\boldsymbol{\mu}$ on

$$\mathbf{R}^d \times S^{d-1} \qquad\qquad \mathbf{R}^d \times P^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and

$$\psi \in \mathrm{C}(S^{d-1}) \qquad\qquad \psi \in \mathrm{C}(P^{d-1})$$

one has

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} (\psi \circ p\pi)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar\varphi_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times S^{d-1}} \varphi_1(\mathbf{x})\bar\varphi_2(\mathbf{x})\psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x},\boldsymbol{\xi}) \qquad = \int_{\mathbf{R}^d \times P^{d-1}} \varphi_1(\mathbf{x})\bar\varphi_2(\mathbf{x})\psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x},\boldsymbol{\xi})\,.$$

First commutation lemma

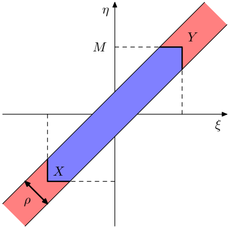
Lemma. (general form of the first commutation lemma) *If* $b \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ *satisfy the condition*

$$(\forall\, \rho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |a(\boldsymbol{\xi}) - a(\boldsymbol{\eta})| \leqslant \varepsilon \;\; (\text{a.e. } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \rho))\;,$$

then $C := [P_a, M_b]$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$*.* ∎

For given $M, \rho \in \mathbf{R}^+$ denote the set

$$Y = Y(M, \rho) = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geqslant M \;\&\; |\boldsymbol{\xi} - \boldsymbol{\eta}| \leqslant \rho\}\;.$$



where X denotes the complement of Y in the diagonal strip of width ρ.

In both cases discussed above, this lemma can also be proven directly, based on elementary inequalities.

Proof of the First commutation lemma

I. Assume additionally that $|\operatorname{supp}\hat{b}| \leqslant \rho$; for $u \in \mathcal{S}(\mathbf{R}^d)$

$$\widehat{Cu}(\boldsymbol{\xi}) = a(\boldsymbol{\xi})\big(\hat{b} * u\big)(\boldsymbol{\xi}) - \Big(\hat{b} * \big(a\hat{u}\big)\Big)(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} k(\boldsymbol{\xi},\boldsymbol{\eta})\hat{u}(\boldsymbol{\eta})\,d\boldsymbol{\eta}\ ,$$

where $k(\boldsymbol{\xi},\boldsymbol{\eta}) = \big(a(\boldsymbol{\xi}) - a(\boldsymbol{\eta})\big)\hat{b}(\boldsymbol{\xi}-\boldsymbol{\eta})$. For this ρ and arbitrary $\varepsilon > 0$ we can find an M such that $|a(\boldsymbol{\xi}) - a(\boldsymbol{\eta})| \leqslant \varepsilon\ \ (\text{a.e. } (\boldsymbol{\xi},\boldsymbol{\eta}) \in Y(M,\rho))$.
Next we decompose $C = D + E$, where

$$\widehat{Du}(\boldsymbol{\xi}) = \int_X k(\boldsymbol{\xi},\boldsymbol{\eta})\hat{u}(\boldsymbol{\eta})\,d\boldsymbol{\eta}\ ,$$
$$\widehat{Eu}(\boldsymbol{\xi}) = \int_Y k(\boldsymbol{\xi},\boldsymbol{\eta})\hat{u}(\boldsymbol{\eta})\,d\boldsymbol{\eta}\ .$$

As X is bounded, $\widehat{D\cdot}$ is an integral operator with compactly supported and bounded kernel, therefore a Hilbert-Schmidt operator, and compact. As $\mathcal{F}$ is an isometry, D is also a Hilbert-Schmidt operator, and therefore compact.
On Y we have $|k(\boldsymbol{\xi},\boldsymbol{\eta})| \leqslant \varepsilon|\hat{b}(\boldsymbol{\xi}-\boldsymbol{\eta})|$, thus by Young's inequality $\|\widehat{Eu}\|_2 \leqslant \varepsilon\|\hat{b}\|_1\|\hat{u}\|_2$ for $u \in \mathcal{S}(\mathbf{R}^d)$. As $\mathcal{S}(\mathbf{R}^d)$ is dense in $\mathrm{L}^2(\mathbf{R}^d)$, by the Plancherel theorem, we have also for the operator norm $\|E\| \leqslant \varepsilon\|\hat{b}\|_1$.
Taking a sequence $\varepsilon \longrightarrow 0$, we get that C is a limit (in the operator norm topology) of a sequence of Hilbert-Schmidt operators, so C is also compact.

Proof of the First commutation lemma (cont.)

II. For any $b \in \mathrm{C}_0(\mathbf{R}^d)$, C is a bounded operator, with norm $\leqslant 2\|a\|_\infty \|b\|_\infty$. In particular, if $b_m \longrightarrow b$ in $\mathrm{L}^\infty(\mathbf{R}^d)$, then $C_m := [P_a, M_{b_m}]$ converges in the operator (uniform) topology to C. Therefore, C will be a compact operator, if C_m are such.

Finally, we can approximate b by a sequence of functions as in part I of the proof, obtaining a sequence of compact operators converging to C.

Indeed, for given $b \in \mathrm{C}_0(\mathbf{R}^d)$ we can find a sequence (f_n) in $\mathcal{S}(\mathbf{R}^d)$ uniformly converging to b. Then $g_n := \hat{f}_n \in \mathcal{S}(\mathbf{R}^d) \subseteq \mathrm{L}^1(\mathbf{R}^d)$, so for any fixed n we can find a sequence $(g_n^m)_m$ in $\mathrm{C}_c^\infty(\mathbf{R}^d)$ converging in L^1 norm to g_n. The sequence $(\bar{\mathcal{F}} g_n^m)_m$ now uniformly converges to f_n, and after applying the Cantor diagonal procedure we find that $b_n := \bar{\mathcal{F}} g_n^{m(n)}$ uniformly converges to b, as claimed.

In particular ...

Lemma. *Let* $\pi : \mathbf{R}^d{}_* \longrightarrow \Sigma$ *be a smooth projection to a smooth compact hypersurface, such that* $|\nabla\pi(\boldsymbol{\xi})| \longrightarrow 0$ *for* $|\boldsymbol{\xi}| \longrightarrow \infty$*, and* $a \in \mathrm{C}(\Sigma)$*. Then (the extended)* a *satisfies the assumptions of previous lemma.*

Dem. Taking C resulting from uniform continuity of a on compact Σ:

$$\Big|a(\boldsymbol{\xi})-a(\boldsymbol{\eta})\Big| = \Big|a(\pi(\boldsymbol{\xi}))-a(\pi(\boldsymbol{\eta}))\Big| \leqslant C\Big|\pi(\boldsymbol{\xi})-\pi(\boldsymbol{\eta})\Big| \leqslant |\boldsymbol{\xi}-\boldsymbol{\eta}| \sup_{\boldsymbol{\zeta}\in[\boldsymbol{\xi},\boldsymbol{\eta}]} \Big|\nabla\pi(\boldsymbol{\zeta})\Big| \,,$$

where we applied the Mean value theorem to projection π.

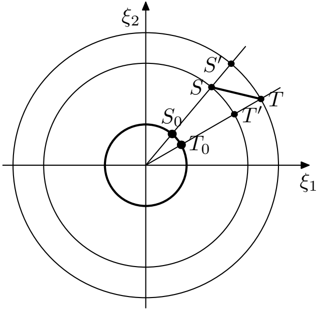
For $|\boldsymbol{\xi}-\boldsymbol{\eta}| \leqslant \rho$ and $\varepsilon > 0$ given, we can find M large enough such that for $|\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geqslant M > \rho$ the above is bounded by $|\varepsilon|$.

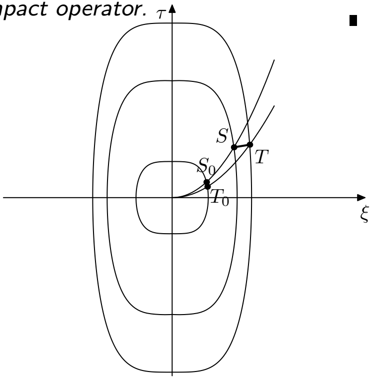
Q.E.D.

Check that this applies to the classical and parabolic H-measures.
In both cases we have a continuous function a defined on a smooth compact surface Σ (S^{d-1} or P^{d-1}), then extended to $\mathbf{R}^d{}_*$ taking constant values along certain curves, which transversally intersect Σ and cover the whole space (rays from the origin, or parts of quadratic parabolas in the parabolic case). It only remains to be shown that the projections satisfy $\|\nabla\pi(\boldsymbol{\xi})\| \longrightarrow 0$ for $|\boldsymbol{\xi}| \longrightarrow \infty$.
It is a matter of straightforward calculation to check that $\|\nabla\pi\| \leqslant 1/|\boldsymbol{\xi}|$ in the first case, and $\|\nabla\pi\| \leqslant c\rho^{-2}$ in the second (c being some positive constant).

Alternative proof

Lemma. *The commutator* $[P_a, M_b]$ *is a compact operator.* ∎





$$\frac{d(S_0, T_0)}{1} = \frac{d(S, T')}{r_S} = \frac{d(S', T)}{r_T}$$

$$d(S_0, T_0) \leqslant 2\frac{d(S, T)}{r_S + r_T}$$

$$d(S_0, T_0) \leqslant C\frac{d(S, T)}{\rho_S + \rho_T}$$

($\rho_S := \rho(\tau_S, \xi_S)$ etc.)

Representation of bilinear functionals

Recall the Riesz representation theorem:
A positive functional L on $\mathrm{C}_c(X)$ defines the unique Radon measure μ on X

$$(\forall\, f \in \mathrm{C}_c(X)) \quad Lf = \int_X f\, d\mu\,.$$

Boundedness in L^∞ norm allows extension of L from $\mathrm{C}_c(X)$ to $\mathrm{C}_0(X)$ by continuity ($\mathcal{M}_b(X) := \mathrm{C}_0(X)'$).

We need such a representation for positive continuous *bilinear* forms on $\mathrm{C}_0(X) \times \mathrm{C}_0(Y)$.

Lemma. (representation of bilinear functionals) *Let X, Y be open and bounded in $\mathbf{R}^d$, $\mathbf{R}^r$, and B a continuous bilinear form on $\mathrm{C}_0(X) \times \mathrm{C}_0(Y)$. If for $f \in \mathrm{C}_0(X)$ and $g \in \mathrm{C}_0(Y)$, $f, g \geqslant 0$ implies $B(f,g) \geqslant 0$, then there exists a bounded Radon measure μ on $X \times Y$ such that for any $f \in \mathrm{C}_0(X)$ and $g \in \mathrm{C}_0(Y)$ the following representation is valid:*

$$B(f,g) = {}_{\mathcal{M}_b(X\times Y)}\langle\, \mu, f \boxtimes g \,\rangle_{\mathrm{C}_0(X\times Y)}\,.$$

∎

The representation is valid on manifolds as well, even on any locally compact Hausdorff spaces.

Proof of existence

Recall that $P_{\bar{\psi}}$ stands for the Fourier multiplier associated to $\overline{\psi}$; so by the Plancherel formula the limit reads

$$\lim_n \int_{\mathbf{R}^{1+d}} (\phi_1 \mathsf{u}_n)(\tau, \boldsymbol{\xi}) \otimes \big(P_{\bar{\psi}} \phi_2 \mathsf{u}_n\big)(\tau, \boldsymbol{\xi}) d\tau d\boldsymbol{\xi}.$$

By the First commutation lemma $(P_{\bar{\psi}}\phi_2 - \phi_2 P_{\bar{\psi}})\mathsf{u}_n = K\mathsf{u}_n$, with K being a compact operator on L^2, so the limit can be written as

$$\lim_n \int_{\mathbf{R}^{1+d}} (\phi_1 \overline{\phi_2} \mathsf{u}_n)(\tau, \boldsymbol{\xi}) \otimes \big(P_{\bar{\psi}} \mathsf{u}_n\big)(\tau, \boldsymbol{\xi}) d\tau d\boldsymbol{\xi}.$$

The above sequence is bounded by $C\|\phi_1\|_{\mathrm{L}^\infty}\|\phi_2\|_{\mathrm{L}^\infty}\|\psi\|_{\mathrm{L}^\infty}$, with $C = \sup_n \|\mathsf{u}_n\|^2_{\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)}$.

Cantor diagonal procedure

As $\mathrm{C}_0(\mathbf{R}^{1+d})$ and $\mathrm{C}(P)$ are separable, we denote their countable dense subsets with S and T respectively. The Cantor diagonal procedure results in a subsequence u_{n_r} such that the above limit is valid for each $(\phi_1, \phi_2, \psi) \in S \times S \times T$. To this end we index by $m \in \mathbf{N}$ all ordered triples in $S \times S \times T$. As the above sequence is bounded in $\mathrm{M}_{r\times r}$, for $m = 1$ there is a subsequence $\mathsf{u}_{n_1(n)}$ such that the sequence

$$\int_{\mathbf{R}^{1+d}} (\phi_1^1 \overline{\phi_2^1} \mathsf{u}_{n_1(n)})(\tau, \boldsymbol{\xi}) \otimes \left(P_{\bar{\psi}^1} \mathsf{u}_{n_1(n)}\right)(\tau, \boldsymbol{\xi}) d\tau d\boldsymbol{\xi}$$

converges. For this subsequence, there is another subsequence $\mathsf{u}_{n_2(n)}$, for which analogous sequence as above converges with $m = 2$ instead of $m = 1$. Continuing the procedure, in such a way we construct a subsequence $\mathsf{u}_{n_1(1)}, \mathsf{u}_{n_2(2)}, \dots,$ for which the sequence

$$\int_{\mathbf{R}^{1+d}} (\phi_1^m \overline{\phi_2^m} \mathsf{u}_{n_r(r)})(\tau, \boldsymbol{\xi}) \otimes \left(P_{\bar{\psi}^m} \mathsf{u}_{n_r(r)}\right)(\tau, \boldsymbol{\xi}) d\tau d\boldsymbol{\xi}$$

converges for any m, i.e. for any choice of functions (ϕ_1, ϕ_2, ψ) from dense set $S \times S \times T$. This subsequence we shall denote in the same way as the original sequence, for simplicity.

Extension by density

In the next step we prove that the above sequence converges for arbitrary $(\phi_1, \phi_2, \psi) \in \mathrm{C}_0(\mathbf{R}^{1+d}) \times \mathrm{C}_0(\mathbf{R}^{1+d}) \times \mathrm{C}(P)$. Indeed, let $(\phi_1^k, \phi_2^k, \psi^k) \in S \times S \times T$ be a sequence converging to (ϕ_1, ϕ_2, ψ). Then

$$\int_{\mathbf{R}^{1+d}} \Big((\phi_1\overline{\phi_2}\mathsf{u}_n)(\tau,\boldsymbol{\xi}) \otimes (P_{\bar\psi}\mathsf{u}_n)(\tau,\boldsymbol{\xi}) - (\phi_1\overline{\phi_2}\mathsf{u}_m)(\tau,\boldsymbol{\xi}) \otimes (P_{\bar\psi}\mathsf{u}_m)(\tau,\boldsymbol{\xi})\Big) d\tau d\boldsymbol{\xi}$$

$$= \int \Big(((\phi_1\overline{\phi_2} - \phi_1^k\overline{\phi_2^k})\mathsf{u}_n)(\tau,\boldsymbol{\xi}) \otimes (P_{\bar\psi}\mathsf{u}_n)(\tau,\boldsymbol{\xi})$$

$$+(\phi_1^k\overline{\phi_2^k}\mathsf{u}_n)(\tau,\boldsymbol{\xi}) \otimes ((P_{\bar\psi} - P_{\bar\psi^k})\mathsf{u}_n)(\tau,\boldsymbol{\xi})\Big)\, d\tau d\boldsymbol{\xi}$$

$$- \int \Big(((\phi_1\overline{\phi_2} - \phi_1^k\overline{\phi_2^k})\mathsf{u}_m)(\tau,\boldsymbol{\xi}) \otimes (P_{\bar\psi}\mathsf{u}_m)(\tau,\boldsymbol{\xi})$$

$$+(\phi_1^k\overline{\phi_2^k}\mathsf{u}_m)(\tau,\boldsymbol{\xi}) \otimes ((P_{\bar\psi} - P_{\bar\psi^k})\mathsf{u}_m)(\tau,\boldsymbol{\xi})\Big)\, d\tau d\boldsymbol{\xi}$$

$$+ \int \Big((\phi_1^k\overline{\phi_2^k}\mathsf{u}_n)(\tau,\boldsymbol{\xi}) \otimes (P_{\bar\psi^k}\mathsf{u}_n)(\tau,\boldsymbol{\xi})$$

$$-(\phi_1^k\overline{\phi_2^k}\mathsf{u}_m)(\tau,\boldsymbol{\xi}) \otimes (P_{\bar\psi^k}\mathsf{u}_m)(\tau,\boldsymbol{\xi})\Big)\, d\tau d\boldsymbol{\xi}\ .$$

First two integrals on the right hand side are bounded by

$$C\Big(\|\phi_1\overline{\phi_2}-\phi_1^k\overline{\phi_2^k}\|_{\mathrm{L}^\infty}\|\psi\|_{\mathrm{L}^\infty}+\|\phi_1^k\overline{\phi_2^k}\|_{\mathrm{L}^\infty}\|\psi-\psi^k\|_{\mathrm{L}^\infty}\Big)\ ,$$

where $C=\sup_n\|\mathsf{u}_n\|^2_{\mathrm{L}^2}$, thus arbitrary small, for k large enough.

The remaining integral represents the difference between $m-$th and $n-$th term in a convergent sequence, so we have a Cauchy sequence, therefore convergent. We have thus defined a mapping $\mathrm{C}_0(\mathbf{R}^{1+d})\times\mathrm{C}_0(\mathbf{R}^{1+d})\times\mathrm{C}(P)\to\mathrm{M}_{r\times r}$, which is linear in each of its arguments. It is also continuous, as

$$\|\phi_1\overline{\phi}_2\mathsf{u}_n\otimes(P_{\bar\psi}\mathsf{u}_n)\|_{\mathrm{L}^1}\leqslant\|\phi_1\overline{\phi}_2\|_{\mathrm{L}^\infty}\|\mathsf{u}_n\|_{\mathrm{L}^2}\|(P_{\bar\psi}\mathsf{u}_n)\|_{\mathrm{L}^2}\leqslant C\|\phi_1\overline{\phi}_2\|_{\mathrm{L}^\infty}\|\psi\|_{\mathrm{L}^\infty}\ .$$

This mapping depends only on the product $\phi_1\overline{\phi}_2$, and function ψ, so for any $i,j\in 1..r$ a bilinear form $\mu_{ij}:=\boldsymbol{\mu}\mathsf{e}_i\cdot\mathsf{e}_j$ on $\mathrm{C}_0(\mathbf{R}^{1+d})\times\mathrm{C}(P)$ is given:

$$\begin{aligned}\langle\mu_{ij},\phi_1\overline{\phi}_2\boxtimes\psi\rangle:&=\lim_n\int\limits_{\mathbf{R}^{1+d}}(\phi_1\overline{\phi}_2u_{in})(\tau,\boldsymbol{\xi})\overline{\big(P_{\bar\psi}u_{jn}\big)}(\tau,\boldsymbol{\xi})d\tau d\boldsymbol{\xi}\\&=\lim_n\int\limits_{\mathbf{R}^{1+d}}\mathcal{F}\Big(\phi_1u_{in}\Big)(\tau,\boldsymbol{\xi})\overline{\mathcal{F}\Big(\phi_2u_{jn}\Big)}(\tau,\boldsymbol{\xi})\,\psi\Big(\tau_0,\boldsymbol{\xi}_0\Big)d\tau d\boldsymbol{\xi}\ ,\end{aligned}$$

where $u_{in}:=\mathsf{u}_n\cdot\mathsf{e}_i$.By interchanging the indices i and j in the last equality, and choosing real functions ϕ_1,ϕ_2,ψ, we obtain that $\mu_{ij}=\overline{\mu}_{ji}$ for any $i,j\in 1..r$, i.e. matrix measure $\boldsymbol{\mu}$ is hermitian.

Positivity and conclusion

Furthermore, for $\phi, \psi \geqslant 0$ we take $\phi_1 = \phi_2 = \sqrt{\phi}$. Then for any $\boldsymbol{\lambda} \in \mathbf{C}^r$ we have

$$\Big\langle \sum_{i,j} \lambda_i \overline{\lambda}_j \mu_{ij}, \phi_1 \overline{\phi}_2 \boxtimes \psi \Big\rangle = \lim_n \int_{\mathbf{R}^{1+d}} \Big| \sum_i \lambda_i \mathcal{F}\Big(\sqrt{\phi} u_{in}\Big) \Big|^2 (\tau, \boldsymbol{\xi})\, \psi(\tau_0, \boldsymbol{\xi}_0) d\tau d\boldsymbol{\xi} \geqslant 0 \,,$$

and bilinear form $B = \boldsymbol{\mu}\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}$ is positively semidefinite. Now we can apply the Lemma on representation of bilinear forms, thus the form B is determined by a Radon measure $m_{\boldsymbol{\lambda}}$. By varying vector $\boldsymbol{\lambda}$ in $m_{\boldsymbol{\lambda}} = \boldsymbol{\mu}\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}$, after taking into account the hermitian character of $\boldsymbol{\mu}$, we identify all components μ_{ij}.

Immediate properties

Corollary. *Parabolic H-measure* $\boldsymbol{\mu}$ *is hermitian and nonnegative:*

$$\boldsymbol{\mu} = \boldsymbol{\mu}^* \qquad \textit{and} \qquad (\forall\, \boldsymbol{\phi} \in \mathrm{C}_0(\mathbf{R}^{1+d}; \mathbf{C}^r)) \quad \langle \boldsymbol{\mu}, \boldsymbol{\phi} \otimes \boldsymbol{\phi} \rangle \geq 0 \;,$$

where $\langle \boldsymbol{\mu}, \boldsymbol{\phi} \otimes \boldsymbol{\phi} \rangle$ *is considered as a Radon measure on* P^d*.* ∎

Indeed, let $(\mathsf{e}_1, \dots, \mathsf{e}_r)$ be an orthonormal basis in $\mathbf{C}^r$. For $\mu_{ij} := \boldsymbol{\mu}\mathsf{e}_i \cdot \mathsf{e}_j$

$$\langle \mu_{ij}, \phi_1 \overline{\phi}_2 \boxtimes \psi \rangle := \lim_n \int_{\mathbf{R}^{1+d}} \mathcal{F}\Big(\phi_1 u_{in}\Big) \overline{\mathcal{F}\Big(\phi_2 u_{jn}\Big)} (\psi \circ \pi) d\tau d\boldsymbol{\xi} \;,$$

where $u_{in} := \mathsf{u}_n \cdot \mathsf{e}_i$. By exchanging indices above, and taking real functions ϕ_1, ϕ_2, ψ it follows $\mu_{ij} = \overline{\mu}_{ji}$ for each pair (i, j).
By taking $\phi_1 := \boldsymbol{\phi} \cdot \mathsf{e}_i$, $\phi_2 := \boldsymbol{\phi} \cdot \mathsf{e}_j$ and ψ real nonnegative in the last equation, summation in i and j gives the second statement.

Simple localisation

For parabolic H-measures we have simple localisation as an immediate consequence of the definition.

Corollary. *Let the sequence* (u_n) *define a parabolic H-measure* $\boldsymbol{\mu}$*. If all the components* $\mathsf{u}_n \cdot \mathsf{e}_i$ *have their supports in closed sets* $K_i \subseteq \mathbf{R}^{1+d}$ *respectively, then the support of the component* $\boldsymbol{\mu}\mathsf{e}_i \cdot \mathsf{e}_j$ *is contained in* $(K_i \cap K_j) \times \mathrm{P}^d$*.* ∎

If u_n were defined on an open set $\Omega \subseteq \mathbf{R}^{1+d}$, we would first extend each u_n by zero to $\mathbf{R}^{1+d}$ (such an extension clearly preserves the weak convergence), and then apply the existence theorem. The resulting parabolic H-measure has its support contained in $\mathrm{Cl}\,\Omega$, by Corollary.

Weak $*$ limits

Parabolic H-measures can be used to describe weak $*$ limits of quadratic quantities.

Corollary. *If* $\mathsf{u}_n \otimes \mathsf{u}_n$ *converges weakly* $*$ *to a measure* $\boldsymbol{\nu}$*, then for every* $\phi \in \mathrm{C}_0(\mathbf{R}^{1+d})$*:*

$$\langle \boldsymbol{\nu}, \phi \rangle = \langle \boldsymbol{\mu}, \phi \boxtimes 1 \rangle \ .$$

∎

Indeed, by choosing $\phi_1, \phi_2 \in \mathrm{C}_0(\mathbf{R}^{1+d})$ such that $\phi = \phi_1 \bar{\phi}_2$, and taking $\psi := 1$ in the defining limit, by Plancherel's theorem we have

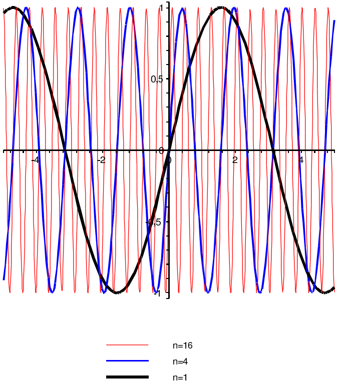
$$\langle \boldsymbol{\mu}, \phi_1 \bar{\phi}_2 \boxtimes 1 \rangle = \lim_n \int_{\mathbf{R}^{1+d}} \phi_1 \mathsf{u}_n \otimes \phi_2 \mathsf{u}_n \, d\mathbf{x} = \int_{\mathbf{R}^{1+d}} \lim_n (\mathsf{u}_n \otimes \mathsf{u}_n) \phi_1 \bar{\phi}_2 \, d\mathbf{x} = \langle \boldsymbol{\nu}, \phi_1 \bar{\phi}_2 \rangle \ .$$

Lemma. *Let* (u_n) *be a pure sequence in* $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$*, and* $\boldsymbol{\mu}$ *the corresponding parabolic H-measure. Then the sequence* $(\overline{\mathrm{u}}_n)$ *is pure with associated parabolic H-measure* $\boldsymbol{\nu}$*, such that* $\boldsymbol{\nu}(t,\mathbf{x},\tau,\boldsymbol{\xi}) = \boldsymbol{\mu}^\top(t,\mathbf{x},-\tau,-\boldsymbol{\xi})$*. In particular, a parabolic H-measure* μ *associated to a real scalar sequence is antipodally symmetric, i. e.* $\mu(t,\mathbf{x},\tau,\boldsymbol{\xi}) = \mu(t,\mathbf{x},-\tau,-\boldsymbol{\xi})$*.* ∎

Oscillation

$$u_n(\mathbf{x}) := v(n\mathbf{x}) \rightharpoonup 0$$

$v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$ periodic function (with the unit period in each of variables), with the zero mean value.



Oscillation (classical H-measures)

The associated H-measure

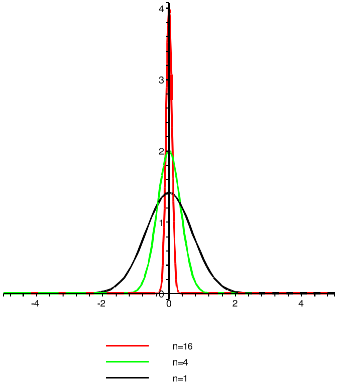
$$\mu(\mathbf{x},\boldsymbol{\xi}) = \sum_{\mathsf{k}\in\mathbf{Z}^d\setminus\{0\}} |v_{\mathsf{k}}|^2 \lambda(\mathbf{x})\,\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}),$$

v_{k} Fourier coefficients of v ($v(\mathbf{x}) = \sum_{\mathsf{k}\in\mathbf{Z}^d} v_{\mathsf{k}} e^{2\pi i \mathsf{k}\cdot\mathbf{x}}$).

Dual variable *preserves* information on the direction of propagation (of oscillation).

Concentration

$$u_n(\mathbf{x}) := n^{\frac{d}{2}} v\,(n\mathbf{x})\,, \qquad \Big(v \in \mathrm{L}^2(\mathbf{R}^d)\Big)\,.$$



Concentration (classical H-measures)

The associated H-measure is of the form $\delta_0(\mathbf{x})\nu(\boldsymbol{\xi})$, where ν is measure on S^{d-1} with surface density

$$\nu(\boldsymbol{\xi}) = \int_0^\infty |\hat{v}(t\boldsymbol{\xi})|^2 t^{d-1} dt,$$

i.e.

$$\mu(\mathbf{x}, \boldsymbol{\xi}) = \int_{\mathbf{R}^d} |\hat{v}(\boldsymbol{\eta})|^2 \delta_{\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}}(\boldsymbol{\xi}) \delta_0(\mathbf{x}) \, d\boldsymbol{\eta},$$

where $\hat{v}$ denotes the Fourier transformation of v.

Oscillation (parabolic H-measures)

Let $v \in \mathrm{L}^2(Z)$ be a periodic function

$$v(t, \mathbf{x}) = \sum_{(\omega,\mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega,\mathsf{k}} \, e^{2\pi i(\omega t + \mathsf{k}\cdot\mathbf{x})} \ ,$$

where $\hat{v}_{\omega,\mathsf{k}}$ denotes Fourier coefficients. Further, assume that v has mean value zero, i.e. $\hat{v}_{0,0} = 0$.

For $\alpha, \beta \in \mathbf{R}^+$, we have a sequence of periodic functions with period tending to zero:

$$u_n(t, \mathbf{x}) := v(n^\alpha t, n^\beta \mathbf{x}) = \sum_{(\omega,\mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega,\mathsf{k}} \, e^{2\pi i(n^\alpha \omega t + n^\beta \mathsf{k}\cdot\mathbf{x})} \ .$$

Their Fourier transforms are:

$$\hat{u}_n(\tau, \boldsymbol{\xi}) = \sum_{(\omega,\mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega,\mathsf{k}} \, \delta_{n^\alpha \omega}(\tau) \delta_{n^\beta \mathsf{k}}(\boldsymbol{\xi}) \ .$$

Oscillation (cont.)

(u_n) is a pure sequence, and the corresponding parabolic H-measure $\mu(t, \mathbf{x}, \tau, \boldsymbol{\xi})$ is

$$\lambda(t,\mathbf{x}) \begin{cases} \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \omega\neq 0}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}) + \sum_{\mathsf{k}\in\mathbf{Z}^d} |\hat{v}_{0,\mathsf{k}}|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}), & \alpha > 2\beta \\ \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \mathsf{k}\neq 0}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}) + \sum_{\omega\in\mathbf{Z}} |\hat{v}_{\omega,0}|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}), & \alpha < 2\beta \\ \displaystyle\sum_{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{\left(\frac{\omega}{\rho^2(\omega,\mathsf{k})},\frac{\mathsf{k}}{\rho(\omega,\mathsf{k})}\right)}(\tau,\boldsymbol{\xi}), & \alpha = 2\beta, \end{cases}$$

where λ denotes the Lebesgue measure.

Concentration (parabolic H-measures)

For $v \in \mathrm{L}^2(\mathbf{R}^{1+d})$ and $\alpha, \beta \in \mathbf{R}^+$

$$u_n(t, \mathbf{x}) := n^{\alpha+\beta d} v(n^{2\alpha} t, n^{2\beta} \mathbf{x}),$$

is bounded in $\mathrm{L}^2(\mathbf{R}^{1+d})$ with the norm $\|u_n\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} = \|v\|_{\mathrm{L}^2(\mathbf{R}^{1+d})}$ which does not depend on n, and weakly converges to zero.

(u_n) is a pure sequence, with the parabolic H-measure $\langle \boldsymbol{\mu}, \phi \boxtimes \psi \rangle =$

$$\phi(0,0) \begin{cases} \int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi(\frac{\sigma}{|\sigma|}, 0) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}^d} |\hat{v}(0, \boldsymbol{\eta})|^2 \psi(0, \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|})\, d\boldsymbol{\eta}, & \alpha > 2\beta \\ \int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi(0, \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}} |\hat{v}(\sigma, 0)|^2 \psi(\frac{\sigma}{|\sigma|}, 0)\, d\sigma, & \alpha < 2\beta \\ \int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi\left(\frac{\sigma}{\rho^2(\sigma, \boldsymbol{\eta})}, \frac{\boldsymbol{\eta}}{\rho(\sigma, \boldsymbol{\eta})}\right) d\sigma d\boldsymbol{\eta}, & \alpha = 2\beta. \end{cases}$$

From examples we learn ...

Actually, any non-negative Radon measure on $\Omega \times P^{d-1}$, of total mass A^2, can be described as a parabolic H-measure of some sequence $u_n \rightharpoonup 0$, with $\|u_n\|_{\mathrm{L}^2} \leqslant A + \varepsilon$.

Both for oscillation and concentration, for $\alpha > 2\beta$ the measure μ is supported in *poles*, while for $\alpha < 2\beta$ on the *equator* of the surface P^d, regardless of the choice of v.

When $\alpha = 2\beta$ the parabolic H-measure can be supported in any point of the surface P^d.

Symmetric systems

$$\partial_k(\mathbf{A}^k \mathsf{u}) + \mathbf{B}\mathsf{u} = \mathsf{f} \ , \ \mathbf{A}^k \in \mathrm{C}_b(\mathbf{R}^d; \mathrm{M}_{r\times r}) \text{ Hermitian}$$

Assume:

$$\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0 \ , \qquad \text{and defines } \boldsymbol{\mu}$$

$$\mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \ .$$

Theorem. (localisation principle) If u^n satisfies:

$$\partial_k\big(\mathbf{A}^k \mathsf{u}^n\big) \longrightarrow 0 \ \text{ in space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \ ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$ on $\Omega \times S^{d-1}$ one has:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})\bar{\boldsymbol{\mu}} = \mathbf{0} \ .$$

Thus, the support of H-measure $\boldsymbol{\mu}$ is contained in the set $\big\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\big\}$ of points where $\mathbf{P}$ is a singular matrix.

The localisation principle is behind the applications to the small-amplitude homogenisation, which can be used in optimal design.

It is a generalisation of compactness by compensation to variable coefficients.

Localisation principle for parabolic H-measures

In the parabolic case the details become more involved.

Anisotropic Sobolev spaces ($s \in \mathbf{R}$; $k_p(\tau, \boldsymbol{\xi}) := (1 + \sigma^4(\tau, \boldsymbol{\xi}))^{1/4}$)

$$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) := \left\{ u \in \mathcal{S}' : k_p^s \hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d}) \right\} .$$

Theorem. (localisation principle) Let $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{C}^r)$, uniformly compactly supported in t, satisfy ($s \in \mathbf{N}$)

$$\sqrt{\partial_t}^s (\mathsf{u}_n \cdot \mathsf{b}) + \sum_{|\boldsymbol{\alpha}|=s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} (\mathsf{u}_n \cdot \mathsf{a}_{\boldsymbol{\alpha}}) \longrightarrow 0 \quad \text{in} \quad \mathrm{H}_{\mathrm{loc}}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d}) ,$$

where $\mathsf{b}, \mathsf{a}_{\boldsymbol{\alpha}} \in \mathrm{C}_b(\mathbf{R}^{1+d}; \mathbf{C}^r)$, while $\sqrt{\partial_t}$ is a pseudodifferential operator with polyhomogeneous symbol $\sqrt{2\pi i \tau}$, i.e.

$$\sqrt{\partial}_t u = \overline{\mathcal{F}} \left(\sqrt{2\pi i \tau}\, \hat{u}(\tau) \right) .$$

For parabolic H-measure $\boldsymbol{\mu}$ associated to sequence (u_n) one has

$$\boldsymbol{\mu} \Big((\sqrt{2\pi i \tau})^s \bar{\mathsf{b}} + \sum_{|\boldsymbol{\alpha}|=s} (2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}}\, \bar{\mathsf{a}}_{\boldsymbol{\alpha}} \Big) = 0.$$

How to use such a relation? — the heat equation

$$\begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}\nabla u_n) = \mathsf{div}\,\mathsf{f}_n \\ u_n(0) = \gamma_n \ , \end{cases}$$

$\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d};\mathbf{R}^d)$, $\gamma_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$

continuous, bounded and positive definite: $\mathbf{A}(t,\mathbf{x})\,\mathsf{v}\cdot\mathsf{v} \geqslant \alpha\mathsf{v}\cdot\mathsf{v}$

Localise in time: take θu_n, for $\theta \in \mathrm{C}^1_c(\mathbf{R}^+)$, ...
Now we can apply the localisation principle (we still denote the localised solutions by u_n).

Furthermore, $\sqrt{\partial_t}\,(u_n) := \left(\sqrt{2\pi i\tau}\,\widehat{u_n}\right)^\vee \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^{1+d})$.

The heat equation (cont.)

Take

$$\tilde{\mathsf{v}}_n = (v_n^0, \mathsf{v}_n, \mathsf{f}_n) := (\sqrt{\partial_t} u_n, \nabla u_n, \mathsf{f}_n) \rightharpoonup 0$$

in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{R}^{1+2d})$, which (on a subsequence) defines H-measure

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \mu_0 & \boldsymbol{\mu}_{01} & \boldsymbol{\mu}_{02} \\ \boldsymbol{\mu}_{10} & \boldsymbol{\mu} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{20} & \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_f \end{bmatrix}.$$

The localisation principle gives us:

$$\begin{aligned} \mu_0 \sqrt{2\pi i \tau} - 2\pi i \boldsymbol{\mu}_{01} \cdot \mathbf{A}^\top \boldsymbol{\xi} - 2\pi i \boldsymbol{\mu}_{02} \cdot \boldsymbol{\xi} &= 0 \\ \boldsymbol{\mu}_{10} \sqrt{2\pi i \tau} - 2\pi i \boldsymbol{\mu} \mathbf{A}^\top \boldsymbol{\xi} - 2\pi i \boldsymbol{\mu}_{12} \boldsymbol{\xi} &= 0 \\ \boldsymbol{\mu}_{20} \sqrt{2\pi i \tau} - 2\pi i \boldsymbol{\mu}_{21} \mathbf{A}^\top \boldsymbol{\xi} - 2\pi i \boldsymbol{\mu}_f \boldsymbol{\xi} &= 0. \end{aligned}$$

After some calculation (linear algebra) ...

Expression for H-measure — from given data

$$\mathrm{tr}\boldsymbol{\mu} = \frac{(2\pi\boldsymbol{\xi})^2}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi},$$

$$\boldsymbol{\mu} = \frac{(2\pi)^2}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}(\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi}.$$

$$\mu_0 = \frac{|2\pi\tau|}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi}.$$

Thus, from the H-measures for the right hand side term f one can calculate the H-measure of the solution.

However, the oscillation in initial data dies out (the equation is hypoelliptic). Only the right hand side affects the H-measure of solutions.

The situation is different for the Schrödinger equation and for the vibrating plate equation.

Classical H-measures

Theorem. (propagation principle for symmetric systems, N.A. 1992–6)
Let $\mathbf{A}^k \in \mathrm{C}^1_0(\Omega; \mathrm{M}_{r\times r})$ *be hermitian and* $\mathbf{B} \in \mathrm{C}_0(\Omega; \mathrm{M}_{r\times r})$*.*
If for every n *the pair* $(\mathsf{u}_n, \mathsf{f}_n)$ *satisfies the system*

$$\partial_k(\mathbf{A}^k \mathsf{u}_n) + \mathbf{B}\mathsf{u}_n = \mathsf{f}_n \ ,$$

and $\mathsf{u}_n, \mathsf{f}_n \longrightarrow 0$ *in* $\mathrm{L}^2(\Omega;\, \mathbf{C}^r)$*, then any H-measure*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix}$$

associated to (a subsequence of) the sequence $(\mathsf{u}_n, \mathsf{f}_n)$ *satisfies, in the sense of distributions on* $\Omega \times S^{d-1}$*, the following first order pde:*

$$\partial_l(\partial^l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) - \partial^l_T(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) + (d-1)(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11})\xi^l + 2\mathbf{S} \cdot \boldsymbol{\mu}_{11} = 2\mathsf{Re}\,\mathsf{tr}\boldsymbol{\mu}_{12} \ ,$$

where $\partial^l_T := \partial^l - \xi^l \xi_k \partial^k$ *is the (*l*-th component of) tangential gradient on the unit sphere, while* $\mathbf{S}$ *is the hermitian part of matrix* $\mathbf{B}$*.* ∎

This is based on the Second commutation lemma.

Some function spaces ... in the parabolic case

We take: $k_p(\tau,\boldsymbol{\xi}) := \sqrt[4]{1+(2\pi\tau)^2+(2\pi|\boldsymbol{\xi}|)^4}$ and define

$$\mathrm{X}^{\frac{m}{2},m}(\mathbf{R}^{1+d}) := \left\{ b \in \mathcal{S}' : k_p^m\, \hat{b} \in \mathrm{L}^1(\mathbf{R}^{1+d}) \right\} .$$

$\mathrm{X}^{\frac{m}{2},m}(\mathbf{R}^{1+d})$ is a vector space, a Banach space when equipped with the norm:

$$\|b\|_{\mathrm{X}^{\frac{m}{2},m}} := \int_{\mathbf{R}^{1+d}} k_p^m\, |\hat{b}|\, d\tau d\boldsymbol{\xi} \; .$$

Furthermore, $\mathcal{S} \hookrightarrow \mathrm{X}^{\frac{m}{2},m}(\mathbf{R}^{1+d}) \hookrightarrow \mathcal{S}'$, dense and continuous embeddings.

For $s \in \mathbf{R}$, $s > m + d/2 + 1$, we also have:

$$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) \hookrightarrow \mathrm{X}^{\frac{m}{2},m}(\mathbf{R}^{1+d}) \; .$$

Homogeneous case: $k(\tau,\boldsymbol{\xi}) := \sqrt{1+4\pi^2|(\tau,\boldsymbol{\xi})|^2}$.
For $m \in \mathbf{R}$, $\rho \in \langle 0,1]$ and $\delta \in [0,1\rangle$ define $S^m_{\rho,\delta}$ as the set of all $a \in \mathrm{C}^\infty(\mathbf{R}^{1+d} \times \mathbf{R}^{1+d})$ satisfying:

$$(\forall\,\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbf{N}_0^{1+d})(\exists\,C_{\boldsymbol{\alpha},\boldsymbol{\beta}} > 0) \quad \left|\partial_{\boldsymbol{\beta}}\partial^{\boldsymbol{\alpha}} a\right| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}} k^{m-\rho|\boldsymbol{\alpha}|+\delta|\boldsymbol{\beta}|} \ .$$

$S^m_{\rho,\delta}$ is a vector space of *symbols* of order m and type ρ,δ. (we need $S^m_{\frac{1}{2},0}$)
These symbols define operators on $\mathcal{S}$ (and by transposition also on $\mathcal{S}'$):

$$a(\cdot;D)\varphi := \bar{\mathcal{F}}(a\hat{\varphi}) \ .$$

If $a \in S^k_{\frac{1}{2},0}$ and $b \in S^m_{\frac{1}{2},0}$,then the symbol of the composition of operators $a(\cdot,D)b(\cdot,D)$ is in $S^{k+m}_{\frac{1}{2},0}$, and is given as an asymptotic expansion:

$$\sum_{|\boldsymbol{\alpha}|\geqslant 0} \frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}}_{t,\mathbf{x}} a\, D^{\boldsymbol{\alpha}}_{\tau,\boldsymbol{\xi}} b \ .$$

Second commutation lemma

Lemma. *(**Martin Lazar & N.A.**) Let P_ψ and M_ϕ be operators on $\mathrm{L}^2(\mathbf{R}^{1+d})$, with symbols $\psi \in \mathrm{C}^1(\mathrm{P}^d)$ and $\phi \in \mathrm{X}^{\frac{1}{2},1}(\mathbf{R}^{1+d})$.*
Then for $K = [P_\psi, M_\phi] = P_\psi M_\phi - M_\phi P_\psi$ one has:
a) $K \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^{1+d}); \mathrm{H}^{\frac{1}{2},1}(\mathbf{R}^{1+d}))$.

b) Parabolic extension ψ^p, up to a compact operator on $\mathrm{L}^2(\mathbf{R}^{1+d})$, satisfies:

$$\partial_j K = \partial_j \left(P_\psi M_\phi - M_\phi P_\psi\right) = P_{\xi_j \nabla_{\boldsymbol{\xi}} \psi^p} M_{\nabla_{\mathbf{x}} \phi} \,.$$

Similarly also for $\sqrt{\partial_t}\Big(P_\psi M_\phi - M_\phi P_\psi\Big)$, with ξ_j replaced by $\sqrt{\frac{\tau}{2\pi i}}$. ∎

This result allows for the treatment of nonhypoelliptic parabolic equations (like the Schrödinger equation or the equation for vibrating elastic plate), as it was done for the wave equation.

Calculations in the smooth case

Denote by $\psi^p = \psi \circ p$ the parabolic extension of $\psi \in \mathrm{C}^\infty(\mathrm{P}^d)$ to $\mathbf{R}_*^{1+d}$, and further $\tilde{\psi} := (1-\theta)\psi^p$, for $\theta \in \mathrm{C}_c^\infty(\mathbf{R}^{1+d})$ equal to 1 around the origin, which vanishes for $\rho \geqslant 1$.

Lemma. $\tilde{\psi} \in S^0_{\frac{1}{2},0}$, *while* $\rho^{-m} \in S^{-\frac{m}{2}}_{\frac{1}{2},0}$ *for* $k \geqslant 0$. ∎

For $\tilde{K} := [P_{\tilde{\psi}}, M_\phi]$ one has

$$2\pi i \xi_j \sigma(\tilde{K}) + \partial_j \sigma(\tilde{K}) = \sum_{|(k,\boldsymbol{\alpha})| \geqslant 1} \frac{1}{k!\boldsymbol{\alpha}!} D_\tau^k D_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} \tilde{\psi}\, \partial_t^k \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \left(2\pi i \xi_j \phi + \partial_j \phi\right),$$

$\partial_\tau^k \partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} \tilde{\psi}$ behaves like $\rho^{-(2k+|\boldsymbol{\alpha}|)}$ for large $(\tau, \boldsymbol{\xi})$, while $t, \mathbf{x}$ derivatives of ϕ remain in $S^0_{\frac{1}{2},0}$, so the symbol of the commutator $\tilde{K} \in S^{-\frac{1}{2}}_{\frac{1}{2},0}$, as well as $\partial_j \sigma(\tilde{K})$.
Therefore in the asymptotic expansion the terms of the form $\xi_j \partial_\tau^k \partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} \tilde{\psi}$ belong to $S^{-\frac{1}{2}}_{\frac{1}{2},0}$ for $k \geqslant 1$ or $|\boldsymbol{\alpha}| \geqslant 2$, so as the principal symbol of $\partial_j \tilde{K}$ remains $\xi_j \nabla_{\boldsymbol{\xi}} \tilde{\psi}^p \, \nabla_{\mathbf{x}} \phi$.
As it is parabolicly homogeneous (outside of the compact $\rho \leqslant 1$), so $S^0_{\frac{1}{2},0}$ and determines a bounded operator on $\mathrm{L}^2(\mathbf{R}^{1+d})$.
$\partial_j \tilde{K} - P_{\xi_j \nabla_{\boldsymbol{\xi}} \tilde{\psi}} M_{\nabla_{\mathbf{x}} \phi} \in S^{-\frac{1}{2}}_{\frac{1}{2},0}$, leading to an operator from $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^{1+d}); \mathrm{H}^{\frac{1}{2}}(\mathbf{R}^{1+d}))$.

The Schrödinger equation

$$\begin{cases} i\partial_t u_n + \operatorname{div}(\mathbf{A}\nabla u_n) = f_n \\ u_n(0,\cdot) = u_n^0 \, , \end{cases}$$

where $u_n^0 \longrightarrow 0$ in $\mathrm{H}^1(\mathbf{R}^d)$, $f_n \longrightarrow 0$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbf{R}^d))$, with $(\partial_t f_n)$ being bounded in $\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbf{R}^d))$.

These assumptions assure that (on a subsequence) $\begin{bmatrix} \nabla u_n \\ f_n \end{bmatrix}$ determines the parabolic H-measure of the block form

$$\begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \mu_f \end{bmatrix},$$

where $\boldsymbol{\mu} = \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\nu$, while $\boldsymbol{\mu}_{12} = \boldsymbol{\xi}\nu_{12}$. The localisation principle also gives that $\boldsymbol{\mu}$ is supported within the closed set of $\mathbf{R}^{1+d} \times \mathrm{P}^d$ determined by the relation $Q(t,\mathbf{x};\tau,\boldsymbol{\xi}) = 0$, which is disjoint with the set where $\boldsymbol{\xi} = 0$.

$Q(t,\mathbf{x};\tau,\boldsymbol{\xi}) = 2\pi\tau + 4\pi^2\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi}$ is the symbol of the Schrödinger operator.

Propagation for the Schrödinger equation

Theorem. *If we additionally assume that* $\mathbf{A} \in \mathrm{X}^{\frac{1}{2},1}(\mathbf{R}_0^+ \times \mathbf{R}^d; \mathrm{M}_{d\times d})$ *(i.e. that* $\mathbf{A} \in \mathrm{C}^1 \cap \mathrm{X}^{\frac{1}{2},1}$*), the trace of parabolic H-measure* $\nu = \mathrm{tr}\,\boldsymbol{\mu}$ *satisfies the equation*

$$\langle \nu, \{\Psi, Q\}\rangle + \left\langle \nu, \Psi\,\frac{\alpha^2}{4}\,\frac{3-\alpha^2}{\alpha^2-1}\boldsymbol{\xi}\cdot\nabla_{\mathbf{x}}Q \right\rangle = \langle 2\mathrm{Re}\,\nu_{12}, 4\pi^2|\boldsymbol{\xi}|^2\Psi\rangle\,,$$

where $\Psi = \phi \boxtimes \psi, \phi \in \mathrm{C}_c^1(\mathbf{R}^+ \times \mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^1(\mathrm{P}^d)$*.*

We use the Poisson bracket *(only in* $\mathbf{x}$ *and* $\boldsymbol{\xi}$*, not in* t *and* τ*):*

$$\{\Psi, Q\} := \nabla^{\boldsymbol{\xi}}\Psi\cdot\nabla_{\mathbf{x}}Q - \nabla_{\mathbf{x}}\Psi\cdot\nabla^{\boldsymbol{\xi}}Q\,,$$

while $\alpha^2 = \frac{4}{4-|\boldsymbol{\xi}|^2}$ *on* P^d*.* ∎

After integration by parts on P^d

Theorem. *For H-measure* $\boldsymbol{\mu}$ *associated to* ∇u_n*, with* $\mathbf{A} \in \mathrm{C}^1(\mathbf{R}_0^+ \times \mathbf{R}^d; \mathrm{M}_{d\times d}) \cap \mathrm{X}^{\frac{1}{2},1}(\mathbf{R}_0^+ \times \mathbf{R}^d; \mathrm{M}_{d\times d})$*, the trace* $\nu = \mathrm{tr}\boldsymbol{\mu}$ *satisfies the transport equation*

$$\nabla_{\mathbf{x}}\nu \cdot \left(\nabla^{\boldsymbol{\xi}}Q - \frac{\alpha^2}{4}\left(\alpha^2 + d + \frac{2}{\alpha^2-1}\right)Q\boldsymbol{\xi}\right) - \nabla^{\tau,\boldsymbol{\xi}}\nu \cdot \left(\begin{bmatrix} 0 \\ \nabla_{\mathbf{x}}Q \end{bmatrix} - \left(\begin{bmatrix} 0 \\ \nabla_{\mathbf{x}}Q \end{bmatrix} \cdot \mathsf{n}\right)\mathsf{n}\right) = |2\pi\boldsymbol{\xi}|^2\, 2\mathrm{Re}\,\nu_{12}.$$

■

The characteristics are

$$\frac{d}{ds}\begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \nabla^{\boldsymbol{\xi}}Q - \frac{\alpha^2}{4}\left(\alpha^2 + d + \frac{2}{\alpha^2-1}\right)Q\boldsymbol{\xi} \end{bmatrix}$$

$$\frac{d}{ds}\begin{bmatrix} \tau \\ \boldsymbol{\xi} \end{bmatrix} = -\left(\mathbf{I} - \mathsf{n}\otimes\mathsf{n}\right)\begin{bmatrix} 0 \\ \nabla_{\mathbf{x}}Q \end{bmatrix},$$

with initial conditions

$$t(0) = t_0\,, \quad \mathbf{x}(0) = \mathbf{x}_0\,, \qquad \tau(0) = \tau_0\,, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0\,.$$

This system has a solution, which might not be unique (if we have additional smoothness of $\mathbf{A}$, like $\mathbf{A} \in \mathrm{C}^1(\mathbf{R}_0^+ \times \mathbf{R}^d; \mathrm{M}_{d\times d})$, the solution will be unique).

Propagation of singularities

Take $(\tau_0, \boldsymbol{\xi}_0) \in \mathrm{P}^d$, and multiply the second ode by $\mathsf{n}/\alpha = (\tau, \boldsymbol{\xi}/2)$:

$$\frac{1}{2}\frac{d}{ds}(\tau^2 + |\boldsymbol{\xi}|^2/2) = -(1 - \mathsf{n}^2)\begin{bmatrix} 0 \\ \nabla_{\mathbf{x}} Q \end{bmatrix} \cdot \frac{\mathsf{n}}{\alpha} = 0\,.$$

Then $(\tau, \boldsymbol{\xi})$ remains on P^d over the interval of existence.

By the Theorem, for a homogeneous equation (i.e. when $\nu_{12} = 0$), measure ν remains constant along the integral curves on $\mathbf{R}^{1+d} \times \mathrm{P}^d$.

Furthermore, if $Q(t_0, \mathbf{x}_0; \tau_0, \boldsymbol{\xi}_0) = 0$, then along any characteristic we have $q(s) := Q(t(s), \mathbf{x}(s); \tau(s), \boldsymbol{\xi}(s)) = 0$, for $s \geqslant 0$.

This result generalises the localisation principle $Q\nu = 0$, and shows that Q vanishes along integral curves that pass through the support of ν.

As the H-measure describes the density of microlocal energy, this can be interpreted as its propagation.

A general view

We can unify the results: consider equations of the form

$$P_0 \varrho P_0 u_n + \mathsf{P}_1 \cdot \mathbf{A} \mathsf{P}_1 u_n = 0\,,$$

where P_0 and P_1 stand for (pseudo)differential operators in time and space variables, with (principal) symbols p_0 and p_1, and $Q = \varrho p_0^2 + \mathbf{A}\mathsf{p}_1 \cdot \mathsf{p}_1$ being the symbol of the differential operator defining the left-hand side of the equation. For the parabolic H-measure $\tilde{\boldsymbol{\mu}}$ associated to $(P_0 u_n, \mathsf{P}_1 u_n)$, converging weakly in L^2 to 0, $\tilde{\boldsymbol{\mu}}$ is of the form

$$\tilde{\boldsymbol{\mu}} = \overline{\frac{\mathsf{p} \otimes \mathsf{p}}{|\mathsf{p}|^2}} \tilde{\nu}\,,$$

where $\tilde{\nu} := \mathrm{tr}\tilde{\boldsymbol{\mu}}$ is a scalar measure, and the localisation principle reads

$$Q\tilde{\nu} = 0\,.$$

Finally, the propagation principle states

$$\Big\langle \frac{\xi_m \tilde{\nu}}{|\mathsf{p}|^2}, \{\phi, Q\} \Big\rangle + \Big\langle \frac{\nu}{|\mathsf{p}|^2}, p\,\partial_m Q \Big\rangle = 0\,.$$

This covers both the classical and the parabolic case.

Other variants

— Evgenij Jurjevič Panov (ultraparabolic H-measures)
— Darko Mitrović, Ivan Ivec, Martin Lazar, Marko Erceg (fractional orders)

With the characteristic length — semiclassical measures:
— Patrick Gérard, Pierre-Louis Lions, Thierry Paul (semiclassical/Wigner measures)
— Luc Tartar (variants with a characteristic length, multiscale H-measures)
— Marko Erceg, Martin Lazar, N.A. (one-scale H-measures)

L^p theory
— Darko Mitrović, Marko Erceg, Marin Mišur, N.A. (H-distributions)
— Ivan Ivec, N.A. (mixed-norm variants)
— Marko Erceg and N.A. (semiclassical distributions)

We have a number of tools, and know how to create new ones, adjusted to problems at hand.