H-measures and variants

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Solving partial differential equations

In broad terms, you can try to solve equation

$$A(u) = f$$

by an approximation

$$A_n(u_n) = f_n \ .$$

Usually, when you get a sequence of approximate solutions (u_n), you can only obtain (after taking a subsequence) that $u_n \rightharpoonup u$ in some weak sense (weak topology on a function space).
It is usually a considerable task to prove that from that weak convergence you can infer

$$A_n(u_n) \rightharpoonup A(u) \ ,$$

which completes the argument of approximation.

This approach can be simplified when the equations have a variational structure.

A very simple example

Before more interesting examples it might be good to look at a very simple example.

$$F : [0, 2] \longrightarrow \mathbf{R} \; , \quad F(x) = (x-1)^2 + 1 = x^2 - 2x + 2 \; .$$

F is bounded, $F[0, 2]$ is a bounded set and has an infimum in $\mathbf{R}$.
Therefore there is a minimising sequence (y_n), $\lim y_n = \inf F$.

As the sequence is in the image, we have $x_n \in [0, 2]$, $y_n = F(x_n)$ and therefore:

$$\lim F(x_n) = \lim y_n = \inf F \; .$$

1. Can we conclude that (x_n) has an accumulation point in $[0, 2]$?
   Yes! By compactness of $[0, 2]$. $\quad x_{n_k} \longrightarrow x_0$
2. Can we pass to the limit?
   Yes! F is continuous, so $\lim F(x_{n_k}) = F(x_0) = \inf F$.
   A continuous function attains its minimum and maximum on a compact (a segment on a line).
   This is in essence of the direct approach in the calculus of variations (David Hilbert, Leonida Tonelli).

The direct approach in the calculus of variations

How to find the minimum of a nonlinear integral functional, such as

$$F(u) := \int_\Omega f(u(x))\,dx \quad \text{or} \qquad G(u) := \int_\Omega g(\nabla u(x))\,dx\ ?$$

If we assume that f and g grow as q-th power, then it is natural to seek the minimiser among the functions in:

$$\mathrm{L}^q(\Omega)\ , \quad \text{or} \qquad \mathrm{W}^{1,q}(\Omega)\ ,$$

noting that in the second case we can add a boundary condition.

Of course, we could have taken a space of smoother functions, but it will be immediately clear that this will not solve the problem.
We start with a minimising sequence (u_n) for F or G, say:

$$F(u_n) \longrightarrow \inf F(\mathrm{L}^q(\Omega))\ .$$

Direct approach (continued)

By the assumed coercivity ($F(\xi) \geqslant \alpha|\xi|^q - \beta$), the sequence u_n is bounded in $\mathrm{L}^q(\Omega)$, so it has a weakly convergent subsequence ($q \in \langle 1, \infty \rangle$)

$$u_{n_k} \rightharpoonup u_0 \ .$$

In order to prove that the accumulation point u_0 is actually a minimiser, we need:

$$F(u_0) \leqslant \liminf F(u_{n_k}) \ ,$$

which is a weaker property than continuity (i.e. weak sequential lower semicontinuity).
If it is satisfied, then we have found a minimiser and solved the problem.

Here becomes clear why the metrisability is important: lower semicontinuity, as a topological property, can be characterised by sequences (we have minimising sequences, as the codomain of the functional is $\mathbf{R}$).
Furthermore, the above theory depends on the completeness of the space of functions.
A sufficient condition for semicontinuity of F, G is the convexity of f, g.

Sketch of the Tartar programme

Physical laws are often expressed as systems of partial differential equations, of which some equations can be nonlinear.

It turned out that it is useful to distinguish between two types of physical laws:

(linear) conservation laws . . . mass, energy, momentum, charge etc.
These are generally valid physical laws.

(nonlinear) constitution laws . . . elastic fluids, electrodynamics of continua
These laws characterise particular types of materials.

How to describe the interaction of nonlinear constitutive assumptions and linear conservation laws?

Example: electrostatics

D – electric induction, E – total electric field, ρ – charge density

Maxwell: $\operatorname{div} \mathsf{D} = \rho, \quad \operatorname{rot} \mathsf{E} = 0$
These are general conservation laws (system of linear pde-s)

A particular material is characterised by the relation: $\mathsf{D} = \mathsf{A}(\mathsf{E})$, where A is generally nonlinear.
In vacuum: $\mathsf{A}(\mathsf{E}) = \varepsilon_0 \mathsf{E}$, generally linearised $\mathsf{A}(\mathsf{E}) = \mathbf{A}\mathsf{E}$, where $\mathbf{A}$ depends on the space variable.

On a simply connected domain $\mathsf{E} = -\nabla u$ (a gradient of a potential), so by eliminating D from the system in general we get a nonlinear pde:

$$-\operatorname{div}\left(\mathsf{A}(\nabla u)\right) = \rho \,.$$

Connection between microscopic and macroscopic scale

What are the known mathematical models?

Probabilistic approach: $\omega \in \Pi$ - probability space, $\mathbf{x} \in \Omega \subseteq \mathbf{R}^d$
$u(\mathbf{x}, \omega)$ — microscopic, while expectation $E[u(\mathbf{x}, \cdot)]$ macroscopic quantity
Conceptual weakness: away from the atomic scale we know that physical laws are deterministic.

Periodic modulation: $u : \Omega \times T \longrightarrow \mathbf{R}$, $T := \mathbf{R}^d / \mathbf{Z}^d$ a torus
$u(\mathbf{x}, \mathbf{y})$ — microscopic, while $\int_T u(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$ macroscopic quantity
Good for crystals and some man-made materials, but the periodicity is in general to strong assumption.

Tartar's approach: (u_n) oscillating sequence of functions on Ω
(i.e. (u_n) converges weakly, but not strongly)
The passage from micro- to macro-scale is given by the limit: $u_n \rightharpoonup u$.

What is the connection with physical passage from one scale to the other?

Oscillating solutions of pde-s

A generalisation of periodic modulation.

$u : \Omega \times T \longrightarrow \mathbf{R}$ (periodic in $\mathbf{y}$), $u_n(\mathbf{x}) := u(\mathbf{x}, n\mathbf{y})$
Then $u_n \rightharpoonup u_\infty$, $u_\infty(\mathbf{x}) := \int_T u(\mathbf{x}, \mathbf{y}) d\mathbf{y}$.

If we have a periodicity in the second (fast) variable, and we are interested what happens on the limit when period tends to zero (e.g. the crystal lattice becomes finer and finer—the limit to continuum), we obtain indeed the weak limit of the given sequence.
In physics, this transition from discrete to continuous would be written as:

$$\sum_{i=1}^{n} \frac{L}{n} f\left(\frac{iL}{n}\right) \longrightarrow \int_0^L f(x)\, dx \ ,$$

while in mathematics we assume f to be continuous on $[0, L]$ and write

$$\sum_{i=1}^{n} \frac{L}{n} \delta_{\frac{iL}{n}} \xrightarrow{\mathcal{M}_b *} \chi_{[0,L]} \ .$$

Here δ denotes the Dirac mass, while $\mathcal{M}_b$ the space of bounded Radon measures on $[0, L]$, which is the dual of $C[0, L]$ (the construction of the weak $*$ topology we are going to describe can be applied; the spaces are not reflexive!).

What can be said about nonlinear constraints?

Weak convergence is well behaved with respect to linear operators. However, we would like to consider nonlinear laws as well.

For simplicity, take L^∞ with weak $*$ topology and $F : \mathbf{R}^r \longrightarrow \mathbf{R}$ continuous (so that $F \circ u_n$ is again a bounded sequence, if u_n is such in L^∞).

Theorem. Let $K \subseteq \mathbf{R}^r$ be a bounded set, (u_n) a sequence in $\mathrm{L}^\infty(\Omega; K)$, $\mathsf{u}_n \stackrel{*}{\longrightarrow} \mathsf{u}$.

Then $\mathsf{u}(\mathbf{x}) \in \mathsf{Cl}\,\mathrm{conv} K$ (a.e. $\mathbf{x}$).

Conversely, for $\mathsf{u} \in \mathrm{L}^\infty(\Omega; \mathsf{Cl}\,\mathrm{conv} K)$ there is a *sequence* $\mathsf{u}_n \in \mathrm{L}^\infty(\Omega; K)$ such that $\mathsf{u}_n \stackrel{*}{\longrightarrow} \mathsf{u}$.

[If K is not bounded, the converse is not true.]

An example

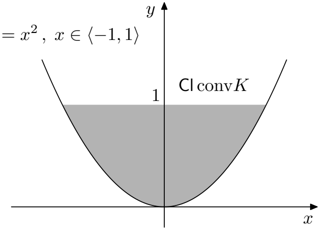
For the sequence $u_n(x) = \sin nx\,,\ x \in \Omega = \langle -\pi, \pi \rangle$ we have (in L^∞)

$$u_n \xrightharpoonup{*} 0$$
$$u_n^2 \xrightharpoonup{*} \frac{1}{2}$$

In general, if $u_n \xrightharpoonup{*} u$ then also $F \circ u_n \xrightharpoonup{*} F \circ u$ for a linear function F, but not necessarily for a nonlinear ... We need Young measures for that.

Another approach is based on the previous theorem ... $\mathsf{v}_n := \begin{bmatrix} u_n \\ u_n^2 \end{bmatrix}$

$$\mathsf{v}_n \in \mathrm{L}^\infty(\Omega; K)$$
$$\mathsf{v}_n \xrightharpoonup{*} ?$$



Young measures

Theorem. Let (u_n) be a sequence in $\mathrm{L}^\infty(\Omega; K)$.
Then there is a subsequence (u_{n_k}) and a weakly $*$ measurable family of probability measures $(\nu_x, x \in \Omega)$ supported on $\mathsf{Cl}\, K$, such that for any continuous function F on $\mathsf{Cl}\, K$ one has

$$F \circ \mathsf{u}_{n_k} \xrightarrow{\;*\;} \langle \nu_\cdot, F \rangle = \int_{\mathsf{Cl}\, K} F(\lambda)\, d\nu_\cdot(\lambda)\,.$$

If K is bounded, the converse is also true.

(More precisely: $\nu \in \mathrm{L}^\infty_*(\Omega; \mathcal{M}_b(\mathsf{Cl}\, K))$, as $\mathcal{M}_b(\mathsf{Cl}\, K)$ is not reflexive.)

Young measures — an application

Let us see how the above can be applied in describing the limit of $F \circ \mathsf{u}_n$ (at least on a subsequence). In an earlier example:

$$\mathsf{E}_n \xrightharpoonup{*} \mathsf{E} \quad \Longrightarrow \quad \mathsf{D}_n \xrightharpoonup{*} \int \mathsf{A}(\lambda) d\nu.(\lambda) \ .$$

On a subsequence we get that

$$u_n \xrightharpoonup{*} \int \lambda d\nu.(\lambda) \ .$$

Conversely, if for any continuous F holds:

$$F \circ \mathsf{u}_n \xrightharpoonup{*} \int F(\lambda)\, d\nu.(\lambda) \, ,$$

then necessarily $\nu_x = \delta_{u(x)}$, and the sequence converges strongly.

div $-$ rot lemma: example in electrostatics

On the microscopic level the fields obey the Maxwell system: $\operatorname{div} \mathsf{D}_n = \rho$ and $\operatorname{rot} \mathsf{E}_n = 0$, and we have the electrostatic energy $\int \mathsf{E}_n \cdot \mathsf{D}_n$.

What can we say about that energy on the macroscopic scale?

$$\mathsf{E}_n \xrightarrow{\mathrm{L}^2} \mathsf{E} \qquad \text{and} \qquad \mathsf{D}_n \xrightarrow{\mathrm{L}^2} \mathsf{D}\ .$$

$$\mathsf{E}_n \cdot \mathsf{D}_n \xrightarrow{\mathcal{M}_b*} \mathsf{E} \cdot \mathsf{D}\ .$$

This is the consequence of the famous div-rot lemma (Murat, Tartar), and the physical meaning is that there is no hidden electrostatic energy.

An example with the wave equation ($d = 2$)

$$\rho_n u_n'' - \mathrm{div}\,(\mathbf{A}_n u_n) = 0 \qquad \text{with corresponding initial/boundary conditions}$$

kinetic energy: $\frac{1}{2}\rho|u'|^2$
potential energy: $\frac{1}{2}\mathbf{A}\nabla u \cdot \nabla u$
From $u_n \xrightarrow{\mathrm{H}^1} 0$ it follows that

$$\frac{1}{2}\rho|u'|^2 - \frac{1}{2}\mathbf{A}\nabla u \cdot \nabla u \xrightarrow{\mathcal{M}_{b*}} 0\,,$$

which gives the macroscopic equipartition of energy (while the total energy is not zero, as that would imply the strong convergence).

Compactness by compensation

$\mathsf{u}_n \longrightarrow \mathsf{u}_0$ in $\mathrm{L}^2(\Omega;\mathbf{R}^r)$, $\mathcal{A}\nabla\mathsf{u}_n = \mathbf{A}^k\partial_k\mathsf{u}_n$ precompact in $\mathrm{H}^{-1}_{\mathrm{loc}}(\Omega;\mathbf{R}^r)$
($\mathcal{A}$ is a third rank tensor, with constant coefficients).

A characteristic set:

$$\mathcal{V} := \left\{(\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathbf{R}^r \times S^{d-1} : \mathcal{A}(\boldsymbol{\xi}\otimes\boldsymbol{\lambda}) = \mathbf{A}^k\boldsymbol{\lambda}\xi_k = 0\right\} ,$$

and its projection to the physical space:

$$\Lambda := \left\{\boldsymbol{\lambda} \in \mathbf{R}^r : (\exists\, \boldsymbol{\xi} \in S^{d-1})\ (\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathcal{V}\right\} .$$

Theorem. *For any quadratic form Q, for which $Q(\Lambda) \geqslant 0$, any weak $*$ accumulation point l of sequence $Q(\mathsf{u}_n)$ satisfies $l \geqslant Q(\mathsf{u}_0)$.* ∎

Example. $\mathsf{u}_n \longrightarrow \mathsf{u}_0$ in $\mathrm{L}^2(\mathbf{R}^2;\mathbf{R}^2)$, while $(\partial_1 u_n^1)$ and $(\partial_2 u_n^2)$ are bounded in $\mathrm{L}^2(\mathbf{R}^2)$ (therefore precompact in $\mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^2)$). The characteristic set is
$\mathcal{V} = \{(\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathbf{R}^2 \times S^1 : \xi_1\lambda^1 = \xi_2\lambda^2 = 0\}$, and its projection
$\Lambda = \{\boldsymbol{\lambda} \in \mathbf{R}^2 : \lambda^1\lambda^2 = 0\}$.
$Q(\boldsymbol{\lambda}) := \lambda^1\lambda^2$ annuls on Λ ($\pm Q(\Lambda) \geqslant 0$).
Therefore any accumulation point of $u_n^1 u_n^2$ is equal to $u_0^1 u_0^2$ (weak $*$ in measures).

Weak convergence in L^p and Sobolev spaces

The Sobolev spaces, which form the framework for modern study of pde-s, are based on L^p spaces.

We take $\frac{1}{p} + \frac{1}{p'} = 1$; $p \in [1, \infty]$.

For $p \in [1, \infty\rangle$ one has $(\mathrm{L}^p)' = \mathrm{L}^{p'}$, so the weak convergence is:

$$f_n \rightharpoonup f \qquad \Longleftrightarrow \qquad (\forall \varphi \in \mathrm{L}^{p'}) \int f_n \varphi \longrightarrow \int f \varphi$$

For $p \in \langle 1, \infty \rangle$ we have nice results:

- By density, it is enough to test the convergence only on $\varphi \in \mathrm{C}_c^\infty$.
- Weak compactness: $\|f_n\|_{\mathrm{L}^p} \leqslant C \quad \Longrightarrow \quad f_{n_k} \rightharpoonup f$.
- $f_n \rightharpoonup f$ and $\|f_n\|_{\mathrm{L}^p} \longrightarrow \|f\|_{\mathrm{L}^p}$ implies $f_n \longrightarrow f$.

Starting from L^p we define the Sobolev spaces (on $\Omega \subseteq \mathbf{R}^d$)

$$\mathrm{W}^{m,p}(\Omega) := \{u \in \mathrm{L}^p(\Omega) : \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^p(\Omega), |\boldsymbol{\alpha}| \leqslant m\} .$$

These spaces have the same reflexivity and separability properties as the corresponding L^p spaces.

There are also loc variants, spaces of functions with compact support, ...

Vector spaces in duality

$a : V \times W \longrightarrow F$ ($\mathbf{R}$ or $\mathbf{C}$) given billinear (or sesquilinear) form.
Vector spaces V and W are *in duality* with respect to a if:

1) $(\forall\, w \in W)\ \ a(v,w) = 0 \quad \Longrightarrow \quad v = 0$
2) $(\forall\, v \in V)\ \ a(v,w) = 0 \quad \Longrightarrow \quad w = 0$.

Then for any $w \in W$ we can define a seminorm on V by formula $|v|_w := |a(v,w)|$. This family of seminorms $(|v|_w, w \in W)$ defines a locally convex Hausdorff topology on V, which we denote by $\sigma(V,W)$.
Similarly we define the topology $\sigma(W,V)$ on W.
Locally convex (Hausdorff) topological vector space V and its dual V' are in duality, with respect to pairing:

$$a(v,v') = \langle v, v' \rangle = v'(v)\ .$$

(2) is trivially satisfied, while for (1) we need the Hahn-Banach theorem.
In such a way we get the weak topology $\sigma(V,V')$ on V,
and the weak * topology $\sigma(V',V)$ on V'.
In particular, this works for unitary (Hilbert) spaces, normed (Banach) and metrisable locally convex (Fréchet) spaces; in each case we apply the above construction.
(N.B. Usually, weak topologies are not metrisable, so sequences do not characterise their properties.)

$p = \infty$ and $p = 1$

What is the dual of L^∞? *a huge mess*
Practically, we can use the weak * convergence on L^∞, for which we have the Banach compactness theorem.
So, $\|f_n\|_{\mathrm{L}^\infty} \leqslant C \quad \Longrightarrow \quad f_{n_k} \xrightarrow{\;*\;} f$.
(bounded sets are even metrisable in the weak $*$ topology)

It remains L^1; it is known that for weak convergence we additionally need equiintegrability (the Dunford-Pettis theorem); furthermore, L^∞ is not separable, neither is C_c^∞ dense in it, all of which precludes the above conclusions.

However, $\mathrm{L}^1 \hookrightarrow \mathcal{M}_b$ (bounded Radon measures), with $\|f\|_{\mathcal{M}_b} = \|f\|_{\mathrm{L}^1}$.
As $\mathcal{M}_b = (\mathrm{C}_0)'$, where C_0 is a Banach space, the closure of C_c^∞ in the norm of L^∞.

We again have: $\|f_n\|_{\mathrm{L}^1} \leqslant C \quad \Longrightarrow \quad f_{n_k} \xrightarrow{\;*\;} \mu$ (in the space $\mathcal{M}_b$).

What are H-measures?

Mathematical objects introduced by:

- Luc Tartar, motivated by intended applications in homogenisation (H), and
- Patrick Gérard, whose motivation were certain problems in kinetic theory (and who called these objects *microlocal defect measures*).

Start from $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$, and take the Fourier transform:

$$\widehat{\varphi u_n}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} (\varphi u_n)(\mathbf{x}) d\mathbf{x}$$

As φu_n is supported on a fixed compact set K, so $|\widehat{\varphi u_n}(\boldsymbol{\xi})| \leqslant C$.
Furthermore, $u_n \rightharpoonup 0$, and from the definition $\widehat{\varphi u_n}(\boldsymbol{\xi}) \longrightarrow 0$ pointwise.
By the Lebesgue dominated convergence theorem applied on bounded sets, we get
$\widehat{\varphi u_n} \longrightarrow 0$ strong, i.e. strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$.

On the other hand, by the Plancherel theorem: $\|\widehat{\varphi u_n}\|_{\mathrm{L}^2(\mathbf{R}^d)} = \|\varphi u_n\|_{\mathrm{L}^2(\mathbf{R}^d)}$.

If $\varphi u_n \not\longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, then $\widehat{\varphi u_n} \not\longrightarrow 0$; some information must go to infinity.

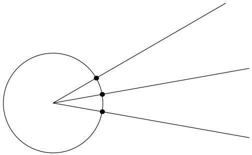
Limit is a measure

How does it go to infinity in various directions? Take $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, and consider:

$$\lim_n \int_{\mathbf{R}^d} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)|\widehat{\varphi u_n}|^2 d\boldsymbol{\xi} = \int_{\mathrm{S}^{d-1}} \psi(\boldsymbol{\xi}) d\nu_\varphi(\boldsymbol{\xi}) \; .$$

The limit is a linear functional in ψ, thus an integral over the sphere of some nonnegative Radon measure (a bounded sequence of Radon measures has an accumulation point), which depends on φ.

How does it depend on φ?



The inverse Fourier transform:

$$\check{\mathsf{v}}(\mathbf{x}) := \bar{\mathcal{F}}\mathsf{v}(\mathbf{x}) := \int_{\mathbf{R}^d} e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} \mathsf{v}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \; .$$

Existence of H-measures

Theorem. (u^n) *a sequence in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, $\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there is a subsequence* $(\mathsf{u}^{n'})$ *and* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that:*

$$\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{F}\Big(\varphi_1 \mathsf{u}^{n'}\Big) \otimes \mathcal{F}\Big(\varphi_2 \mathsf{u}^{n'}\Big) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi}) \ .$$

∎

Notation:

$$\mathsf{v} \cdot \mathsf{u} := \sum v_i \bar{u}_i$$

$$(\mathsf{v} \otimes \mathsf{u})\mathsf{a} := (\mathsf{a} \cdot \mathsf{u})\mathsf{v}$$

$$\mathbf{A} \cdot \mathbf{B} := \mathrm{tr}(\mathbf{A}\mathbf{B}^*)$$

$$\partial_j := \frac{\partial}{\partial x^j}$$

$$\partial^j := \frac{\partial}{\partial \xi_j}$$

A class of symbols

Linear partial differential operator in the classical theory:

$$Lu := P(\mathbf{x}, D)u ,$$

where

$$P(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\alpha|\leq m} c_\alpha(\mathbf{x})\boldsymbol{\xi}^\alpha , \qquad D_j := \frac{1}{2\pi i}\partial_j ,$$

$$P(\mathbf{x}, D)u(x) = \sum_{|\alpha|\leq m} c_\alpha(\mathbf{x})\bar{\mathcal{F}}\big(\boldsymbol{\xi}^\alpha \hat{u}(\boldsymbol{\xi})\big) .$$

We are interested in similar operators:

$$A\mathsf{u}(\mathbf{x}) := \bar{\mathcal{F}}\left(a\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)\hat{\mathsf{u}}(\boldsymbol{\xi})\right) \qquad \text{the Fourier multiplier}$$

$$B\mathsf{u}(\mathbf{x}) := b(\mathbf{x})\mathsf{u}(\mathbf{x}) \qquad \text{multiplication (the Sobolev multiplier)} .$$

The first commutation lemma

Lemma. *For* $a \in \mathrm{C}(S^{d-1})$ *and* $b \in \mathrm{C}_0(\mathbf{R}^d)$ *the operators* A *and* B *are in* $\mathcal{H} := \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$,
and their norms coincide with the sup norms of functions a *and* b.
Commutator $C := [A, B] = AB - BA$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$ *(denoted by* $C \in \mathcal{K}(\mathrm{L}^2(\mathbf{R}^d))$*).*

∎

Admissible symbol is a function $P \in \mathrm{C}(\mathbf{R}^d \times S^{d-1})$ of the form:
$P(\mathbf{x}, \boldsymbol{\xi}) = \sum_k b_k(\mathbf{x}) a_k(\boldsymbol{\xi})$; with $a_k \in \mathrm{C}(S^{d-1}), b_k \in \mathrm{C}_0(\mathbf{R}^d)$ satisfying:
$\sum_k \|a_k\|_\infty \|b_k\|_\infty < \infty$.
$L \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$ has an admissible symbol P if it can be written in the form:
$L = \sum_k A_k B_k (\mathrm{mod}\ \mathcal{K}(\mathrm{L}^2(\mathbf{R}^d)))$.
Among all operators corresponding to the given symbol P we choose the *standard*one: $L_0 := \sum_k A_k B_k$, satisfying (for $u \in \mathrm{L}^2(\mathbf{R}^d) \bigcap \mathrm{L}^1(\mathbf{R}^d)$):

$$\mathcal{F}(L_0 u)(\boldsymbol{\xi}) = \sum_k a_k\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) \int_{\mathbf{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} b_k(\mathbf{x}) u(\mathbf{x})\, d\mathbf{x}$$
$$= \int_{\mathbf{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} P\left(\mathbf{x}, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) u(\mathbf{x})\, d\mathbf{x}\ .$$

Therefore, L_0 is well defined—it does not depend on the choice of admissible symbol P.

Quantisation

The above definitions lead to the correspondence between multiplication in $\mathcal{H}/\mathcal{K}$ and the multiplication of symbols.

Consider $L := \sum_k B_k A_k$, where A_k and B_k are as in the decomposition of standard operator L_0. For $u \in \mathrm{L}^2(\mathbf{R}^d) \cap \mathcal{F}\big(\mathrm{L}^1(\mathbf{R}^d)\big)$ we have:

$$
\begin{aligned}
Lu(\mathbf{x}) &= \sum_k b_k(\mathbf{x}) \int_{\mathbf{R}^d} e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} a_k\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) \hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi} \\
&= \int_{\mathbf{R}^d} e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} P\left(\mathbf{x}, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) \hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi}
\end{aligned}
$$

and this is exactly the operator with symbol P in the linear theory.

Note that L and L_0 differ to a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$:

$$
L - L_0 = \sum_k \big(B_k A_k - A_k B_k\big) = \sum_k \left[B_k, A_k\right] ,
$$

as by the First commutation lemma every term is compact, with the norm estimated by $2\|a_k\|_\infty \|b_k\|_\infty$
(here we use the fact that the uniform limit of compact operators is again compact).

Symbols and operators

$$Au(\mathbf{x}) = \int_{\mathbf{R}^d} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} a(\mathbf{x},\boldsymbol{\xi})\hat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi} \;.$$

Classically, $a \in \mathrm{S}^m$, a $A \in \Psi^m$ (Hörmander's classes):

$$(\forall\, \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbf{N}_0^d)(\exists\, C_{\boldsymbol{\alpha}\boldsymbol{\beta}} > 0) \quad |\partial_{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}} a(\mathbf{x},\boldsymbol{\xi})| \leq C_{\boldsymbol{\alpha}\boldsymbol{\beta}}\Big(\sqrt{1+4\pi^2|\boldsymbol{\xi}|^2}\Big)^{m-|\boldsymbol{\beta}|} \;,$$

$\Psi_c^m(\mathbf{R}^d)$ is a subspace of $\Psi^m(\mathbf{R}^d)$ consisting of operators with compactly supported symbols in variable $\mathbf{x}$, which gives us operator $A : \mathcal{S}' \to \mathcal{E}'$.
The theorem on existence of H-measures we can rephrase ($\mathbf{p}$ is the principal symbol of $\mathbf{P}$):

$$(\forall\, \mathbf{P} \in \Psi_c^0(\mathbf{R}^d; \mathbf{M}^{r\times r})) \quad \lim_{n'} \int_{\mathbf{R}^d} \mathbf{P}\mathsf{u}_{n'} \cdot \mathsf{u}_{n'} \, d\mathbf{x} = \langle \boldsymbol{\mu}, \mathbf{p} \rangle \;.$$

In applications we are interested in less regular classes of symbols.

Symmetric systems

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \ , \ \mathbf{A}^k \ \text{Hermitian}$$

Assume:

$$\mathsf{u}^n \xrightharpoonup{\mathrm{L}^2} 0 \quad \text{(weakly)},$$

$$\mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \quad \text{(strongly)}.$$

The supports of $\mathsf{u}^n, \mathsf{f}^n$ are contained in a compact inside Ω; then we extend them by zero to $\mathbf{R}^d$.

Theorem. (localisation property) *If* $\mathsf{u}^n \rightharpoonup 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)^r$ *defines* $\boldsymbol{\mu}$*, and if* u^n *satisfies:*

$$\partial_k\left(\mathbf{A}^k \mathsf{u}^n\right) \to 0 \ \text{ in the space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \ ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$ *on* $\Omega \times S^{d-1}$ *it holds:*

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu} = \mathbf{0} \ .$$

∎

Thus, the support of H-measure $\boldsymbol{\mu}$ is contained in the set $\left\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\right\}$ of points where $\mathbf{P}$ is a singular matrix.)
The localisation property is behind known applications in homogenisation of small amplitudes.

Second commutation lemma

$$X^m := \left\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d)\ |\boldsymbol{\alpha}| \le m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \right\}$$

is a Banach space with the norm:

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \left(1 + 4\pi^2 |\boldsymbol{\xi}|^2\right)^{m/2} \left|\hat{w}(\boldsymbol{\xi})\right| d\boldsymbol{\xi} \ .$$

$X^m \subseteq \mathrm{C}^m(\mathbf{R}^d)$, and the derivatives up to order m vanish in infinity (they are in $\mathrm{C}_0(\mathbf{R}^d)$).

On the other hand, $\mathrm{H}^s(\mathbf{R}^d) \subseteq X^m$, for $s > m + \frac{d}{2}$.

X^m is an algebra with respect to the multiplication of functions; it holds:

$$\|f * g\|_{\mathrm{L}^1} \le \|f\|_{\mathrm{L}^1} \|g\|_{\mathrm{L}^1}$$

$$\|\hat{f} \cdot \hat{g}\|_{X^0} \le \|\hat{f}\|_{X^0} \|\hat{g}\|_{X^0}$$

$X^m_{\mathrm{loc}}(\Omega)$: the space of all functions u such that $\varphi u \in X^m$, for $\varphi \in \mathrm{C}_c^\infty(\Omega)$.

Lemma. *A, B standard operators, with symbols a, b, with one of the assumptions:*

a) $a \in \mathrm{C}^1(S^{d-1})$ *and* $b \in X^1$.

b) $a \in X^1_{loc}(\mathbf{R}^d_*)$ *and* $b \in \mathrm{C}^1_0(\mathbf{R}^d)$.

Then $C := [A, B] \in \mathcal{L}\big(\mathrm{L}^2(\mathbf{R}^d), \mathrm{H}^1(\mathbf{R}^d)\big)$, *and* ∇C *has a symbol* $(\nabla_{\boldsymbol{\xi}} a \cdot \nabla_{\mathbf{x}} b)\boldsymbol{\xi}$.

(in (a) we extend a to a homogeneous function on $\mathbf{R}^d_* := \mathbf{R}^d \setminus \{\mathbf{0}\}$) ∎

Propagation property for symmetric systems

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad , \qquad \mathbf{A}^k \text{ Hermitian}$$

Theorem. *Let* $\mathbf{A}^k \in \mathrm{C}_0^1(\Omega; \mathrm{M}_{r\times r})$*.*
If $(\mathsf{u}^n, \mathsf{f}^n)$ *satisfy the above for* $n \in \mathbf{N}$*, and* $\mathsf{u}^n, \mathsf{f}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\Omega)$*, then for any* $\psi \in \mathrm{C}_0^1(\Omega \times S^{d-1})$*, the H-measure associated to sequence* $(\mathsf{u}^n, \mathsf{f}^n)$*:*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix},$$

satisfies:

$$\left\langle \boldsymbol{\mu}_{11}, \{\mathbf{P}, \psi\} + \psi \partial_k \mathbf{A}^k - 2\psi \mathbf{S} \right\rangle + \langle 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12}, \psi \rangle = 0 \ ,$$

where $\mathbf{S} := \frac{1}{2}(\mathbf{B} + \mathbf{B}^*)$*, while the Poisson bracket is:*
$\{\phi, Q\} = \nabla_{\boldsymbol{\xi}} \phi \cdot \nabla_{\mathbf{x}} Q - \nabla_{\mathbf{x}} \phi \cdot \nabla_{\boldsymbol{\xi}} Q.$ ∎

$\boldsymbol{\mu}$ is associated to the pair of sequences $(\mathsf{u}^n, \mathsf{f}^n)$, the block $\boldsymbol{\mu}_{11}$ is determined by u^n, $\boldsymbol{\mu}_{22}$ with f^n, while the non-diagonal blocks correspond to the product of u^n and f^n.

The equation for H-measure

Corollary. *In the sense of distributions on* $\Omega \times S^{d-1}$ *the H-measure* $\boldsymbol{\mu}$ *satisfies:*

$$\begin{aligned}\partial^l \mathbf{P} \cdot \partial_l \boldsymbol{\mu}_{11} - \partial_t^l(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) + (d-1)(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11})\xi^l \\ + (2\mathbf{S} - \partial_l \mathbf{A}^l) \cdot \boldsymbol{\mu}_{11} = 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12} \,,\end{aligned}$$

where $\partial_t^l := \partial^l - \xi^l \xi_k \partial^k$ *is the tangential gradient on the unit sphere.* ∎

This allows us to investigate the behaviour of H-measures as solutions of initial-value problems, with appropriate initial conditions.
Besides the wave equations, there are applications to Maxwell's and Dirac's systems, even to the equations that change their type (like the Tricomi equation).

The wave equation

$$(\rho u')' - \text{div}\,(\mathbf{A}\nabla u) = g\ .$$

It can be written as an equivalent symmetric system ($t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$):

$$\partial_0(\rho\partial_0 u) - \sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u) = g\ .$$

By introducing: $v_j := \partial_j u$, for $j \in 0..d$, we obtain (Schwarz' symmetries!):

$$\begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A} & \\ 0 & & & \end{bmatrix} \partial_0 \mathsf{v} + \sum_{i=1}^{d} \begin{bmatrix} 0 & -a^{i1} & \cdots & -a^{id} \\ -a^{i1} & & & \\ \vdots & & \mathbf{0} & \\ -a^{id} & & & \end{bmatrix} \partial_i \mathsf{v}$$

$$+ \begin{bmatrix} b^0 & b^1 & \cdots & b^d \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \mathsf{v} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix}\ .$$

The symbol of differential operator is:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = \xi_k \mathbf{A}^k(\mathbf{x}) = \begin{bmatrix} \xi_0 \rho & -(\mathbf{A}\boldsymbol{\xi}')^\top \\ -\mathbf{A}\boldsymbol{\xi}' & \xi_0 \mathbf{A} \end{bmatrix}\ .$$

Transport of H-measures associated to the wave equation

From the localisation property we can conclude that $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu$. For the right hand side of the equation we have:

$$\langle \gamma, \varphi_1 \bar{\varphi}_2 \psi \rangle := \lim_n \int_{\mathbf{R}^{d+1}} \widehat{\varphi_1 v_{0,n}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 g_n}(\boldsymbol{\xi})} \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} \; .$$

Theorem. *On $\mathbf{R}^{d+1} \times S^d$ measure ν satisfies ($Q := \rho\xi_0^2 - \mathbf{A}\boldsymbol{\xi}' \cdot \boldsymbol{\xi}'$):*

$$\nabla_{\boldsymbol{\xi}} Q \cdot \nabla_{\mathbf{x}}(\xi_0 \nu) - Q\partial_0 \nu + \big(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \mathbf{I}\big)\nabla_{\mathbf{x}} Q \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu) + (d+2)\big(\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}\big)(\xi_0 \nu) = 2\mathrm{Re}\,\gamma$$

∎

Theorem follows from the Corollary, if we write out the terms, e.g.:

$$\partial^l \mathbf{P} \cdot \partial_l \boldsymbol{\mu} = \nabla_{\boldsymbol{\xi}} Q \cdot \nabla_{\mathbf{x}}(\xi_0 \nu) - Q\partial_0 \nu$$
$$\partial_l \mathbf{P} \cdot \partial^l \boldsymbol{\mu} = \partial_0 Q \nu + \nabla_{\mathbf{x}} Q \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu)$$
$$\dots$$

The equation can be written in a nicer form:

$$\{Q, \xi_0 \nu\} + (\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi})\big[\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu) + (d+2)(\xi_0 \nu)\big] - Q\partial_0 \nu = 2\mathrm{Re}\,\gamma \; .$$

An explicit example

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\cdot) = v \\ u_t(0,\cdot) = w \ . \end{cases}$$

D'Alembert's formula for solution:

$$u(t,x) = \frac{v(x+t)+v(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} w(y)\,dy \ .$$

Physically important quantity is *energy density:*

$$d(t,x) := \frac{1}{2}(u_t^2 + u_x^2) \ ,$$

as well as the *energy* at time t: $$e(t) := \int_{\mathbf{R}} d(t,x)\,dx.$$

After simple calculations we get

$$4d(t,x) = \Big(v'(x+t)+w(x+t)\Big)^2 + \Big(v'(x-t)-w(x-t)\Big)^2 \ .$$

Assume that the physical system is modelled by the above wave equation on the microscale. In order to pass to the macroscale, in the spirit of Tatar's programme, we have to pass to the weak limit.

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightharpoonup{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightharpoonup{\mathrm{L}^2(\mathbf{R})} 0 \ .$$

It follows that

$$u_n \rightharpoonup 0 \ ,$$

but $d_n \rightharpoonup d \geqslant 0$ weakly $*$ in the space of Radon measures; in general d is not zero.

Applying the div-rot lemma we arrive at *equipartition of energy*, i.e.
$u_t^2 - u_x^2 \rightharpoonup 0$;
the kinetic and potential energy are balanced at the macroscopic level.

In order to determine the solution completely, let us take periodically modulated initial conditions (we work in spaces $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R})$ and $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})$):

$$v_n(x) := \frac{1}{n}\sin(nx) \qquad \text{and} \qquad w_n(x) := \sin(nx) \ .$$

Simple calculations lead us to: $d_n(t,x) = 1 + \cos 2nx \sin 2nt \rightharpoonup 1$,
weak $*$ in the space of Radon measures, therefore in the space of distributions as well.

Even though the sequence of solutions (u_n) weakly converges to zero, the energy density is 1, equally distributed to kinetic and potential energy.

How this can be computed in general?

Two interesting quadratic forms:

$$q(x;\mathsf{v}) := \frac{1}{2}\left[\rho(x)v_0^2 + \mathbf{A}(x)v\cdot v\right]\,,$$
$$Q(x;\mathsf{v}) := \frac{1}{2}\left[\rho(x)v_0^2 - \mathbf{A}(x)v\cdot v\right]\,.$$

Convergence of initial data and uniformly compact support imply:

$$u_n \overset{*}{\rightharpoonup} 0 \quad \text{in} \quad \mathrm{L}^\infty(\mathbf{R};\mathrm{H}^1)\cap \mathrm{W}^{1,\infty}(\mathbf{R};\mathrm{L}^2).$$

The energy density is $d_n = q(\nabla u_n)$.
Goal: compute the distributional limit d_n, i.e. the limit

$$D_n = \int_{\langle 0,T\rangle\times\mathbf{R}^d} d_n\phi\, dtdx\;.$$

Results:

— Gilles Francfort & François Murat: in linear case, C^∞ coefficients
— Patrick Gérard: for constant coefficients, nonlinearity with term $u^p, p\leqslant 5$
— Martin Lazar and N.A.: for semilinear case ($d=3, p=3$), variable coefficients

Expressing H-measure via a scalar measure

By multiplying with a cutoff function $\theta = 1$ on $[0, T]$, we define L^2 functions
$\mathsf{V}^n(t, x) := \theta(t)\nabla u_n(t, x)$;
$\mathsf{V}^n \rightharpoonup 0$ u $\mathrm{L}^2(\mathbf{R}^{d+1})$; also take ΨDO P with principal symbol

$$\mathbf{p}^0(\mathbf{x}) = \begin{bmatrix} \rho\phi & 0 \\ 0 & \phi\mathbf{A} \end{bmatrix}.$$

The limit can be expressed via H-measure $\boldsymbol{\mu}$ of sequence V^n

$$\lim_n D_n = \lim_n \frac{1}{2} \int_{\mathbf{R}^{d+1}} P\mathsf{V}^n \cdot \mathsf{V}^n d\mathbf{x} = \frac{1}{2}\langle \boldsymbol{\mu}, \mathbf{p}^0 \rangle,$$

Theorem. *The H-measure is of the form* $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu$, *where* $\nu \geqslant 0$ *is a scalar Radon measure satisfying* $Q(x, \boldsymbol{\xi})\nu = 0$.
If $\boldsymbol{\mu}'$ *is an H-measure of the sequence* $(\nabla u_n, f_n)$, *where* f_n *is a nonlinear (or nonhomogeneous) term (*$f_n = u_n^3$*), then*

$$\boldsymbol{\mu}' = \begin{bmatrix} (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu & \boldsymbol{\xi}\nu^{12} \\ (\boldsymbol{\xi}\nu^{12})^* & \times \end{bmatrix},$$

and we have the transport equation

$$\langle \xi_0\nu, \{\phi, Q\} \rangle = 2\langle \xi_0 \mathrm{Re}\, \nu^{12}, \phi \rangle, \qquad \phi \in \mathrm{C}_c^\infty(\mathbf{R}^{d+1} \times S^d),$$

∎

Express ν from the initial conditions

Corollary. *Weak $*$ limit of energy densities d_n is a measure d on $\mathbf{R}^{d+1}$ given by*

$$d = \rho \int_{S^d} (\xi_0)^2 d\nu(\cdot, \boldsymbol{\xi}) \ .$$

∎

It remains to determine the H-measure ν; by integration by parts in the propagation property:

$$\nabla_{\mathbf{x}}(\xi_0\nu) \cdot \Big(\nabla_{\boldsymbol{\xi}} Q - (d+2)Q\boldsymbol{\xi}\Big) - \nabla_{\boldsymbol{\xi}}(\xi_0\nu) \cdot \Big(\nabla_{\mathbf{x}} Q - (\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi})\boldsymbol{\xi}\Big) = 2\xi_0 \mathsf{Re}\, \nu_{12}$$

For $\Phi \in \mathrm{C}_c^\infty([0,\infty\rangle \times \mathbf{R}^d \times S^d))$ we define:

$$\langle\!\langle \nu, \Phi \rangle\!\rangle := \int_{\mathbf{R}^{d+1}} \int_{S^d} \Phi(t, x, \boldsymbol{\xi}) \chi_{\langle 0,\infty\rangle}(t) d\nu,$$

and have

$$\langle\!\langle \xi_0\nu, \{\Phi, Q\} \rangle\!\rangle - \langle\!\langle 2\xi_0 \mathsf{Re}\, \nu^{12}, \Phi \rangle\!\rangle = \int_{\mathbf{R}^d} \int_{S^d} \left[\frac{\partial Q}{\partial \xi_0}(\xi_0\nu) \right]_{t=0} \Phi(0, x, \xi) d\xi dx$$

What is the trace of $\frac{\partial Q}{\partial \xi_0}(\xi_0\nu)$ on $t = 0$?
As $\frac{\partial Q}{\partial \xi_0}(\xi_0\nu) = \rho(x){\xi_0}^2\nu$ holds, while $\xi_0 \neq 0$ for $(\xi_0, \cdot) \in \operatorname{supp}\nu$, so the same quantity defines the trace of ν on $t = 0$.

Compute ν by following integral curves

The equations of projections of bicharacteristics on $\mathbf{R}^{d+1} \times S^d$ (a system of ode-s)

$$\begin{cases} \frac{d\mathbf{x}}{ds} = \nabla_{\boldsymbol{\xi}} Q(\mathbf{x}, \boldsymbol{\xi}) - (d+2) Q(\mathbf{x}, \boldsymbol{\xi}) \boldsymbol{\xi} \\ \frac{d\boldsymbol{\xi}}{ds} = -(\mathbf{I} - \boldsymbol{\xi} \otimes \boldsymbol{\xi}) \nabla_{\mathbf{x}} Q(\mathbf{x}, \boldsymbol{\xi}) \\ \quad \mathbf{x}(0) = \mathbf{x}_0, \qquad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \qquad\qquad Q(\mathbf{x}_0, \boldsymbol{\xi}_0) = 0 \; . \end{cases}$$

Theorem. *The trace of ν on $t = 0$ is defined by*
$\langle\!\langle \xi_0 \nu, \{\Phi, Q\} \rangle\!\rangle - \langle\!\langle 2\xi_0 \mathsf{Re}\, \nu^{12}, \Phi \rangle\!\rangle$,
for $\Phi \in \mathrm{C}_c^\infty([0, T\rangle \times \mathbf{R}^d \times S^d))$. The above expression is equal to

$$\frac{1}{4} \Big(\Big\langle \tilde{\nu}_+, \Phi_{(t=0, x, \mathsf{S}^-(x,\xi))} \Big\rangle + \Big\langle \tilde{\nu}_-, \Phi_{(t=0, x, \mathsf{S}^+(x,\xi))} \Big\rangle \Big) \; ,$$

where $\mathsf{S}^\pm : \mathbf{R}^3 \times S^{3-1} \longrightarrow S^3$,

$$\mathsf{S}^\pm(x, \xi) := \left(\pm \sqrt{\frac{\rho(x)}{\rho(x) + \mathbf{A}(x)\xi \cdot \xi}}, \sqrt{\frac{\mathbf{A}(x)\xi \cdot \xi}{\rho(x) + \mathbf{A}(x)\xi \cdot \xi}} \boldsymbol{\xi} \right) \; ,$$

while $\tilde{\nu}_\pm$ are H-measures associated to $u_n^\pm = \zeta \sqrt{\rho} w_n \pm i \Lambda v_n$.
Above we have $\zeta(x) \in \mathrm{C}_c^\infty(\mathbf{R}^3; \mathbf{R}_0^+)$, s.t. $\zeta(x) = 1$ on the common support of v_n and w_n, while Λ is an operator with compactly supported (in $\mathbf{x}$) principal symbol $\lambda(x, \xi) := \zeta(x) \sqrt{\mathbf{A}(x)\xi \cdot \xi}$.

In our example ...

From formulas:

$$\partial_t u_n(t,x) = \sin 2nx(\cos 2nt - \sin 2nt)$$
$$\partial_x u_n(t,x) = \cos 2nx(\cos 2nt + \sin 2nt)\ ,$$

and (four Dirac masses in $\boldsymbol{\xi}$, independent of $\mathbf{x}$)

$$\nu = \frac{1}{4}\sum \lambda^2 \otimes \delta_{\frac{(\pm 1, \pm 1)}{\sqrt{2}}}\ ,$$

which gives the same limit for d_n as before ($= 1$).
By following the described procedure, first compute H-measures for initial data; by theorem:

$$\tilde{\nu}_{\pm} = \frac{1}{2}\lambda^1 \otimes \left(\delta_1 + \delta_{-1}\right)\ .$$

The trace of ν is

$$\nu\big|_{t=0} = \frac{1}{8\xi_0^2}\lambda^1 \otimes \delta_{\frac{(\pm 1, \pm 1)}{\sqrt{2}}} = \frac{1}{4}\lambda^1 \otimes \delta_{\frac{(\pm 1, \pm 1)}{\sqrt{2}}}\ .$$

Transport along integral curves of the system:

$$\nu(t,x,\boldsymbol{\xi}) = \nu\big|_{t=0}(x + \frac{\xi_1}{\xi_0}t, \boldsymbol{\xi})\ ,$$

gives the same result as explicit computation.

The limit microlocal energy density is equal for linear and semilinear equation.