Propagation principle for parabolic H-measures

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Applications of Generalized Functions in Harmonic Analysis, Mechanics, Stochastics and PDE

WeConMApp

H-measures were introduced independently by Luc Tartar and Patrick Gérard in the late 1980s and their existence is established by the following theorem.

Theorem 1. *If* (u_n) *is a sequence in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$ *such that* $\mathsf{u}_n \rightharpoonup 0$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and an* $r\times r$ *Hermitian complex matrix Radon measure* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d\times\mathrm{S}^{d-1}$ *such that for any* $\varphi_1,\varphi_2\in\mathrm{C}_0(\mathbf{R}^d)$ *and* $\psi\in\mathrm{C}(\mathrm{S}^{d-1})$ *one has:*

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$$\begin{aligned}\lim_{n'}\int_{\mathbf{R}^d}(\varphi_1\mathsf{u}_{n'})\otimes\mathcal{A}_\psi(\varphi_2\mathsf{u}_{n'})\,d\mathbf{x} &= \langle\boldsymbol{\mu},(\varphi_1\overline{\varphi_2})\boxtimes\overline{\psi}\rangle\\ &=\int_{\mathbf{R}^d\times\mathrm{S}^{d-1}}\varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})\psi(\boldsymbol{\xi})}\,d\boldsymbol{\mu}(\mathbf{x},\boldsymbol{\xi})\,,\end{aligned}$$

where $\mathcal{F}(\mathcal{A}_\psi v)(\boldsymbol{\xi})=\psi(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})\mathcal{F}v(\boldsymbol{\xi})$. ∎

Parabolic projections

$\mathrm{P}^d \ldots$ a rotational ellipsoid in $\mathbf{R}^{1+d}$ defined by $\tau^2 + \frac{|\boldsymbol{\xi}|^2}{2} = 1$
$p \ldots$ a projection to the manifold P^d, along projection curves (parabolas)

$$\varphi_{\boldsymbol{\nu}}(s) = (s^2\tau_0,\, s\boldsymbol{\xi}_0)\,, \quad s > 0\,, \quad \boldsymbol{\nu} = (\tau_0,\, \boldsymbol{\xi}_0) \in \mathrm{P}^d\,,$$

given by

$$p(\tau, \boldsymbol{\xi}) = \left(\frac{\tau}{\kappa^2(\tau, \boldsymbol{\xi})}, \frac{\boldsymbol{\xi}}{\kappa(\tau, \boldsymbol{\xi})} \right),$$

where $\kappa^2(\tau, \boldsymbol{\xi}) := |\boldsymbol{\xi}/2|^2 + \sqrt{|\boldsymbol{\xi}/2|^4 + \tau^2}$.

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κ takes a constant values $s \in \mathbf{R}^+$ on each ellipsoid $\tau^2 + |s\boldsymbol{\xi}|^2/2 = s^4$

Specifically, P^d can alternatively be characterised with $\kappa(\tau, \boldsymbol{\xi}) = 1$.

Theorem 2. *If* (u_n) *is a sequence in* $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$ *such that* $\mathsf{u}_n \longrightarrow 0$*, then there exist a subsequence* $(\mathsf{u}_{n'})$ *and an* $r \times r$ *Hermitian matrix Radon measure* $\boldsymbol{\mu}$ *on* $\mathbf{R}^{1+d} \times \mathrm{P}^d$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^{1+d})$ *and* $\psi \in \mathrm{C}(\mathrm{P}^d)$ *one has:*

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^{1+d}} \mathcal{F}(\varphi_1 \mathsf{u}_{n'}) \otimes \mathcal{F}(\varphi_2 \mathsf{u}_{n'})(\psi \circ p)\, d\boldsymbol{\eta} &= \langle \boldsymbol{\mu}, (\varphi_1 \overline{\varphi_2}) \boxtimes \psi \rangle \\ &= \int_{\mathbf{R}^{1+d} \times \mathrm{P}^d} \varphi_1(\mathbf{y}) \overline{\varphi_2(\mathbf{y})} \psi(\boldsymbol{\eta})\, d\boldsymbol{\mu}(\mathbf{y}, \boldsymbol{\eta})\,,\end{aligned}$$

where we use notation $\mathbf{y} = (t, \mathbf{x})$ *and* $\boldsymbol{\eta} = (\tau, \boldsymbol{\xi})$*.* ∎

Hörmander classes

For $m \in \mathbf{R}$, $\rho \in \langle 0, 1]$ and $\delta \in [0, 1\rangle$, the Hörmander symbol class $\mathrm{S}^m_{\rho,\delta}$ is defined as the set:

$$\Big\{a(\mathbf{x}, \boldsymbol{\xi}) \in \mathrm{C}^\infty(\mathbf{R}^d \times \mathbf{R}^d) : \\ (\forall\, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{N}_0^d)(\exists\, C_{\boldsymbol{\alpha},\boldsymbol{\beta}} > 0) \quad \big|\partial_{\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} a\big| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}} k^{m - \rho|\boldsymbol{\alpha}| + \delta|\boldsymbol{\beta}|}\Big\} ,$$

where $k(\boldsymbol{\xi}) := \sqrt{1 + 4\pi^2 |\boldsymbol{\xi}|^2}$ and $\partial_{\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} a := \partial^{\boldsymbol{\beta}}_{\mathbf{x}} \partial^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}} a$.

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To each $a \in \mathrm{S}^m_{\rho,\delta}$ one can assign an operator A defined for $\varphi \in \mathcal{S}$ as

$$A\varphi(\mathbf{x}) = \int_{\mathbf{R}^d} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} a(\mathbf{x}, \boldsymbol{\xi}) \hat{\varphi}(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

where $A\varphi$ is again a function from Schwartz space $\mathcal{S}$. We write $a = \sigma(A)$ and $A = \mathrm{Op}(a)$.

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$\rho \geqslant \delta \ldots$ a continuous operator from $\mathrm{H}^s(\mathbf{R}^d)$ to $\mathrm{H}^{s-m}(\mathbf{R}^d)$

Anisotropic generalisation

For $\boldsymbol{\gamma} \in \langle 0,1]^d$ such that $\gamma_{\max} = \max\{\gamma_1, \ldots, \gamma_d\} = 1$ we introduce anisotropic classes of symbols:

$$\mathrm{S}^{m\boldsymbol{\gamma}} = \Big\{ a(\mathbf{x}, \boldsymbol{\xi}) \in \mathrm{C}^\infty(\mathbf{R}^d \times \mathbf{R}^d) : \\ (\forall\, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{N}_0^d)(\exists\, C_{\boldsymbol{\alpha},\boldsymbol{\beta}} > 0) \quad \big|\partial_{\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} a\big| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}} k_{\boldsymbol{\gamma}}^{m - \sum_{i=1}^d \frac{\alpha_i}{\gamma_i}} \Big\},$$

where $k_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) := 1 + \sum_{i=1}^d (2\pi|\xi_i|)^{\gamma_i}$.

Inclusions to classical spaces

We can prove that

$$k^2(\boldsymbol{\xi}) \leqslant (1+d)k_{\boldsymbol{\gamma}}(\boldsymbol{\xi})^{\frac{2}{\gamma_{\min}}},$$

where $\gamma_{\min} = \min\{\gamma_1, \ldots, \gamma_d\}$, and also

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From the above inequalities we easily obtain

$$\mathrm{S}^{m\boldsymbol{\gamma}} \subseteq \begin{cases} \mathrm{S}^{m}_{\gamma_{\min},0}, & m \geq 0 \\ \mathrm{S}^{m\gamma_{\min}}_{\gamma_{\min},0}, & m < 0 \end{cases},$$

which means that we are able to apply many results from classical pseudodifferential calculus also to anisotropic classes of symbols and operators.

As a special case we have the following parabolic classes of symbols

$$\begin{aligned}\mathrm{S}_p^m = \{&a(\mathbf{y},\boldsymbol{\eta}) \in \mathrm{C}^\infty(\mathbf{R}^{1+d}\times\mathbf{R}^{1+d}) \\ &: (\forall\,\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbf{N}_0^{1+d})(\exists\,C_{\boldsymbol{\alpha},\boldsymbol{\beta}}>0) \quad \left|\partial_{\boldsymbol{\beta}}\partial^{\boldsymbol{\alpha}} a\right| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}} k_p^{m-2\boldsymbol{\alpha}_0-|\boldsymbol{\alpha}'|}\Big\}\,,\end{aligned}$$

where $k_p(\boldsymbol{\eta}) = k_p(\tau,\boldsymbol{\xi}) := 1 + (2\pi|\tau|)^{\frac{1}{2}} + \sum_{i=1}^d 2\pi|\xi_i|$, and $\boldsymbol{\alpha} = (\alpha_0,\boldsymbol{\alpha}')$.

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$$\begin{aligned}\mathrm{S}^m_p &= \{a(\mathbf{y},\boldsymbol{\eta}) \in \mathrm{C}^\infty(\mathbf{R}^{1+d}\times\mathbf{R}^{1+d}) \\ &\quad : (\forall\, \boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbf{N}_0^{1+d})(\exists\, C_{\boldsymbol{\alpha},\boldsymbol{\beta}}>0) \quad \left|\partial_{\boldsymbol{\beta}}\partial^{\boldsymbol{\alpha}} a\right| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}} k_p^{m-2\boldsymbol{\alpha}_0-|\boldsymbol{\alpha}'|}\Big\},\end{aligned}$$

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Inclusion to classical spaces now reads

$$\mathrm{S}^m_p \subseteq \begin{cases} \mathrm{S}^m_{\frac{1}{2},0}\,, & m \geq 0 \\ \mathrm{S}^{\frac{m}{2}}_{\frac{1}{2},0}\,, & m < 0 \end{cases}.$$

$p^m \in \mathrm{C}^\infty(\mathbf{R}^{1+d} \times \mathbf{R}^{1+d} \setminus \{\mathbf{0}\})$ is a parabolic homogeneous function with respect to $\boldsymbol{\eta}$ of order m if

$$p^m(\mathbf{y}, \varepsilon^2\tau, \varepsilon\boldsymbol{\xi}) = \varepsilon^m p^m(\mathbf{y}, \tau, \boldsymbol{\xi}), \quad \varepsilon > 0.$$

For such functions we have the following lemma.

Lemma 1. *Let $p^m(\mathbf{y}, \boldsymbol{\eta})$ be an arbitrary parabolic homogeneous function with respect to $\boldsymbol{\eta}$ of order m, which, as well as all its partial derivatives, is uniformly bounded in $\mathbf{y}$ for $\boldsymbol{\eta}$ inside a compact set, and let $\chi(\boldsymbol{\eta}) \in \mathrm{C}^\infty(\mathbf{R}^{1+d})$ equals to 0 around the origin and equals to 1 outside the ball $\mathrm{K}[0,1]$. Then $p^m\chi$ belongs to S^m_p.* ∎

Basic properties

By direct calculation we have

$$a \in \mathrm{S}_p^m,\, b \in \mathrm{S}_p^n \Longrightarrow ab \in \mathrm{S}_p^{m+n},$$

and

$$a \in \mathrm{S}_p^m \Longrightarrow \partial_{\xi_i} a \in \mathrm{S}_p^{m-1},\, \partial_\tau a \in \mathrm{S}_p^{m-2},\, \partial_{x_i} a \in \mathrm{S}_p^m,\, \partial_t a \in \mathrm{S}_p^m.$$

Class of operators assigned to symbols from S_p^m we denote by Σ_p^m.

Continuity results

As for nonnegative m the class S_p^m is embedded into $\mathrm{S}^m_{\frac{1}{2},0}$, according to classical theory A can be considered as a continuous operator from $\mathrm{H}^s(\mathbf{R}^{1+d})$ to $\mathrm{H}^{s-m}(\mathbf{R}^{1+d})$.

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Actually, we prove a better result

Lemma 2. *An operator* $A \in \Sigma_p^m$ *is a continuous operator from* $\mathrm{H}^{\frac{s}{2},\,s}(\mathbf{R}^{1+d})$ *into* $\mathrm{H}^{\frac{s-m}{2},\,s-m}(\mathbf{R}^{1+d})$. ∎

Polyhomogeneous parabolic symbols

Now we define classes of polyhomogeneous parabolic symbols, and corresponding operators:

$$\Psi_p^m = \{P \in \Sigma_p^m : \sigma(P)(\mathbf{y},\boldsymbol{\eta}) = p^m(\mathbf{y},\boldsymbol{\eta})\chi(\boldsymbol{\eta}) + p^{m-1}(\mathbf{y},\boldsymbol{\eta})\}\,,$$

where $p^m \in \mathrm{C}^\infty(\mathbf{R}^{1+d} \times \mathbf{R}^{1+d} \setminus \{\mathbf{0}\})$ is a parabolic homogeneous function with respect to $\boldsymbol{\eta}$ of order m, while $\chi \in \mathrm{C}^\infty(\mathbf{R}^{1+d})$ equals to 0 around the origin and equals to 1 outside $\mathrm{K}[0,1]$, and $p^{m-1} \in \mathrm{S}_p^{m-1}(\mathbf{R}^{1+d})$.

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By $\sigma^m(P) := p^m$ we denote the principal symbol of the operator $P \in \Psi_p^m$.

When we use matrix-valued symbols with values in $\mathrm{M}_{r\times r}(\mathbf{C})$ we write $\Psi_{p,\,r}^m$ instead of Ψ_p^m and $\mathrm{S}_{p,\,r}^m$ instead of S_p^m.

We also write $\Psi_{p,\,r}^{m,\,c}$ for symbols compactly supported in $\mathbf{y}$, and similarly for the class $\Sigma_{p,\,r}^{m,\,c}$.

It follows from the definition that $\sigma^m(P)\chi \in \mathrm{S}_p^m$ for $P \in \Psi_p^m$.

Lemma 3. *If A^* denotes the adjoint operator to $A \in \Sigma^{m}_{p,\,r}$, and $B \in \Sigma^{n}_{p,\,r}$, then*

a) $\sigma(A^*) - \sigma(A)^* - \frac{1}{2\pi i} \sum_{j=1}^{d} \frac{\partial^2 \sigma(A)^*}{\partial x_j \partial \xi_j} \in \mathrm{S}^{m-2}_{p,\,r}$,

b) $\sigma(AB) - \sigma(A)\sigma(B) - \frac{1}{2\pi i} \sum_{j=1}^{d} \frac{\partial \sigma(A)}{\partial \xi_j} \frac{\partial \sigma(B)}{\partial x_j} \in \mathrm{S}^{m+n-2}_{p,\,r}$.

Thus, if $\sigma(A)$ commutes with $\sigma(B)$, then $[A, B] := AB - BA \in \Sigma^{m+n-1}_{p,\,r}$ with symbol $1/(2\pi i)\{\sigma(A), \sigma(B)\}_{\mathbf{x},\,\boldsymbol{\xi}}$ (up to the symbol of lower order), where

$$\{\sigma(A), \sigma(B)\}_{\mathbf{x},\,\boldsymbol{\xi}} = \sum_{j=1}^{d} \frac{\partial \sigma(A)}{\partial \xi_j} \frac{\partial \sigma(B)}{\partial x_j} - \sum_{j=1}^{d} \frac{\partial \sigma(A)}{\partial x_j} \frac{\partial \sigma(B)}{\partial \xi_j}$$

is the Poisson bracket of $\sigma(A)$ and $\sigma(B)$ in variables $\mathbf{x}$ and $\boldsymbol{\xi}$. ∎

Further consequences

Corollary. *If* $A \in \Sigma^m_{p,\,r}$ *and* $B \in \Sigma^n_{p,\,r}$*, then*
a) $\sigma(A^*) - \sigma(A)^* \in \mathrm{S}^{m-1}_{p,\,r}$,
b) $\sigma(AB) - \sigma(A)\sigma(B) \in \mathrm{S}^{m+n-1}_{p,\,r}$. ∎

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b) $\sigma(AB) - \sigma(A)\sigma(B) \in \mathrm{S}^{m+n-1}_{p,\,r}$.

∎

Lemma 4. *If $A \in \Psi^m_{p,\,r}$ and $B \in \Psi^n_{p,\,r}$, then*

a) $A^* \in \Psi^m_{p,\,r}$ *and* $\sigma^m(A^*) = \overline{\sigma^m(A)}^t =: \sigma^m(A)^*$,

b) $AB \in \Psi^{m+n}_{p,\,r}$ *and* $\sigma^{m+n}(AB) = \sigma^m(A)\sigma^n(B)$.

∎

Extended definition of parabolic H-measures

Definition of parabolic H-measures given in Theorem 2 can be extended to operators from $\Psi^{0,\,c}_{p,\,r}$.

Lemma 5. *If* (u_n) *is a sequence in* $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$ *such that* $\mathsf{u}_n \rightharpoonup 0$*, then there exist a subsequence* $(\mathsf{u}_{n'})$ *and an* $r\times r$ *Hermitian matrix Radon measure* $\boldsymbol{\mu}$ *on* $\mathbf{R}^{1+d}\times \mathrm{P}^d$ *such that for any* $A\in\Psi^{0,\,c}_{p,\,r}$ *one has:*

$$\lim_{n'}\int_{\mathbf{R}^{1+d}} A\mathsf{u}_{n'}\cdot\mathsf{u}_{n'}\,d\mathbf{y} =: \lim_{n'}\langle A\mathsf{u}_{n'},\mathsf{u}_{n'}\rangle = \langle\boldsymbol{\mu},\sigma^0(A)\rangle$$

$$:= \sum_{i,j=1}^{r}\int_{\mathbf{R}^{1+d}\times\mathrm{P}^d}\sigma^0(A)_{ij}\,d\mu_{ij}(\mathbf{y},\boldsymbol{\eta})\,.$$

∎

Localisation principle

Theorem 3. (localisation principle) *Let* (u_n) *be a sequence in* $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$ *such that* $\mathsf{u}_n \rightharpoonup 0$, *and let* $R\mathsf{u}_n \longrightarrow 0$ *strongly in* $\mathrm{H}_{\mathrm{loc}}^{-\frac{m}{2},-m}(\mathbf{R}^{1+d};\mathbf{C}^r)$ *for some* $R \in \Psi^m_{p,\,r}$. *Then, for the associated parabolic H-measure* $\boldsymbol{\mu}$, *it holds*

$$\sigma^m(R)\boldsymbol{\mu}^\top = \mathbf{0}\,.$$

∎

Propagation principle

Theorem 4. (propagation principle) *Let* (u_n) *be a sequence in* $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$ *such that* $\mathsf{u}_n \rightharpoonup 0$, *and let* $R\mathsf{u}_n \longrightarrow 0$ *strongly in* $\mathrm{H}_{\mathrm{loc}}^{\frac{-m+1}{2},-m+1}(\mathbf{R}^{1+d};\mathbf{C}^r)$ *for some* $R \in \Psi^m_{p,\,r}$, *such that* $\sigma^m(R)$ *is self-adjoint and the lower order symbol of* R *defines an element in* $\Psi^{m-1}_{p,\,r}$ *with principal symbol* $\sigma^{m-1}(R)$.
Then, for the associated parabolic H-measure $\boldsymbol{\mu}$ *and for any* $a \in \mathrm{C}^\infty_c(\mathbf{R}^{1+d}\times \mathrm{P}^d)$, *it holds*

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Then, for the associated parabolic H-measure $\boldsymbol{\mu}$ *and for any* $a\in\mathrm{C}^\infty_c(\mathbf{R}^{1+d}\times\mathrm{P}^d)$, *it holds*

$$\Big\langle\boldsymbol{\mu},\{\sigma^m(R),a\}_{\mathbf{x},\boldsymbol{\xi}}+\Big[2\pi i\Big((\sigma^{m-1}(R))^*-\sigma^{m-1}(R)\Big)\\ +\sum_{j=1}^d\frac{\partial^2\sigma^m(R)}{\partial x_j\partial\xi_j}+\frac{m-1}{2(1+\tau^2)}\boldsymbol{\xi}\cdot\nabla_{\mathbf{x}}\sigma^m(R)\Big]a\Big\rangle=0\,.$$

If $\sigma^m(R)$ *is anti-self-adjoint we can still apply this theorem, by simply using* iR *instead of* R.

∎

Application to the Schrödinger equation

We consider a sequence of initial value problems

$$\begin{cases} i\partial_t u_n + \mathsf{div}\,(\mathbf{A}\nabla u_n) = f_n \\ u_n(0,\cdot) = u_n^0\,, \end{cases}$$

where $u_n^0 \rightharpoonup 0$ in $\mathrm{H}^1(\mathbf{R}^d)$, while $f_n \rightharpoonup 0$ in $W(0,T;\mathrm{L}^2(\mathbf{R}^d),\mathrm{H}^{-1}(\mathbf{R}^d))$, where

$$\|f\|_W^2 = \|f\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbf{R}^d))} + \|\partial_t f\|^2_{\mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^d))}\,.$$

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We also take $\mathbf{A} \in \mathrm{C}^\infty(\mathbf{R}^+\times\mathbf{R}^d;\mathrm{M}_{d\times d}(\mathbf{R}))\cap\mathrm{L}^\infty(\mathbf{R}^+\times\mathbf{R}^d;\mathrm{M}_{d\times d}(\mathbf{R}))$ to be a symmetric matrix field such that $\mathbf{A}\mathsf{v}\cdot\mathsf{v} \geqslant \alpha|\mathsf{v}|^2$ (a.e.) for an $\alpha > 0$. Furthermore, we suppose that all partial derivatives $\partial^{\boldsymbol{\alpha}}\sqrt{\mathbf{A}}$ belong to $\mathrm{C}_b(\mathbf{R}^+\times\mathbf{R}^d;\mathrm{M}_{d\times d}(\mathbf{R}))$.

Theorem 5. *Under the above assumptions and additional assumption than $f_n \longrightarrow 0$ strongly in $\mathrm{L}^2(\mathbf{R}^{1+d})$ an H-measure $\tilde{\boldsymbol{\mu}}$ associated with the sequence*

$$(v_0^n, \mathsf{v}_1^n) = (Pu_n, \sqrt{\mathbf{A}}\,\nabla u_n)\,,$$

where $\sigma(P) = i\kappa(\tau,\boldsymbol{\xi})\chi(\tau,\boldsymbol{\xi})$, satisfies (for every $a \in \mathrm{C}_c^\infty(\mathbf{R}^{1+d}\times \mathrm{P}^d)$)

$$\langle \tilde{\boldsymbol{\mu}}, \tilde{\mathbf{B}}\rangle = 0\,,$$

where

$$\tilde{\mathbf{B}} = \begin{bmatrix} b_{00} & (b_j)^\top \\ (b_j) & \mathbf{B} \end{bmatrix}$$

with $b_{00} = \frac{2\pi\tau\nabla_{\mathbf{x}}a\cdot\nabla_{\boldsymbol{\xi}}\kappa}{\kappa^2}$, $b_j = 2\pi(\sum_{k,l}\xi_l\partial_k\tilde{a}_{jl}\partial_{\xi_k}a - \sum_k \tilde{a}_{jk}\partial_{x_k}a)$ and

$$\mathbf{B} = (\nabla_{\mathbf{x}}a\;\cdot\nabla_{\boldsymbol{\xi}}\kappa)\mathbf{I} + 2a(\nabla_{\boldsymbol{\xi}}\kappa\cdot\nabla_{\mathbf{x}}\sqrt{\mathbf{A}})\mathbf{A}^{-1/2}\,.$$

Here $\tilde{a}_{ij}$ are elements of $\sqrt{\mathbf{A}}$. Moreover, we have

$$\Big\langle \frac{\tilde{\nu}}{|\mathsf{q}|^2}, \nabla_{\mathbf{x}}a\cdot\Big((\tau + 2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}}\kappa - 4\pi\kappa\mathbf{A}\boldsymbol{\xi}\Big) \\ + 4\pi\Big(((\kappa\nabla_{\boldsymbol{\xi}}a + a\nabla_{\boldsymbol{\xi}}\kappa)\cdot\nabla_{\mathbf{x}}\sqrt{\mathbf{A}})\boldsymbol{\xi}\cdot\sqrt{\mathbf{A}}\boldsymbol{\xi}\Big)\Big\rangle = 0\,,$$

where $\tilde{\nu} = \mathrm{tr}\tilde{\boldsymbol{\mu}}$ and $\mathsf{q} = (i\kappa, 2\pi i\sqrt{\mathbf{A}}\,\xi)$. ∎

Remark. We would like to express the claim of Theorem 5 in terms of the main symbol of Schrödinger equation

$$Q(t, \mathbf{x}; \tau, \boldsymbol{\xi}) := 2\pi\tau + (2\pi)^2 \mathbf{A}(t, \mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi}\,,$$

hoping that we can get results similar to those obtained in [AL]. However, we know how to do this only in a special case when $\sqrt{\mathbf{A}}$ and its derivatives commute ($p \in 1..d$):

$$\sqrt{\mathbf{A}} \cdot \partial_{x_p} \sqrt{\mathbf{A}} = \partial_{x_p} \sqrt{\mathbf{A}} \cdot \sqrt{\mathbf{A}}\,.$$

In that case we have

$$\nabla_{\mathbf{x}} \mathbf{A} = 2\sqrt{\mathbf{A}} \cdot \nabla_{\mathbf{x}} \sqrt{\mathbf{A}} = 2\nabla_{\mathbf{x}} \sqrt{\mathbf{A}} \cdot \sqrt{\mathbf{A}}\,,$$

and the claim of Theorem 5 can now be rewritten as

$$\Big\langle \frac{\tilde{\nu}}{|\mathsf{q}|^2}, \kappa\{a, Q\}_{\mathbf{x},\boldsymbol{\xi}} + (Q\nabla_{\mathbf{x}} a + a\nabla_{\mathbf{x}} Q) \cdot \nabla_{\boldsymbol{\xi}} \kappa \Big\rangle = 0\,.$$

∎

Application to the vibrating plate equation

We consider a sequence of initial value problems

$$\begin{cases} \partial_t(\rho\partial_t u_n) + \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u_n) & = f_n \\ u_n(0,\cdot) & = u_n^0 \\ \partial_t u_n(0,\cdot) & = u_n^1\,, \end{cases}$$

where $u_n^0 \rightharpoonup 0$ in $\mathrm{H}^2(\mathbf{R}^d)$, $u_n^1 \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$ and $f_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d)$.

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where $u_n^0 \longrightarrow 0$ in $\mathrm{H}^2(\mathbf{R}^d)$, $u_n^1 \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d)$ and $f_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d)$.

We take $\rho \in \mathrm{C}^\infty(\mathbf{R}^+;\mathrm{L}^\infty(\mathbf{R}^d)) \cap \mathrm{C}^\infty(\mathbf{R}^+ \times \mathbf{R}^d)$ such that $\rho \geqslant \rho_0$, where $\rho_0 \in \mathbf{R}^+$ is a given constant, and $\mathbf{M} \in \mathrm{C}^\infty(\mathbf{R}^+ \times \mathbf{R}^d;\mathcal{L}(\mathrm{M}_{d\times d}))$ to be a real symmetric tensor field of order four such that $\mathbf{MA}\cdot\mathbf{A} \geqslant \alpha\mathbf{A}\cdot\mathbf{A}$ for given $\alpha > 0$ and every $\mathbf{A} \in \mathrm{M}_{d\times d}$, with the following symmetries: $M_{klij} = M_{ijkl} = M_{jikl} = M_{ijlk}$.

$\sqrt{\mathbf{M}}$ is well-defined and satisfies the same properties.

Furthermore, we suppose
$\mathbf{M}$, $\partial_t\mathbf{M}$, $\sqrt{\mathbf{M}}$, $\partial_t\sqrt{\mathbf{M}}$, $\nabla\nabla\sqrt{\mathbf{M}} \in \mathrm{C}_b(\mathbf{R}^+ \times \mathbf{R}^d;\mathcal{L}(\mathrm{M}_{d\times d}))$.

Theorem 6. *Under the above assumptions an H-measure $\tilde{\boldsymbol{\mu}}$ associated with the sequence*

$$(v_0^n, \mathsf{v}_1^n) = (Pu_n, \sqrt{\mathbf{M}}\,\nabla\nabla u_n)\,,$$

where $\sigma(P) = 2\pi\tau$, satisfies (for every $a \in \mathrm{C}_c^\infty(\mathbf{R}^{1+d} \times \mathrm{P}^d)$)

$$\langle \tilde{\boldsymbol{\mu}}, \mathbf{N} \rangle = 0\,,$$

where

$$\mathbf{N} = \begin{bmatrix} N_{00} & (\tilde{N}_{ij})^\top \\ (\tilde{N}_{ij}) & \mathbf{0} \end{bmatrix}$$

with $N_{00} = (\frac{a}{2(1+\tau^2)}\boldsymbol{\xi} - \nabla_{\boldsymbol{\xi}} a) \cdot \frac{\tau \nabla_{\mathbf{x}} \rho}{2\pi}$ and

$$\tilde{N}_{ij} = 2\sum_{k,l} \xi_k \tilde{m}_{ijkl} \partial_{x_l} a + \sum_{k,l,p} \xi_k \xi_l \partial_{x_p} \tilde{m}_{ijkl} (\frac{a}{2(1+\tau^2)}\xi_p - \partial_{\xi_p} a)\,.$$

Here $(\tilde{N}_{ij})$ is the vector column indexed by i, j, while $\sqrt{\mathbf{M}} = (\tilde{m}_{ijkl})$. Moreover, we have

$$\Big\langle \frac{\tilde{\nu}}{|\mathsf{q}|^2}, \tau^3 (\frac{a}{2(1+\tau^2)}\boldsymbol{\xi} - \nabla_{\boldsymbol{\xi}} a) \cdot \nabla_{\mathbf{x}} \rho - 8\pi^2 \tau \sqrt{\mathbf{M}}\,(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \cdot \Big(2\sqrt{\mathbf{M}}\,(\nabla_{\mathbf{x}} \bar{a} \otimes \boldsymbol{\xi}) + (\frac{\bar{a}}{2(1+\tau^2)}\boldsymbol{\xi} - \nabla_{\boldsymbol{\xi}} \bar{a}) \cdot \nabla_{\mathbf{x}} \sqrt{\mathbf{M}} (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \Big) \Big\rangle = 0\,,$$

where $\tilde{\nu} = \mathrm{tr}\tilde{\boldsymbol{\mu}}$ and $\mathsf{q} = (2\pi\tau, -4\pi^2\sqrt{\mathbf{M}}\,\xi \otimes \xi)$. ∎

Remark. We would like to express the claim of Theorem 7 in terms of the main symbol of vibrating plate equation

$$Q(t,\mathbf{x};\tau,\boldsymbol{\xi}) := -(2\pi\tau)^2\rho + (2\pi)^4\mathbf{M}(t,\mathbf{x})(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\,,$$

hoping again to get a result similar to one obtained in [AL]. However, we know how to do this only in a special case when $\sqrt{\mathbf{M}}$ and its derivatives commute ($p \in 1..d$):

$$\sqrt{\mathbf{M}}\cdot\partial_{x_p}\sqrt{\mathbf{M}} = \partial_{x_p}\sqrt{\mathbf{M}}\cdot\sqrt{\mathbf{M}}\,.$$

In that case we have

$$\nabla_{\mathbf{x}}\mathbf{M} = 2\sqrt{\mathbf{M}}\cdot\nabla_{\mathbf{x}}\sqrt{\mathbf{M}} = 2\nabla_{\mathbf{x}}\sqrt{\mathbf{M}}\cdot\sqrt{\mathbf{M}}\,,$$

and the claim of Theorem 7 can now be rewritten as

$$\left\langle \frac{\tau\tilde{\nu}}{|\mathsf{q}|^2}, \{a,Q\}_{\mathbf{x},\boldsymbol{\xi}} - \frac{a}{2(1+\tau^2)}\boldsymbol{\xi}\cdot\nabla_{\mathbf{x}}Q\right\rangle = 0\,.$$

∎