Fully nonlinear mean field games

Indranil Chowdhury
Zagreb

Espen R. Jakobsen
Trondheim

Miłosz Krupski
Wrocław, Zagreb

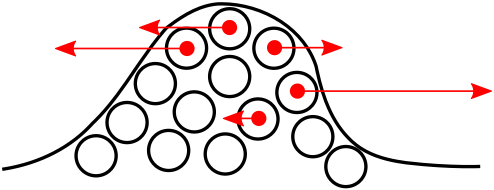
https://arxiv.org/abs/2104.06985

Wien, September 2022

MiTPDE Croatian Science Foundation IP-2018-01-2449

$$\begin{cases} -\partial_t u = \Delta u + H(\nabla u) + \mathfrak{f}(m) & \text{on } [0,T]\times\mathbb{R}^d, \\ u(T) = \mathfrak{g}\big(m(T)\big) & \text{on } \mathbb{R}^d, \\ \partial_t m = \Delta m + \operatorname{div}(H'(\nabla u)\, m) & \text{on } [0,T]\times\mathbb{R}^d, \\ m(0) = m_0 & \text{on } \mathbb{R}^d. \end{cases}$$

- Agents control (individually, but interchangeably) **the drift** of a Wiener process describing their positions.



Fully nonlinear (parabolic, local/nonlocal) MFG

$$
\begin{cases}
-\partial_t u = F(\mathcal{L}u) + \mathfrak{f}(m) & \text{on } [0,T]\times\mathbb{R}^d,\\
u(T) = \mathfrak{g}\big(m(T)\big) & \text{on } \mathbb{R}^d,\\
\partial_t m = \mathcal{L}^*\big(F'(\mathcal{L}u)\, m\big) & \text{on } [0,T]\times\mathbb{R}^d,\\
m(0) = m_0 & \text{on } \mathbb{R}^d.
\end{cases}
$$

- Agents control **the time rate** θ of any Lévy process ($\mathcal{L}$)
- θ is a stochastic process such that $\theta(t)$ is a stopping time
- "Local-in-time generator" $\theta'(t)\mathcal{L}$ — not Lévy, but Markov (inhomog.)
- Same for any number of Lévy processes
- To get the classical model: Δ, $dx_1, \ldots dx_d, -dx_1, \ldots - dx_d$

Lévy operators

- Lévy $\Leftrightarrow$ maximum principle $\Leftrightarrow$ pseudodifferential with symbol

$$p(\xi) = ic \cdot \xi + \langle A\xi, \xi \rangle + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{iz\cdot\xi} - 1 + \mathbb{1}_{B_1}(iz \cdot \xi) \right) \nu(dz),$$

$$\int_{\mathbb{R}^d} (1 \wedge |z|^2)\, \nu(dz) < \infty, \qquad \nu(\{0\}) = 0.$$

- Example: fractional Laplacian $\nu(dz) = \dfrac{1}{|z|^{d+2\sigma}} dz$, $\sigma \in (0,1)$, $p(\xi) = |\xi|^{2\sigma}$

- Order $2\sigma \Rightarrow \mathcal{L} : C^{2\sigma+\alpha} \to C^{\alpha}$
- 1) Non-degenerate $\Leftrightarrow \nu \asymp |x|^{-d-2\sigma}\, dz$
- 2) Degenerate $\Leftrightarrow \nu \leq |x|^{-d-2\sigma}\, dz$ (or analogue if ν has a singular part)

MFG – uniqueness

- Take (m_1, u_1), (m_2, u_2) and "test" m's against u's (put $m = m_1 - m_2$, $u = u_1 - u_2$, $m[\phi] = \int_{\mathbb{R}^d} \phi \, dm$):

$$m(T)\big[u(T)\big] - m(0)\big[u(0)\big]$$
$$= \int_0^T \Big(m_1\big[\partial_t u + F'(\mathcal{L}u_1)\mathcal{L}u\big] - m_2\big[\partial_t u + F'(\mathcal{L}u_2)\mathcal{L}u\big]\Big)(\tau)\, d\tau = \ldots = 0$$

- F —convex, non-decreasing, $C^{1+\gamma}(\mathbb{R})$, $\mathfrak{f}$, $\mathfrak{g}$ — monotone
- Then

$$m_1 = \mathcal{L}^*(b\, m_1) \text{ and } m_2 = \mathcal{L}^*(b\, m_2), \quad m_1(0) = m_2(0) = m_0,$$

where

$$b(t,x) = \begin{cases} \dfrac{F\big(\mathcal{L}u_1(t,x)\big) - F\big(\mathcal{L}u_2(t,x)\big)}{\mathcal{L}u_1(t,x) - \mathcal{L}u_2(t,x)}, & \text{if } \mathcal{L}u_1(t,x) \neq \mathcal{L}u_2(t,x), \\ F'\big(\mathcal{L}u_1(t,x)\big), & \text{if } \mathcal{L}u_1(t,x) = \mathcal{L}u_2(t,x) \end{cases}$$

- We need: uniqueness of FPK, regularity of HJB.

Fokker–Planck–Kolmogorov

$$\begin{cases} \partial_t m = \mathcal{L}^*(bm) & \text{on } [0,T] \times \mathbb{R}^d, \\ m(0) = m_0 & \text{on } \mathbb{R}^d. \end{cases} \qquad \text{(FPK)}$$

$b = F'(\mathcal{L}u)$ or $b = \int_0^1 F'\Big(s\mathcal{L}u_1 + (1-s)\mathcal{L}u_2\Big)\,ds$ (previous slide)

- $b \in C\big([0,T] \times \mathbb{R}^d\big)$ and $b \geq 0$
- Natural space to look for solutions: $m \in C\big([0,T], \mathcal{P}(\mathbb{R}^d)\big)$:

$$m(t)[\phi(t)] = m_0[\phi(0)] + \int_0^t m(\tau)\big[\partial_t \phi(\tau) + b(\tau)(\mathcal{L}\phi)(\tau)\big]\,d\tau.$$

- Existence: "easy" – set of solutions is convex, **compact** and non-empty.
- Uniqueness by Holmgren: existence of classical solutions to the dual equation

$$\partial_t w = -b\,\mathcal{L}w, \quad w(t) = \psi \in C_c^\infty(\mathbb{R}^d)$$

- Non-deg: $b \in C^\alpha$, $b \geq \kappa > 0$, Mikulevičius & Pragarauskas PotAn14
- Deg: $b \in C^\alpha$, $b \geq 0$, $\mathcal{L}$ of order at most $2\sigma < \frac{7-\sqrt{33}}{4}$
- If $b_n \to b$ locally uniformly, then $\mathcal{M}_n \to \mathcal{M}$ as closed sets ("$K - \limsup$")

Hamilton–Jacobi–Bellman

$$\begin{cases} -\partial_t u = F(\mathcal{L}u) + f(t,x) & \text{on } [0,T]\times\mathbb{R}^d, \\ u(T,x) = g(x) & \text{on } \mathbb{R}^d. \end{cases} \tag{HJB}$$

$f = \mathfrak{f}(m), \quad g = \mathfrak{g}\big(m(T)\big)$

- Fully nonlinear equation $\to$ viscosity solutions.
- Comparison principle (VS uniquely exist): Chasseigne & Jakobsen JDE17
- But we need classical solutions and **a bit more**
- Deg: for $2\sigma < 1$ the comparison principle is enough; no regularization

$$f, g \in C^{2\sigma+\alpha} \quad \Rightarrow \quad \partial_t u, \mathcal{L}u \in C^\alpha$$

- Non-deg, local: Schauder–Caccioppoli estimates (interior regularity)

$$f \in C^{\alpha/2,\alpha}(\mathbb{R}^d) \quad \Rightarrow \quad \partial_t u, D^2 u \in C^\alpha(B_1) \quad \text{(Wang CPAM92)}$$

- Non-deg, non-local: **Conjecture**: Schauder estimates as above.
- (we end up assuming $f \in C^{1.\alpha}$ to get global boundedness, but this is bad)

MFG – existence

- We use Kakutani–Glicksberg–Fan fixed point theorem (i.e. Schauder, but for set-valued maps; solutions to FPK are **compact, convex, non-empty sets**)
- Take $\mu \in C\big([0,T], \mathcal{P}(\mathbb{R}^d)\big)$, solve HJB: $\mathcal{K}_1(\mu) = u$.
- Take u and solve FPK: $\mathcal{K}_2(u) = m$
- Look for a fixed point of $\mathcal{K}(\mu) = \mathcal{K}_2(\mathcal{K}_1(\mu))$.
- Compactness of the map is easy (Prohorov theorem)
- For semi-continuity:

$$
\begin{array}{ccccccccc}
\mu_n & \xmapsto{\mathcal{K}_1} & \mathfrak{f}(\mu_n), \mathfrak{g}(\mu_n) & \longmapsto & \mathcal{L}u_n & \longmapsto & b_n = F'(\mathcal{L}u_n) & \xmapsto{\mathcal{K}_2} & \mathcal{M}_n \\
\downarrow \text{weak} & & \downarrow \text{uniform} & & \downarrow \text{loc unif} & & \downarrow \text{loc unif} & & \downarrow K\!-\!\limsup \\
\mu & \xmapsto{\mathcal{K}_1} & \mathfrak{f}(\mu), \mathfrak{g}(\mu) & \longmapsto & \mathcal{L}u & \longmapsto & b = F'(\mathcal{L}u) & \xmapsto{\mathcal{K}_2} & \mathcal{M}
\end{array}
$$

Thank you!

Research conducted while IC and MK were fellows of
ERCIM Alain Bensoussan Program at NTNU