Homogenisation of nonlocal linear elliptic operators

Marko Erceg

Department of Mathematics, Faculty of Science, University of Zagreb

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Homogenisation - physical motivation

- Passage from micro-scale to macro-scale
- Deriving lower-dimensional models
- Averaging highly heterogeneous materials

Homogenisation - mathematical approach

$\Omega \subseteq \mathbb{R}^d$ open and bounded.

$$\begin{cases} -\mathsf{div}\big(\mathbf{A}_n \nabla u_n\big) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

Homogenisation - mathematical approach

$\Omega \subseteq \mathbb{R}^d$ open and bounded.

$$\begin{cases} -\mathsf{div}\big(\mathbf{A}_n \nabla u_n\big) = f & \text{in } \Omega \\ u_n \in H_0^1(\Omega) \end{cases}$$

- $f \in H^{-1}(\Omega)$
- $\mathbf{A}_n \in \mathcal{M}(\alpha, \beta)$, i.e. $\mathbf{A}_n \in L^\infty(\Omega; \mathbb{R}^{d\times d})$

$$\mathbf{A}_n(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \alpha |\boldsymbol{\xi}|^2$$
$$\mathbf{A}_n(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \frac{1}{\beta} |\mathbf{A}_n(\mathbf{x})\boldsymbol{\xi}|^2$$

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$\implies$ unique solution $u_n \in H_0^1(\Omega)$, and (u_n) bounded in $H_0^1(\Omega)$

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$\Longrightarrow$ (up to a subseq.) $u_n \rightharpoonup u_0$ in $H^1_0(\Omega)$

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Question: Which equation u_0 satisfies?

H-convergence

$$\begin{cases} -\mathrm{div}\big(\mathbf{A}_n \nabla u_n\big) = f & \text{in } \Omega \\ u_n \in H^1_0(\Omega) \end{cases}$$

Various methods:

- G-convergence
- H-convergence
- Asymptotic expansion
- Two-scale convergence
- Γ-convergence

Some books:

Allaire G., Shape Optimization by the Homogenization Method (2002)

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$\mathcal{M}(\alpha, \beta) \ni \mathbf{A}_n \xrightharpoonup{H} \mathbf{A}_0 \in \mathcal{M}(\alpha', \beta')$ iff for any $f \in H^{-1}(\Omega)$

$$u_n \rightharpoonup u_0 \quad \text{in } H^1_0(\Omega)$$

$$\mathbf{A}_n \nabla u_n \rightharpoonup \mathbf{A}_0 \nabla u_0 \quad \text{in } L^2(\Omega; \mathbb{R}^d) \, ,$$

where u_0 solves the problem for $\mathbf{A}_0$.

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H-compactness:

There exists $\mathbf{A}_0 \in \mathcal{M}(\alpha, \beta)$ such that (up to a subseq.) $\mathbf{A}_n \xrightarrow{H} \mathbf{A}_0$.

Lower order terms - correctors

$$\begin{cases} -\mathsf{div}\big(\mathbf{A}_n \nabla u_n\big) + \mathsf{b}_n \cdot \nabla u_n = f & \text{in } \Omega \\ u_n \in H^1_0(\Omega) \end{cases}$$

$\mathbf{A}_n \xrightharpoonup{H} \mathbf{A}_0$, b_n bounded in $L^p(\Omega;\mathbb{R}^d)$, $p > d$, $u_n \rightharpoonup u_0$ in $H^1_{loc}(\Omega)$.

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Existence of correctors:
If $\mathbf{A}_n \xrightharpoonup{H} \mathbf{A}_0$, then there exists $P_n \in L^2(\Omega;\mathbb{R}^{d\times d})$ such that

$$\nabla u_n - P_n \nabla u_0 \to 0 \quad \text{in } L^1_{loc}(\Omega;\mathbb{R}^d)\,.$$

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Application of correctors:

Rewrite the equation as

$$-\mathrm{div}\big(\mathbf{A}_n \nabla u_n\big) + P_n^T \mathsf{b}_n \cdot \nabla u_0 = f - \mathsf{b}_n \cdot \big(\nabla u_n - P_n \nabla u_0\big)$$

$$\downarrow n \to \infty$$

$$-\mathrm{div}\big(\mathbf{A}_0 \nabla u_0\big) + \mathsf{b}_0 \cdot \nabla u_0 = f - 0$$

Fractional Sobolev spaces 1/2

$s \in (0,1)$

$u \in H^s(\mathbb{R}^d)$ iff $u \in L^2(\mathbb{R}^d)$ and

$$(D_s u)(\mathbf{x},\mathbf{y}) := \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d/2+s}} \in L^2(\mathbb{R}^d \times \mathbb{R}^d)\,.$$

E.g. $H^1(\mathbb{R}^d) \subseteq H^s(\mathbb{R}^d)$.

$H^s(\mathbb{R}^d)$ equipped with

$$\|u\|_s := \sqrt{\|u\|^2_{L^2(\mathbb{R}^d)} + \|D_s u\|^2_{L^2(\mathbb{R}^{2d})}}$$

is a Hilbert space.

Fractional Sobolev spaces 2/2

For $\Omega \subseteq \mathbb{R}^d$ open and bounded with a smooth boundary we define $H_0^s(\Omega)$ as the completion of $C_c^\infty(\Omega)$ in $H^s(\mathbb{R}^d)$.
It holds:

$$H_0^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^d) : u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega \right\} .$$

Poincaré inequality: $\exists C > 0$ s.t.

$$\|u\|_{L^2(\mathbb{R}^d)} \leq C \|D_s u\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \,, \quad u \in H_0^s(\mathbb{R}^d) \,.$$

Compact embeddings:
For (u_n) bounded in $H_0^s(\Omega)$ there exists a subsequence $(u_{n'})$ and $u_0 \in H_0^s(\Omega)$ such that $u_{n'} \to u_0$ in $L^2(\mathbb{R}^d)$.

Non-local elliptic operators - symmetric case

$$\mathcal{L}_a u(x) := \text{p.v.} \int_{\mathbb{R}^d} a(\mathbf{x}, \mathbf{y}) \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{y} \ , \quad s \in (0, 1)$$

Motivation:

- Modelling: diffusion with long range interactions
- Probability: infinitesimal generator of stable Lévy processses

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$$a \in \mathcal{A}_{sym}(\alpha,\beta) := \left\{ a \in L^\infty(\mathbb{R}^{2d}) : a(\mathbf{x},\mathbf{y}) \in [\alpha,\beta] \text{ for a.e. } (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d} \right\},$$

where $0 < \alpha \leq \beta < \infty$.

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$$\begin{aligned} \langle \mathcal{L}_a u, v \rangle &= \iint_{\mathbb{R}^{2d}} a(\mathbf{x},\mathbf{y}) \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} v(\mathbf{x})\, d\mathbf{y} d\mathbf{x} \\ &= \iint_{\mathbb{R}^{2d}} a(\mathbf{y},\mathbf{x}) \frac{u(\mathbf{y}) - u(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^{d+2s}} v(\mathbf{y})\, d\mathbf{x} d\mathbf{y} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \langle \mathcal{L}_a u, v \rangle &= \frac{1}{2} \iint_{\mathbb{R}^{2d}} a(\mathbf{x},\mathbf{y}) \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \big(v(\mathbf{x}) - v(\mathbf{y}) \big)\, d\mathbf{y} d\mathbf{x} \\ &= \frac{1}{2} \langle a D_s u, D_s v \rangle \end{aligned}$$

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Thus, $\mathcal{L}_a$ bounded and coercive $\implies$ well-posedness of
($f \in H^{-1}(\Omega) := \big(H^s_0(\Omega)\big)'$)

$$\begin{cases} \mathcal{L}_a u = f \\ \quad u \in H^s_0(\Omega) \end{cases} .$$

Non-local div-rot lemma

Recall,

$$(D_s u)(\mathbf{x},\mathbf{y}) := \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d/2+s}}\,, \quad u \in H_0^s(\Omega).$$

Define non-local divergence as

$$(d_s\phi)(\mathbf{x}) := \text{p.v.}\int_{\mathbb{R}^d} \frac{\phi(\mathbf{x},\mathbf{y}) - \phi(\mathbf{y},\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{d/2+s}}\,d\mathbf{y} \qquad , \quad \phi \in L^2(\mathbb{R}^d \times \mathbb{R}^d).$$

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$$\langle \phi, D_s u\rangle = \langle d_s\phi, u\rangle$$

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$$\langle \phi, D_s u\rangle = \langle d_s\phi, u\rangle$$

$$\langle \mathcal{L}_a u, v\rangle = \frac{1}{2}\langle a D_s u, D_s v\rangle \implies \mathcal{L}_a u = \frac{1}{2} d_s(a D_s u)$$

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Lemma (Bonder, Ritorto, Salort, SIAM J. Math. Anal. (2017))

$$\begin{cases} v_n \rightharpoonup v_0 & \textit{in } H^s(\mathbb{R}^d) \\ \phi_n \rightharpoonup \phi_0 & \textit{in } L^2(\mathbb{R}^{2d}) \\ d_s\phi_n \to d_s\phi & \textit{in } H^{-s}_{loc}(\mathbb{R}^d) \end{cases}$$

$\Longrightarrow \ \phi_n D_s v_n \to \phi_0 D_s v_0$ *in* $\mathcal{D}'(\mathbb{R}^{2d})$.

Non-local problem - H-convergence

$$\begin{cases} \mathcal{L}_{a_n} u_n = f \\ \qquad u_n \in H_0^s(\Omega) \end{cases} .$$

$\mathcal{A}(\alpha,\beta) \ni a_n \xrightharpoonup{H} a_0 \in \mathcal{A}(\alpha,\beta)$ iff for any $f \in H^{-s}(\Omega)$

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There exists $a_0 \in \mathcal{A}(\alpha, \beta')$ such that (up to a subseq.) $a_n \xrightharpoonup{H} a_0$.

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Characterisation: (Bellido, Evgrafov, Rev. Mat. Complut. (2021))
$a_n \xrightharpoonup{*} a_0$ in $L^\infty(\mathbb{R}^{2d})$ iff $a_n \xrightharpoonup{H} a_0$.

Non-symmetric case - well-posedness

$a_{sym}(\mathbf{x},\mathbf{y}) := \big(a(\mathbf{x},\mathbf{y}) + a(\mathbf{y},\mathbf{x})\big)/2$
$a_{anti}(\mathbf{x},\mathbf{y}) := \big(a(\mathbf{x},\mathbf{y}) - a(\mathbf{y},\mathbf{x})\big)/2$

$$\mathcal{L}_a = \mathcal{L}_{a_{sym}} + \mathcal{L}_{a_{anti}}$$

Sufficient conditions for well-posedness of $\mathcal{L}_a u = f$ in $H^s_0(\Omega)$:
there exist $\alpha, \beta, \gamma > 0$ s.t.

- $a(\mathbf{x},\mathbf{y}) \in [\alpha, \beta]$ for a.e. $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}$,
- $\sup_{\mathbf{x}\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|a_{anti}(\mathbf{x},\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}} d\mathbf{y} \leq \gamma$,
- $\inf_{\mathbf{x}\in\mathbb{R}^d} \limsup_{\epsilon\to 0+} \int_{\mathbb{R}^d\setminus B_\epsilon(\mathbf{x})} \frac{a_{anti}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} d\mathbf{y} \geq 0$.

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- $\inf_{\mathbf{x}\in\mathbb{R}^d} \limsup_{\epsilon\to 0+} \int_{\mathbb{R}^d\setminus B_\epsilon(\mathbf{x})} \frac{a_{anti}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} d\mathbf{y} \geq 0$.

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Homogenisation:

With the suitable theory of correctors one can pass to a limit as in the local (classical) setting.

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Up to now we have results only for certain cases (work in progress!).

E.g. for $a_n(\mathbf{y}, \mathbf{y}) = b_n(\mathbf{x}) c_n(\mathbf{x}, \mathbf{y})$, where $c \in \mathcal{A}(\alpha, \beta)$ and $\alpha \leq b_n \leq \beta$ we have

$$a_n \xrightarrow{H} c_0 / \bar{b} \, ,$$

where $1/\bar{b}_n$ is the weak-* limit of $1/b_n$ (effect not present in the symmetric case!)

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$$a_n \xrightarrow{H} c_0/\bar{b}\,,$$

where $1/\bar{b}_n$ is the weak-* limit of $1/b_n$ (effect not present in the symmetric case!)

Kassmann, Piatnitski, Zhizhina, SIAM J. Math. Anal. (2019): non-symmetric a, but in the periodic setting.

And...

...thank you for your attention :)