Strong traces of entropy solutions to degenerate parabolic equations on the boundary

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Degenerate parabolic equation - introduction

$$\partial_t u(t,\mathbf{x}) + \mathsf{div}_{\mathbf{x}}\mathsf{f}\big(t,\mathbf{x},u(t,\mathbf{x})\big) = \mathsf{div}_{\mathbf{x}}\Big(a\big(t,\mathbf{x},u(t,\mathbf{x})\big)\nabla_{\mathbf{x}} u(t,\mathbf{x})\Big)\,,$$

where $\mathsf{f} : \Omega \times \mathbb{R} \to \mathbb{R}^d$ and $a : \Omega \times \mathbb{R} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ are given, and $u : \Omega \to \mathbb{R}$ is unknown ($\Omega \subseteq \mathbb{R}^+ \times \mathbb{R}^d$ open).

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- f and a sufficiently smooth;
- Homogeneous case: $\mathsf{f} = \mathsf{f}(u)$, $a = a(u)$.

Motivation for the equation:

- flow in porous media (e.g. $\mathsf{f} = 0$ and $a(u) = mu^{m-1}\mathrm{I}$ – porous media equation)
  - heterogeneous layers $\longrightarrow$ discontinuous flux and a lack of diffusion in some directions
- sedimentation-consolidation process

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Aim:

- Existence of traces of solutions, i.e. give meaning to $u(t,\mathbf{x})$ for $(t,\mathbf{x}) \in \partial\Omega$.

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- Formulation, well-posedness, optimal control, etc., for initial-boundary problems.
- Characterising the limit of hyperbolic relaxation towards a scalar conservation law.

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Challenges:

- heterogeneous flux f and diffusion matrix a;
- degeneracy of a, i.e. $a \geq 0$.

Entropy solutions: $a = 0$

For simplicity we consider homogeneous case.

$$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \Omega \subseteq \mathbb{R}^+ \times \mathbb{R}^d \,.$$

Entropy solutions: (Kružkov) $u \in L^\infty(\Omega)$ s.t. $\forall \lambda \in \mathbb{R}$ and $\forall \varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$,

$$\int_\Omega |u - \lambda| \varphi_t + \operatorname{sgn}(u - \lambda)(\mathsf{f}(u) - \mathsf{f}(\lambda)) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x} dt \geq 0 \,.$$

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Cauchy problem: For $\Omega = \mathbb{R}^+ \times \mathbb{R}^d$ and $u_0 \in L^\infty(\mathbb{R}^d)$ we prescribe:

$$\operatorname{ess\,lim}_{t \to 0^+} u(t, \cdot) = u_0 \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^d)$$

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Kružkov (1970): existence and uniqueness of entropy solutions to Cauchy problems for heterogeneous fluxes f.

Panov (2010): existence of entropy solutions for non-smooth heterogeneous fluxes under non-degeneracy assumptions

Entropy solutions: $a \geq 0$

For simplicity we consider homogeneous case.

$$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = \mathsf{div}_{\mathbf{x}}\big(a(u)\nabla_{\mathbf{x}} u\big) \quad \text{in} \quad \Omega \subseteq \mathbb{R}^+ \times \mathbb{R}^d .$$

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$$\begin{aligned}(\forall \lambda \in \mathbb{R}) \qquad \partial_t |u - \lambda| &+ \mathsf{div}_{\mathbf{x}} \Big(\operatorname{sgn}(u-\lambda)(\mathsf{f}(u) - \mathsf{f}(\lambda))\Big) \\ &- D^2_{\mathbf{x}} \Big(\operatorname{sgn}(u-\lambda)(A(u) - A(\lambda))\Big) \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \,,\end{aligned}$$

where $A' = a$.

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Well-posedness for the Cauchy problem:

Homogeneous case: Chen, Perthame (2003)

Heterogeneous case: Chen, Karlsen (2005)

$\rightarrow$ in both cases a certain chain rule property is required (we will return to this later!)

Strong traces - definition

$$\partial_t u(t,\mathbf{x}) + \mathsf{div}_{\mathbf{x}} \mathsf{f}\big(t,\mathbf{x},u(t,\mathbf{x})\big) = \mathsf{div}_{\mathbf{x}}\Big(a\big(t,\mathbf{x},u(t,\mathbf{x})\big)\nabla_{\mathbf{x}} u(t,\mathbf{x})\Big) \quad \text{in } \Omega$$

Strong trace at $\mathrm{t} = \mathbf{0}$: $\operatorname{ess\,lim}_{t\to 0^+} u(t,\cdot) = u_0 \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^d)$

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Definition (Strong trace)

$u_0 \in L^\infty(\partial\Omega)$ is a strong trace of a solution $u \in L^\infty(\Omega)$ if for any $\mathbf{x} \in \partial\Omega$ we have

$$\operatorname{ess\,lim}_{s\to 0^+} \tilde{u}(s,\cdot) = \tilde{u}_0 \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^d)\,,$$

where $\tilde{u} = u \circ \zeta^{-1}$ and $\tilde{u}_0 = u_0 \circ \zeta^{-1}(0,\cdot)$ are obtained after localising and flattening the boundary.

Strong traces - an overview of the results

$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathsf{f}(u) = 0$

- **Vasseur (2001)**: under a non-degeneracy condition
- **Panov (2005, 2007)**

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$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathsf{f}(u) = \mathrm{div}_{\mathbf{x}}\big(a(u)\nabla u\big)$

- **Kwon (2007)**: for scalar diffusion matrix a, i.e. $a = cI$, $c \geq 0$
- **Aleksić, Mitrović (2013)**: for ultra-parabolic diffusion matrix a, i.e. $a = 0 \oplus b$, where $b > 0$ – only the trace at $t = 0$
- **Frid, Li (2017)**: for $a = b \oplus 0$, where $cI \leq b \leq \Lambda cI$ and $c \geq 0$ – under a non-degeneracy condition
- **E., Mitrović (2021)**: general a – only the trace at $t = 0$

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- **Aleksić, Mitrović (2013)**: ultra-parabolic diffusion matrix a, "partial" heterogeneity – under a non-degeneracy condition (only the trace at $t = 0$)
- **Neves. Panov, Silva (2018)**: rough fluxes, $a = 0$ – under a non-degeneracy condition
- **E., Mišur, Mitrović (2021)**: $a = a(u)$ – under a non-degeneracy condition (only the trace at $t = 0$)

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$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathsf{f}(u) = 0$

- the general result valid without non-degeneracy conditions

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$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathsf{f}(t, \mathbf{x}, u) = \mathrm{div}_{\mathbf{x}}\big(a(t, \mathbf{x}, u)\nabla u\big)$

- a non-degeneracy condition always used

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- the general result valid without non-degeneracy conditions

$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = \mathsf{div}_{\mathbf{x}} \big(a(u) \nabla u \big)$

- no non-degeneracy conditions for the strong trace at the flat boundary (e.g. $t = 0$), but in general they are used

$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(t, \mathbf{x}, u) = \mathsf{div}_{\mathbf{x}} \big(a(t, \mathbf{x}, u) \nabla u \big)$

- a non-degeneracy condition always used

Auxiliary problem

Thus, in order to get the existence of the strong trace on the boundary for entropy solutions to

$$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = \mathsf{div}_{\mathbf{x}}\big(a(u)\nabla u\big)$$

the main step is the following.

Task: To show the existence of the strong trace at $x_1 = 0$ of the solution u to

$$\mathsf{div}_{\mathbf{x}}\big(\mathsf{f}(\mathbf{x}, u)\big) = \mathsf{div}_{\mathbf{x}}\big(a(\mathbf{x}, u)\nabla_{\mathbf{x}} u\big) , \quad \mathbf{x} = (x_1, \mathbf{x}') \in \mathbb{R}^+ \times \mathbb{R}^{d-1} ,$$

under the non-degeneracy condition ($K \subset\subset \mathbb{R}$):

$$\operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^+\times\mathbb{R}^{d-1}} \sup_{|\boldsymbol{\xi}|=1} \mathrm{meas}\Big\{\lambda \in K \,:\, \mathsf{f}(\mathbf{x},\lambda)\cdot\boldsymbol{\xi} \,=\, a(\mathbf{x},\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \,=\, 0\Big\} = 0 \,.$$

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Step I. Blow-up (Vasseur, 2001)

u admits the strong trace $\iff$

$$u_n(x_1, \mathbf{x}', \mathbf{y}) := u\Big(\frac{x_1}{n}, \frac{\mathbf{x}'}{n} + \mathbf{y}'\Big) \text{ is precompact in } \mathrm{L}^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^{2(d-1)})\,.$$

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Step II.

(u_n) is precompact $\iff$ the corresponding microlocal defect functional μ (e.g. H-measures) is zero.

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Step III. μ will be zero if we get (localisation principle)

$$\Big(i\mathsf{f}(\mathbf{x},\lambda)\cdot\boldsymbol{\xi} + a(\mathbf{x},\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\xi}\Big)\mu = 0 \,,$$

by the assumed non-degeneracy condition.

This can be obtained using the rescaled equation that is satisfied by u_n.

However, in this process the choice of μ, i.e. the choice of the scaling, is important!

Auxiliary problem

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Step IV. The choice of the scaling in μ:

- $a = 0$ or $a > 0$: $\boldsymbol{\xi} \mapsto \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$
- $a = 0 \oplus b$, $b > 0$: $\boldsymbol{\xi} = (\boldsymbol{\xi}', \boldsymbol{\xi}'') \mapsto \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}'|+|\boldsymbol{\xi}''|^2}$
- $a = a(\lambda)$: $\boldsymbol{\xi} \mapsto \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|+a(\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\xi}}$
- Problem: $\boldsymbol{\xi} \mapsto \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|+a(\mathbf{x},\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\xi}}$ is too complicated (pseudodifferential operator) and does not give anything...

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$$\mathrm{div}_{\mathbf{x}}\big(\mathsf{f}(\mathbf{x},u)\big) = \mathrm{div}_{\mathbf{x}}\big(a(\mathbf{x},u)\nabla_{\mathbf{x}} u\big)\,, \quad \mathbf{x} = (x_1,\mathbf{x}') \in \mathbb{R}^+ \times \mathbb{R}^{d-1}\,,$$

under the non-degeneracy condition ($K \subset\subset \mathbb{R}$):

$$\operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^+\times\mathbb{R}^{d-1}} \sup_{|\boldsymbol{\xi}|=1} \mathrm{meas}\Big\{\lambda \in K \,:\, \mathsf{f}(\mathbf{x},\lambda)\cdot\boldsymbol{\xi} \,=\, a(\mathbf{x},\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \,=\, 0\Big\} = 0\,.$$

Step V.a

Apply the scaling $\boldsymbol{\xi} \mapsto \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$ to get

$$\Big(a(\mathbf{x},\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\xi}\Big)\mu = 0$$

(only the highest order terms are "visible").

Auxiliary problem

Task: To show the existence of the strong trace at $x_1 = 0$ of the solution u to

$$\mathrm{div}_{\mathbf{x}}\big(\mathsf{f}(\mathbf{x},u)\big) = \mathrm{div}_{\mathbf{x}}\big(a(\mathbf{x},u)\nabla_{\mathbf{x}}u\big)\ , \quad \mathbf{x} = (x_1,\mathbf{x}') \in \mathbb{R}^+ \times \mathbb{R}^{d-1}\ ,$$

under the non-degeneracy condition ($K \subset\subset \mathbb{R}$):

$$\operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^+\times\mathbb{R}^{d-1}} \sup_{|\boldsymbol{\xi}|=1} \mathrm{meas}\Big\{\lambda \in K\ :\ \mathsf{f}(\mathbf{x},\lambda)\cdot\boldsymbol{\xi}\ =\ a(\mathbf{x},\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\xi}\ =\ 0\Big\} = 0\ .$$

Step V.b

Then lower the order of the equation by applying the chain rule from the definition of the entropy solution (Chen, Karlsen, 2005), i.e. the term

$$\mathrm{div}_{\mathbf{x}}\big(a(\mathbf{x},u)\nabla_{\mathbf{x}}u\big)$$

is replaced by a first order term

$\Longrightarrow$ the term $\mathrm{div}_{\mathbf{x}}\big(\mathsf{f}(\mathbf{x},u)\big)$ is "visible", so with the proper analysis we get (in a certain form)

$$\Big(\mathsf{f}(\mathbf{x},\lambda)\cdot\boldsymbol{\xi}\Big)\mu = 0\ ,$$

completing the argument.

Thank you for your attention!