Continuity of pseudodifferential operators with nonsmooth symbols on mixed-norm Lebesgue spaces

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Symbol classes

$S^m_{\rho,\delta,N,N'}$ ... for $|\alpha| \leq N, |\beta| \leq N'$ it holds

$$(\forall x \in \mathbf{R}^d)(\forall \xi \in \mathbf{R}^d) \quad |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|} \, ,$$

where $\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}$

norm: $|\sigma|^{(m,\rho,\delta)}_{N,N'} = \max\limits_{|\alpha|\leq N, |\beta|\leq N'} \sup\limits_{x,\xi\in\mathbf{R}^d} \frac{|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)|}{\langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}}$

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$\dot{S}^{q,m}_{\rho,\delta,N,N'}$ ... for $|\alpha| \leq N, |\beta| \leq N'$ it holds

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Notation

For $N, N' \in \mathbf{N}_0$ we define an equivalent family of semi-norms on $\mathcal{S}(\mathbb{R}^d)$ with

$$|\varphi|_{N,N'} = \sup_{|\alpha|\leq N, |\beta|\leq N'} \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta \varphi(x)|,$$

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By $\lfloor x \rfloor$ we denote the largest integer not greater than x, while $\lfloor x \rfloor_2$ is the largest even integer not greater than x. We also use the standard notation $m^+ = \max\{m, 0\}$.

ΨDO - definition and continuity

For $\sigma \in S^{m}_{\rho,\delta,N,N'}$ or $\sigma \in \dot{S}^{q,m}_{\rho,\delta,N,N'}$ we denote the corresponding pseudodifferential operator T_σ by

$$T_\sigma \varphi(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x,\xi) \hat{\varphi}(\xi) \; đ\xi, \; \varphi \in \mathcal{S}(\mathbf{R}^d),$$

where $đ\xi = (2\pi)^{-d} d\xi$.

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Lemma 1. *$\mathcal{F} : \mathcal{S}_{N,N'}(\mathbf{R}^d) \to \mathcal{S}_{N',N-d-1}(\mathbf{R}^d)$ is a linear bounded mapping for $N \geq d+1$. More precisely, there is a constant $C_{N,N'} > 0$ such that*

$$|\hat{\varphi}|_{N',N-d-1} \leq C_{N,N'} |\varphi|_{N,N'} \quad \textit{for all } \varphi \in \mathcal{S}_{N,N'}(\mathbf{R}^d)\,.$$

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Theorem 1. *Let $\sigma \in S^m_{\rho,\delta,N,N'}$. Then T_σ is a bounded mapping from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}_{N',N}(\mathbf{R}^d)$, and from $\mathcal{S}_{M,M'}(\mathbf{R}^d)$ to $\mathcal{S}_{\min\{N',\, M-d-1\},\min\{N,\, M'-(\lfloor m\rfloor+d+1)^+\}}(\mathbf{R}^d)$, $M \geq d+1$, $M' \geq (\lfloor m\rfloor + d + 1)^+$. More precisely, there is a constant $C_{k,l} > 0$ such that*

$$|T_\sigma \varphi|_{k,l} \leq C_{k,l} |\sigma|^{(m,\rho,\delta)}_{l,k} |\varphi|_{d+1+k,(\lfloor m\rfloor+d+1)^+ + l}\,,$$

for all $k,l \in \mathbf{N}_0$ for which semi-norms are well-defined.

∎

The composition theorem

Theorem 2. *Let* $\sigma_1 \in S^{m_1}_{\rho_1,\delta_1,N_1,N_1'}$, $\sigma_2 \in S^{m_2}_{\rho_2,\delta_2,N_2,N_2'}$, $m_1, m_2 \geq -d$, $m^* = \max\{m_1^+, 2m_1^+ + m_2\}$, $\tilde{m} = \max\{m_1, m_2, m_1 + m_2\}$, $\rho = \min\{\rho_1, \rho_2\}$, $\delta = \max\{\delta_1, \delta_2\}$ *and* $\varphi \in \mathcal{S}_{M,M'}(\mathbf{R}^d)$. *If* $N_1', N_2, N_2', M' \in 2\mathbf{N}_0$ *and*

$$N_2 > \frac{m^*+3d+5}{1-\delta_2},\ N_2' > 3d+5, \quad N_1' > N_2'+3d+5, \quad M > 2d+1,\ M' > \tilde{m}+(1+\delta_2)N_2+3d+5\,,$$

then

$$(T_{\sigma_1} \circ T_{\sigma_2})\varphi(x) = T_{\sigma_1\#\sigma_2}\varphi(x)\,,$$

where

$$\sigma_1\#\sigma_2(x,\xi) = \iint e^{-iy\eta}\sigma_1(x,\xi+\eta)\sigma_2(x+y,\xi)dy\, đ\eta\,.$$

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If additionally $\delta \leq \rho_1$ *and* $N_2 - 2l \geq N_1$, *where* $l \in \mathbf{N}_0$ *is such that*

$$2l > \frac{m_1^+ + d + (\lfloor d \rfloor_2 + 2)\delta}{1-\delta}, \quad 2l \geq \frac{m_1^+ - m_1 + (1-\delta)d + (\lfloor d \rfloor_2 + 2)\delta}{1-\delta}\,,$$

then

$$\sigma_1\#\sigma_2 \in S^{m_1+m_2}_{\rho,\,\delta,\,N_1,\,N_2'}\,.$$

∎

The composition theorem - cont.

Theorem 2. *Moreover, if (for some $K \in \mathbf{N}_0$) $\delta < \rho$,*

$$N_2 \geq \lfloor m_1^+ + K + d + 1 \rfloor_2 + 2, \quad N_2 - 2l - K - 1 \geq N_1, \quad N_1' - K - \lfloor d \rfloor_2 - 3 \geq N_2',$$

where

$$2l > \frac{m_1^+ + d + (\lfloor d \rfloor_2 + 2)\delta + \delta(K+1)}{1-\delta}, \quad 2l \geq \frac{m_1^+ - m_1 + (1-\delta)d + (\lfloor d \rfloor_2 + 2)\delta + \rho(K+1)}{1-\delta},$$

then we have the following asymptotic expansion:

$$\sigma_1 \# \sigma_2(x,\xi) = \sum_{|\gamma| \leq K} \frac{1}{\gamma!} \partial_\xi^\gamma \sigma_1(x,\xi) D_x^\gamma \sigma_2(x,\xi) + r^{(K)}(x,\xi),$$

where

$$r^{(K)} \in S^{m_1+m_2-(\rho-\delta)(K+1)}_{\rho,\,\delta,\,N_1,\,N_2'}.$$

∎

The adjoint

Now we define a formal adjoint of the operator with symbol $\sigma \in S^m_{\rho,\delta,N,N'}$. From Theorem 1 it follows that T_σ maps $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}_{N',N}(\mathbf{R}^d)$. Also, $\mathcal{S}_{N',N}(\mathbf{R}^d) \subseteq L^2(\mathbf{R}^d)$ for $N' > \frac{d}{2}$. This motivates the following definition.

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Definition

Let $\sigma \in S^m_{\rho,\delta,N,N'}$, $\sigma^* \in S^m_{\rho,\delta,M,M'}$, $M', N' > \frac{d}{2}$. Then T_{σ^*} is called a formal adjoint of T_σ if

$$(\forall \varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)) \quad \langle T_\sigma \varphi_1 | \varphi_2 \rangle = \langle \varphi_1 | T_{\sigma^*} \varphi_2 \rangle \,, \tag{1}$$

where $\langle \cdot | \cdot \rangle$ is the standard inner product on $L^2(\mathbf{R}^d)$.

The adjoint theorem

Theorem 3. *Let* $\sigma \in S^m_{\rho,\delta,N,N'}$, $m \geq -d$. *If* $N, N' \in 2\mathbf{N}_0$ *such that*

$$N > \frac{2m^+ + (3-\delta)d + (5-\delta)(1-\delta)}{(1-\delta)^2}, \quad N' > 6d + 12\,,$$

then (1) is satisfied for

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If additionally $\delta \leq \rho$, $N - 2l \geq M$ *and* $N' - \lfloor d \rfloor_2 - 2 \geq M'$, *where* $l \in \mathbf{N}_0$ *is such that*

$$2l > \frac{m^+ + d + (\lfloor d \rfloor_2 + 2)\delta}{1-\delta}, \quad 2l \geq \frac{m^+ - m + (1-\delta)d + (\lfloor d \rfloor_2 + 2)\delta}{1-\delta}\,,$$

then

$$\sigma^* \in S^m_{\rho,\,\delta,\,M,\,M'}\,.$$

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Theorem 3. *Moreover, if (for some $K \in \mathbf{N}_0$) $\delta < \rho$,*

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$$\sigma^*(x,\xi) = \sum_{|\gamma| \leq K} \frac{1}{\gamma!} \partial_\xi^\gamma D_x^\gamma \overline{\sigma(x,\xi)} + r_*^{(K)}(x,\xi),$$

where

$$r_*^{(K)} \in S^{m-(\rho-\delta)(K+1)}_{\rho,\,\delta,\,M,\,M'}.$$

∎

Known continuity results

Our starting point is a famous result by Coifman and Meyer:
for $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ and $m = 0$ it is enough to have $N, N' > \frac{d}{2}$ to obtain the continuity on $L^2(\mathbf{R}^d)$.

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Also, in smooth case we have the following necessary and sufficient condition for continuity on $L^p(\mathbf{R}^d)$ spaces:

$$m \leq -d(1-\rho)\left|\frac{1}{2} - \frac{1}{p}\right|.$$

The general framework

Take $l \in \{0, \ldots, (d-1)\}$ and split $x = (\bar{x}, x') = (x_1, \ldots, x_l; x_{l+1}, \ldots, x_d)$.
Next define $\|f\|_{\bar{\mathbf{p}},\, p} = \|f\|_{(\bar{\mathbf{p}},\, p, \ldots,\, p)}$.
We also define (for each $t > 0$ and $y' \in \mathbf{R}^{d-l}$):

$$\mathcal{F}_{l,t}^{y'} := \left\{ f \in L^1_{loc}(\mathbf{R}^d) : \text{supp } f \subseteq \mathbf{R}^l \times \{x' : |x' - y'|_\infty \leq t\} \;\; \& \;\; \int_{\mathbf{R}^{d-l}} f(\bar{x}, x')\, dx' = 0 \ (\text{ae } \bar{x} \in \mathbf{R}^l) \right\}.$$

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Theorem 4. *Assume that $A, A^* : L_c^\infty(\mathbf{R}^d) \to L^1_{loc}(\mathbf{R}^d)$ are formally adjoint linear operators, i.e. such that*

$$(\forall \varphi, \psi \in C_c^\infty(\mathbf{R}^d)) \quad \int_{\mathbf{R}^d} (A\varphi)\overline{\psi} = \int_{\mathbf{R}^d} \varphi \overline{A^*\psi}.$$

Furthermore, let us assume that (both for $T = A$ and $T = A^*$) there exist constants $N > 1$ and $c_1 > 0$ satisfying

$$(\forall l \in \{0, \dots, (d-1)\})(\forall x_0' \in \mathbf{R}^{d-l})(\forall t > 0)$$
$$\int_{|x' - x_0'|_\infty > Nt} \|Tf(\cdot, x')\|_{\bar{\mathbf{p}}}\, dx' \leq c_1 \|f\|_{\bar{\mathbf{p}},1} \,,$$

for any function $f \in L_c^\infty(\mathbf{R}^d) \cap \mathcal{F}_{l,t}^{x_0'}$ and any $\bar{\mathbf{p}} \in \langle 1, \infty \rangle^l$. ∎

The general framework - cont.

Theorem 4. *If for some $q \in \langle 1, \infty \rangle$ operator A has a continuous extension to an operator from $L^q(\mathbf{R}^d)$ to itself with norm c_q, then A can be extended by the continuity to an operator from $L^{\mathbf{p}}(\mathbf{R}^d)$ to itself for any $\mathbf{p} \in \langle 1, \infty \rangle^d$, with the norm*

$$\begin{aligned} \|A\|_{L^{\mathbf{p}} \to L^{\mathbf{p}}} &\leq \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max(p_{d-j}, (p_{d-j} - 1)^{-1/p_{d-j}})(c_1 + c_q) \\ &\leq c' \prod_{j=0}^{d-1} \max(p_{d-j}, (p_{d-j} - 1)^{-1/p_{d-j}})(c_1 + c_q), \end{aligned}$$

where c and c' are constants depending only on N and d. ∎

Properties of the kernel

We have $\sigma(x,\cdot) \in \mathcal{S}'(\mathbf{R}^d)$ and so there is a $k(x,\cdot) \in \mathcal{S}'(\mathbf{R}^d)$ such that $\widehat{k(x,\cdot)} = \sigma(x,\cdot)$. Then we can write

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Lemma 2. *Let $\sigma \in S^m_{\rho,\delta,N,N'}$, $\rho > 0$. Then the kernel $k(x,z)$ satisfies*

$$|\partial_x^\alpha \partial_z^\beta k(x,z)| \leq C_{\alpha,\beta,L} \cdot |z|^{-d-m-\delta|\alpha|-|\beta|-L}, \quad z \neq 0 \,,$$

for all $|\alpha| \leq N$, $|\beta| \geq 0$ and

$$L \geq (1-\rho)\Big(\Big\lfloor \frac{d+m+\delta|\alpha|+|\beta|}{\rho} \Big\rfloor + 1\Big)^+$$

such that $N' \geq d+m+\delta|\alpha|+|\beta|+L > 0$ and $N' > \frac{d+m+\delta|\alpha|+|\beta|}{\rho}$, and where $C_{\alpha,\beta,L}$ is a constant depending only on α, β and L. ∎

The continuity result

Theorem 5. *Let* $\sigma \in S^m_{\rho,\delta,N,N'}$, $[0,1\rangle \ni \delta \leq \rho \in \langle 0,1]$ *and*

$$m \leq -(1-\rho)(d+1+\rho)\,.$$

If

$$N > \frac{(3-\delta)d + (5-\delta)(1-\delta)}{(1-\delta)^2}, \quad N' > 6d+12\,,$$

then T_σ *is bounded on* $L^{\mathbf{p}}(\mathbf{R}^d)$, $\mathbf{p} \in \langle 1,\infty\rangle^d$. ∎