H-distributions related to Hörmander spaces

Ivana Vojnović

Department of Mathematics and Informatics
University of Novi Sad

Analysis, PDEs and Applications
Dubrovnik, June 19- June 25, 2022

H-measures, H-distributions-motivation

- H-measures, micro-local defect measures (Tartar, Gérard, around 1990) - L^2 space, $\psi \in C(\mathbb{S}^{d-1})$

H-measures, H-distributions-motivation

- H-measures, micro-local defect measures (Tartar, Gérard, around 1990) - L^2 space, $\psi \in C(\mathbb{S}^{d-1})$
- H-distributions (Antonić, Mitrović, 2011) - $L^p - L^q$ spaces, $p = \dfrac{q}{q-1}$, $1 < p < \infty$, $\psi \in C^\kappa(\mathbb{S}^{d-1})$

H-measures, H-distributions-motivation

- H-measures, micro-local defect measures (Tartar, Gérard, around 1990) - L^2 space, $\psi \in C(\mathbb{S}^{d-1})$
- H-distributions (Antonić, Mitrović, 2011) - $L^p - L^q$ spaces, $p = \dfrac{q}{q-1}$,

  $1 < p < \infty$, $\psi \in C^\kappa(\mathbb{S}^{d-1})$
- H-distributions - $W^{-k,p} - W^{k,q}$ spaces, $1 < p < \infty,\ k \in \mathbb{N}_0$ (Aleksić, Pilipović, Vojnović, 2016)

H-measures, H-distributions-motivation

- H-measures, micro-local defect measures (Tartar, Gérard, around 1990) - L^2 space, $\psi \in C(\mathbb{S}^{d-1})$
- H-distributions (Antonić, Mitrović, 2011) - $L^p - L^q$ spaces, $p = \dfrac{q}{q-1}$,

  $1 < p < \infty$, $\psi \in C^\kappa(\mathbb{S}^{d-1})$
- H-distributions - $W^{-k,p} - W^{k,q}$ spaces, $1 < p < \infty,\ k \in \mathbb{N}_0$ (Aleksić, Pilipović, Vojnović, 2016)
- H-distributions on Hörmander B^p_s spaces, $1 < p < \infty, s \in \mathbb{R}$ (Ivec, Vojnović, 2021)

- $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$, $n \to \infty$

- $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$, $n \to \infty$

Existence of H-measure (Tartar, [5])

There exists a subsequence $(u_{n'})$ and a complex Radon measure μ on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ s. t. for all $\varphi_1(x), \varphi_2(x) \in C_0(\mathbb{R}^d)$, $\psi(\xi) \in C(\mathbb{S}^{d-1})$ we have that

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'})(\xi) \overline{\mathcal{F}(\varphi_2 u_{n'})}(\xi) \psi\left(\frac{\xi}{|\xi|}\right) d\xi$$

$$= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \varphi_1(x) \overline{\varphi_2}(x) \psi(\xi) d\mu(x, \xi) = \langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle$$

- $\mathbb{S}^{d-1}$ - unit sphere in $\mathbb{R}^d$

H-distributions, $W^{-k,p} - W^{k,q}$, $1 < p < \infty$

Theorem (H-measures, equivalent formulation)

Let sequences $u_n, v_n \rightharpoonup 0$ *in* $L^2(\mathbb{R}^d)$. *There exist* $(u_{n'}), (v_{n'})$ *and a complex Radon measure* μ *on* $\mathbb{R}^d \times \mathbb{S}^{d-1}$ *such that for all* $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d), \psi \in C(\mathbb{S}^{d-1})$

$$\langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle := \lim_{n' \to \infty} \langle \varphi_1 u_{n'}, \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})} \rangle.$$

- $\mathcal{A}_\psi(u) = \mathcal{F}^{-1}(\psi \mathcal{F}(u))$

H-distributions, $W^{-k,p} - W^{k,q}$, $1 < p < \infty$

Theorem (H-measures, equivalent formulation)

Let sequences $u_n, v_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$. There exist $(u_{n'}), (v_{n'})$ and a complex Radon measure μ on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ such that for all $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d), \psi \in C(\mathbb{S}^{d-1})$

$$\langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle := \lim_{n' \to \infty} \langle \varphi_1 u_{n'}, \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})} \rangle.$$

- $\mathcal{A}_\psi(u) = \mathcal{F}^{-1}(\psi \mathcal{F}(u))$

Theorem

If a sequence $u_n \rightharpoonup 0$ weakly in $W^{-k,p}(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ weakly in $W^{k,q}(\mathbb{R}^d)$, then there exist subsequences $(u_{n'}), (v_{n'})$ and a distribution $\mu \in \mathcal{SE}'(\mathbb{R}^d \times \mathbb{S}^{d-1})$ such that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in C^\kappa(\mathbb{S}^{d-1})$, $\kappa = [d/2] + 1$,

$$\lim_{n' \to \infty} \langle \varphi_1 u_{n'} , \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})} \rangle = \langle \mu, \varphi_1 \bar{\varphi}_2 \psi \rangle.$$

Theorem

Let $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbb{R}^d)$. If for every sequence $v_n \rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^d)$ the corresponding H-distribution is zero, then for every $\theta \in \mathcal{S}(\mathbb{R}^d)$, $\theta u_n \to 0$ strongly in $W^{-k,p}(\mathbb{R}^d)$, $n \to \infty$.

- $1 < q < d$, $u_n \rightharpoonup 0$ in $W^{-k,p}$, $v_n \rightharpoonup 0$ in $W^{k,q}$
- $\sum_{i=1}^{d} \partial_{x_i}(A_i(x)u_n(x)) = f_n(x)$, $A_i \in \mathcal{S}(\mathbb{R}^d)$, $\theta f_n \to 0$ in $W^{-k-1,p}$, $n \to \infty$ for every $\theta \in \mathcal{S}(\mathbb{R}^d)$

Localization property

$$\sum_{j=1}^{d} A_j(x)\xi_j \, \mu(x,\xi) = 0 \text{ in } \mathcal{S}\mathcal{E}'(\mathbb{R}^d \times \mathbb{S}^{d-1})$$

$$\text{supp } \mu \subset \text{char } P$$

- $\mathcal{S}(\mathbb{R}^d)\hat{\otimes}\mathcal{E}(\mathbb{S}^{d-1}) = \mathcal{S}\mathcal{E}(\mathbb{R}^d \times \mathbb{S}^{d-1})$.

Weight functions

A positive function k is called a temperate weight function if there exist constants $C, N > 0$ such that

$$k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^d.$$

Weight functions

A positive function k is called a temperate weight function if there exist constants $C, N > 0$ such that

$$k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^d.$$

It can be derived that k is continuous and the following estimate holds

$$k(0)(1 + C|\xi|)^{-N} \leq k(\xi) \leq k(0)(1 + C|\xi|)^N. \tag{1}$$

Weight functions

A positive function k is called a temperate weight function if there exist constants $C, N > 0$ such that

$$k(\xi+\eta) \leq (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^d.$$

It can be derived that k is continuous and the following estimate holds

$$k(0)(1 + C|\xi|)^{-N} \leq k(\xi) \leq k(0)(1 + C|\xi|)^N. \tag{1}$$

Definition

Let k be a temperate weight function, $1 \leq p < \infty$. We denote by $B_{p,k}$ the space of distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{u}$ is a function and $k\hat{u} \in L^p$. If $u \in B_{p,k}$, then we define $\|u\|_{p,k} = \|k\hat{u}\|_p$.

Basic properties of $B_{p,k}$ spaces:

1. The function $\|\cdot\|_{p,k}$ defines a norm on $B_{p,k}$ and $B_{p,k}$ is a Banach space with this norm.
2. We have that $\mathcal{S}(\mathbb{R}^d)$ is dense in $B_{p,k}$ and
$$\mathcal{S}(\mathbb{R}^d) \subset B_{p,k} \subset \mathcal{S}'(\mathbb{R}^d)$$
continuously.
3. If $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $u \in B_{p,k}$, then $\varphi u \in B_{p,k}$ and $\|\varphi u\|_{p,k} \leq c|\varphi|_{k_0}\|u\|_{p,k}$, for some $k_0 \in \mathbb{N}$.
4. If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $B_{q,1/k}$ is the dual space of $B_{p,k}$.

Algebra property of Hörmander spaces

A basic example of weight functions are functions $k_s(\xi) = (1 + |\xi|^2)^{s/2} = \langle \xi \rangle^s$, $s \in \mathbb{R}$ and we shall consider them in the sequel. We denote by B_s^p the space

$$B_s^p = \{u \in \mathcal{S}'(\mathbb{R}^d) : \ \hat{u} \text{ is a function and } \langle \cdot \rangle^s \hat{u} \in L^p(\mathbb{R}^d)\}.$$

Algebra property of Hörmander spaces

A basic example of weight functions are functions $k_s(\xi) = (1 + |\xi|^2)^{s/2} = \langle \xi \rangle^s$, $s \in \mathbb{R}$ and we shall consider them in the sequel. We denote by B^p_s the space

$$B^p_s = \{u \in \mathcal{S}'(\mathbb{R}^d) : \ \hat{u} \text{ is a function and } \langle \cdot \rangle^s \hat{u} \in L^p(\mathbb{R}^d)\}.$$

Lemma

If $u, v \in B^p_s$ for $1 < p < \infty$ and $s > \frac{d}{q}$, then $uv \in B^p_s$ and for some $C > 0$ it holds

$$\|uv\|_{B^p_s} \leq C \|u\|_{B^p_s} \|v\|_{B^p_s}.$$

Symbol space and existence theorem

Let $m \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Consider the space of all $\psi \in C^N(\mathbb{R}^d)$ for which the norm

$$|\psi|_{S^m_{\infty,N}} := \max_{|\alpha| \le N} \|\partial_\xi^\alpha \psi(\xi)\langle\xi\rangle^{-m+|\alpha|}\|_\infty \tag{2}$$

is finite. This is a Banach space.

Symbol space and existence theorem

Let $m \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Consider the space of all $\psi \in C^N(\mathbb{R}^d)$ for which the norm

$$|\psi|_{s^m_{\infty,N}} := \max_{|\alpha| \leq N} \|\partial^\alpha_\xi \psi(\xi)\langle\xi\rangle^{-m+|\alpha|}\|_\infty \tag{2}$$

is finite. This is a Banach space. If we fix $\psi \in s^m_{\infty,N}$ we can prove the existence of a distribution $\mu_\psi \in \mathcal{S}'(\mathbb{R}^d)$. The following theorem holds.

Symbol space and existence theorem

Let $m \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Consider the space of all $\psi \in C^N(\mathbb{R}^d)$ for which the norm

$$|\psi|_{s^m_{\infty,N}} := \max_{|\alpha|\leq N} \|\partial^\alpha_\xi \psi(\xi)\langle\xi\rangle^{-m+|\alpha|}\|_\infty \tag{2}$$

is finite. This is a Banach space. If we fix $\psi \in s^m_{\infty,N}$ we can prove the existence of a distribution $\mu_\psi \in \mathcal{S}'(\mathbb{R}^d)$. The following theorem holds.

Theorem

Let $1 < p < \infty$, $m, s \in \mathbb{R}$, $N \geq 15d + 2|s| + 25$, $\psi \in s^m_{\infty,N}$ and $u_n \rightharpoonup 0$ in B^p_s, $v_n \rightharpoonup 0$ in B^q_{-s+m}. Then, up to subsequences, there exists a distribution $\mu_\psi \in \mathcal{S}'(\mathbb{R}^d)$ such that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\lim_{n\to\infty} \langle \varphi_1 u_n, \mathcal{A}_\psi(\varphi_2 v_n)\rangle = \langle \mu_\psi, \varphi_1\varphi_2\rangle.$$

Theorem

Let $u_n \rightharpoonup 0$ *in* B^p_s, $s \in \mathbb{R}$ *and assume that*

$$\lim_{n\to\infty} \langle u_n, \mathcal{A}_{\langle \xi \rangle^m}(\varphi v_n)\rangle = 0, \tag{3}$$

for every sequence $v_n \rightharpoonup 0$ *in* B^q_{-s+m}, $m \in \mathbb{R}$ *and every* $\varphi \in \mathcal{S}(\mathbb{R}^d)$. *Then for every* $\theta \in \mathcal{S}(\mathbb{R}^d)$, $\theta u_n \to 0$ *in* B^p_s.

Applications to semilinear equations

Let $1 < p < \infty$, $s > \frac{d}{q}$, $m > 0$ and consider semilinear pseudodifferential equation of the form

$$T_\sigma(u) = u^2, \tag{4}$$

where $\sigma(x, \xi)$ is an elliptic symbol and assume that a solution $u \in B_s^p$ exists. Here

$$T_\sigma u(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x, \xi) \hat{u}(\xi)\, d\xi.$$

Applications to semilinear equations

Let $1 < p < \infty$, $s > \frac{d}{q}$, $m > 0$ and consider semilinear pseudodifferential equation of the form

$$T_\sigma(u) = u^2, \tag{4}$$

where $\sigma(x,\xi)$ is an elliptic symbol and assume that a solution $u \in B_s^p$ exists. Here

$$T_\sigma u(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x,\xi) \hat{u}(\xi)\, d\xi.$$

More precisely, $\sigma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1+|x|)^{-|\alpha|}(1+|\xi|)^{m-|\beta|},\ x,\xi \in \mathbb{R}^d$ and there exist positive constants c, r such that

$$|\sigma(x,\xi)| \geq c(1+|\xi|)^m, \quad |x|+|\xi| \geq r.$$

Applications to semilinear equations

Let $1 < p < \infty$, $s > \frac{d}{q}$, $m > 0$ and consider semilinear pseudodifferential equation of the form

$$T_\sigma(u) = u^2, \tag{4}$$

where $\sigma(x,\xi)$ is an elliptic symbol and assume that a solution $u \in B^p_s$ exists. Here

$$T_\sigma u(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi}\sigma(x,\xi)\hat{u}(\xi)\,d\xi.$$

More precisely, $\sigma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|x|)^{-|\alpha|}(1+|\xi|)^{m-|\beta|},\ x,\xi \in \mathbb{R}^d$ and there exist positive constants c, r such that

$$|\sigma(x,\xi)| \ge c(1+|\xi|)^m, \quad |x|+|\xi| \ge r.$$

Let $u_n \rightharpoonup 0$ in B^p_{s+m} satisfy equation (4) for some elliptic σ.
Then $T_\sigma(u_n) = u_n^2 \rightharpoonup 0$ in B^p_s as $n \to \infty$.

Applications to semilinear equations

Let $1 < p < \infty$, $s > \frac{d}{q}$, $m > 0$ and consider semilinear pseudodifferential equation of the form

$$T_\sigma(u) = u^2, \tag{4}$$

where $\sigma(x,\xi)$ is an elliptic symbol and assume that a solution $u \in B^p_s$ exists. Here

$$T_\sigma u(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi}\sigma(x,\xi)\hat{u}(\xi)\, d\xi.$$

More precisely, $\sigma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|x|)^{-|\alpha|}(1+|\xi|)^{m-|\beta|},\ x,\xi \in \mathbb{R}^d$ and there exist positive constants c, r such that

$$|\sigma(x,\xi)| \ge c(1+|\xi|)^m, \quad |x|+|\xi| \ge r.$$

Let $u_n \rightharpoonup 0$ in B^p_{s+m} satisfy equation (4) for some elliptic σ. Then $T_\sigma(u_n) = u_n^2 \rightharpoonup 0$ in B^p_s as $n \to \infty$. Denote by $\tilde{u}_n = T_\sigma(u_n) = u_n^2 \rightharpoonup 0$ in B^p_s and let $v_n \rightharpoonup 0$ in B^q_{-s}. If $\psi(\xi) = 1$ and if $\mu_1 = 0$, then

$$\theta u_n^2 \to 0 \text{ in } B^p_s \text{ for every } \theta \in \mathcal{S}(\mathbb{R}^d).$$

Bibliography

- Aleksić J.; Pilipović S.; Vojnović I., *H-distributions via Sobolev spaces*, Mediterr. J. Math. 13(5), 3499-3512 (2016)
- Hörmander L., *The Analysis of Linear Partial Differential Operators II. Differential Operators with Constant Coefficients*. Berlin, Springer-Verlag (1983)
- Iancu G.M., *Boundedness of Pseudo-Differential Operators on Hörmander Spaces*. In: Gilbert R.P., Kajiwara J., Xu Y.S. (eds) *Direct and Inverse Problems of Mathematical Physics*. International Society for Analysis, Applications and Computation, vol 5. (2000)
- Ivec I., Vojnović I. - H-distributions on Hörmander spaces, J. Math.Anal.Appl. (2021)
- Tartar, L. *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations.* Proc. Roy. Soc. Edinburgh Sect. A 115, no. 3-4, 193–230 (1990)

Thank you for your attention!