# Microlocal defect functionals and applications


Nenad Antonić

Department of Mathematics
Faculty of Science
University of Zagreb


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# Microlocal defect distributions

## H-measures vs. defect measures

H-measures or microlocal defect measures represent a generalisation of defect measures. Besides the space variables, they depend on the dual variables as well. An H-measure is a Radon measure on the cospherical bundle

$$\Omega \times \mathrm{S}^{d-1} \subseteq T^*\Omega \simeq \Omega \times \mathbf{R}^d$$

over a domain $\Omega \subseteq \mathbf{R}^d$, and it is associated to a weakly converging sequence in $\mathrm{L}^2_{\rm loc}(\Omega)$.

Consider a plain wave:

$$u_n(\mathbf{x}) = \varphi(\mathbf{x}) e^{2\pi i \frac{\mathbf{x}}{\varepsilon_n} \cdot \mathsf{k}},$$

where $\varphi \in \mathrm{L}^2_{\rm loc}(\mathbf{R}^d)$, $\mathsf{k} \in \mathbf{R}^d \setminus \{0\}$, and $\varepsilon_n \to 0^+$. This sequence weakly converges in $\mathrm{L}^2_{\rm loc}(\mathbf{R}^d)$ to $0$ (but not strongly, except in the trivial case $\varphi = 0$).

Defect measure is the limit of $|u_n|^2 = |\varphi|^2$ in the space of (unbounded) Radon measures with respect to the weak-$*$ topology — $|\varphi|^2\lambda^d$.

On the other hand, the H-measure is

$$|\varphi|^2\lambda^d \otimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}},$$

where $\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$ (the Dirac measure at point $\mathsf{k}/|\mathsf{k}|$) is a measures in the dual space (variable $\boldsymbol{\xi}$).

Hence, the direction of oscillation is inherent in the H-measure.

# H-measures vs. semiclassical (Wigner) measures

In the example the H-measure does not distinguish between sequences with different frequencies $\frac{1}{\varepsilon_n}$. We need to incorporate a scale.
That is the case with *semiclassical measures*; the Radon measures on the cotangential bundle $\Omega \times \mathbf{R}^d$. Since they depend upon a characteristic length $(\omega_n)$, $\omega_n \to 0^+$ in the real line, they are more suitable in situations where such a characteristic length naturally appears, often related to highly oscillating problems for partial differential equations.

However, the scale brings new issues: if the characteristic length $(\omega_n)$ of a semiclassical measure is chosen inappropriately, we can lose information.

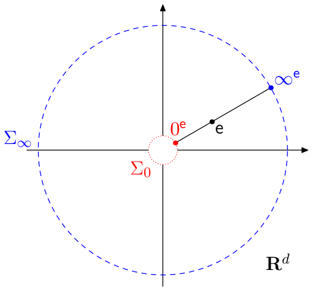
For example, if $\lim_n \frac{\omega_n}{\varepsilon_n} = +\infty$, the semiclassical measure associated to the plane wave is equal to zero measure. This in particular implies that, in contrast to H-measures, a zero semiclassical measure does not necessarily guarantee the strong convergence of the associated sequence (the so-called $(\omega_n)$-oscillatory property needs to be satisfied as well).

H-measures and semiclassical measures are in a general relation (neither is a generalisation of the other) and either has some advantages and disadvantages.

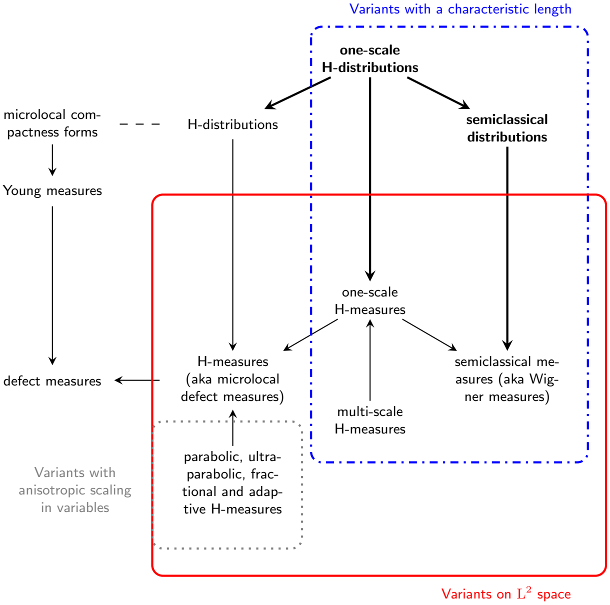
# One-scale H-measures

One-scale H-measures are a true extension of H-measures and semiclassical measures. The localisation principles, both for H-measures and semiclassical measures, can be derived from the localisation principle for one-scale H-measures

The main part in their construction is a proper choice of the domain for dual variables. For a fine tuning with characteristic lengths the set has to be *thick enough*, but we need to allow for directions to be *detected* also at the origin and infinity. This can be achieved with a radial compactification of $\mathbf{R}^d \setminus \{0\}$, denoted by $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$, which is homeomorphic to the $d$-dimensional spherical shell.



# Overview of MDF



# Compactification of $\mathbf{R}^d_* = \mathbf{R}^d \setminus \{0\}$

For the compactifying map $\mathcal{J}$ we take the composition of the translation from the origin in the radial direction for $r_0 > 0$:

$$\mathbf{R}^d_* \ni \boldsymbol{\xi} \overset{\mathcal{T}}{\longmapsto} \frac{|\boldsymbol{\xi}| + r_0}{|\boldsymbol{\xi}|}\boldsymbol{\xi} \in \mathbf{R}^d \setminus \mathrm{K}[0, r_0] \ ,$$

and a compactifying map of the radial compactification.

For the latter, we first identify $\mathbf{R}^d$ with the hypersurface $\xi_0 = 1$ in $\mathbf{R}^{1+d}_{\xi_0,\boldsymbol{\xi}}$, and then apply the modified stereographic projection based on the line through the origin (instead of the South Pole). More precisely, the radial compactification map $\mathcal{R}$ maps $\boldsymbol{\xi}$ to the intersection of $[0,1] \ni t \mapsto (t, t\boldsymbol{\xi})$ (the line through $(1,\boldsymbol{\xi})$ and $(0,0)$ in $\mathbf{R}^{1+d}$) and the upper half of the unit sphere centred at the origin: $\mathrm{S}^d_+ := \{(\zeta_0, \boldsymbol{\zeta}) \in \mathrm{S}^d : \zeta_0 > 0\}$. Since the intersection occurs at $t = (1 + |\boldsymbol{\xi}|^2)^{-\frac{1}{2}}$, we have that $\mathcal{R} : \mathbf{R}^d \to \mathrm{S}^d_+$ is given by

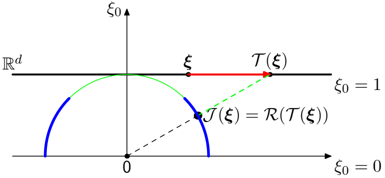
$$\mathcal{R}(\boldsymbol{\xi}) = \left(\frac{1}{\sqrt{1+|\boldsymbol{\xi}|^2}}, \frac{\boldsymbol{\xi}}{\sqrt{1+|\boldsymbol{\xi}|^2}}\right) .$$

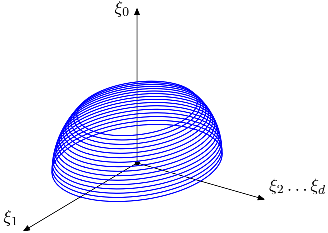
## $\mathcal{J} := \mathcal{R} \circ \mathcal{T} : \mathbf{R}^d_* \to \mathrm{S}^d_{(0,r_1)}$

$$\mathcal{R}(\mathbf{R}^d \setminus \mathrm{K}[\mathbf{0}, r_0]) = \left\{(\zeta_0, \boldsymbol{\zeta}) \in \mathrm{S}^d : 0 < \zeta_0 < r_1\right\} =: \mathrm{S}^d_{(0,r_1)}$$

and

$$\mathcal{J}(\boldsymbol{\xi}) = \left(\frac{1}{\sqrt{1 + (|\boldsymbol{\xi}| + r_0)^2}}, \frac{|\boldsymbol{\xi}| + r_0}{\sqrt{1 + (|\boldsymbol{\xi}| + r_0)^2}} \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right).$$



## Fourier multipliers

Functions from $\mathrm{C}^{\kappa}(\mathrm{S}^{d-1})$, as well as those from $\mathcal{S}(\mathbf{R}^d)$ can be identified as functions on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

**Theorem.** *Any function from $\mathrm{C}^{\lfloor \frac{d}{2} \rfloor + 1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ satisfies Mihlin's condition*

$$|\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})| \leqslant \frac{C}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}} \,, \quad \boldsymbol{\xi} \in \mathbf{R}^d_* \,,$$

*for each $|\boldsymbol{\alpha}| \leqslant \lfloor \frac{d}{2} \rfloor + 1$, when suitably restricted to $\mathbf{R}^d_*$.*
*In particular, for any $p \in (1, \infty)$, it holds ($\mathcal{A}_\psi \mathsf{u} := (\psi \hat{\mathsf{u}})^\vee$)*

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_{d,p} C_d \|\psi\|_{\mathrm{C}^{\lfloor \frac{d}{2} \rfloor + 1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))} \,,$$

*for $\psi \in \mathrm{C}^{\lfloor \frac{d}{2} \rfloor + 1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, where $C_{d,p}$ is the constant from the Mihlin theorem, while $C_d$ is a constant depending only on $d$.* ∎

# First commutation lemma

**Lema.** *Let* $\psi \in \mathrm{C}^{\lfloor \frac{d}{2} \rfloor + 1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, *and denote* $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. *Then the commutator of multiplication* $B_\varphi$ *by* $\varphi$ *and the Fourier multiplier* $\mathcal{A}_{\psi_n}$ *can be expressed as a sum*

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K \,,$$

*where for any* $p \in (1, \infty)$ *we have that* $K$ *is a compact operator on* $\mathrm{L}^p(\mathbf{R}^d)$, *while* $\tilde{C}_n \to 0$ *in the operator norm on* $\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))$. ∎

# Test functions

Let $\Omega \subseteq \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^r_{\mathbf{y}}$, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$.

$$\mathrm{C}^{l,m}(\Omega) := \Big\{ f \in \mathrm{C}(\Omega) : (\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d)(\forall \boldsymbol{\beta} \in \mathbf{N}_0^r) \\ |\boldsymbol{\alpha}| \leqslant l \,\&\, |\boldsymbol{\beta}| \leqslant m \implies \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega) \Big\} ,$$

In a standard way introduce the seminorms using a nested sequence of compacts $K_n$.

$$\mathrm{C}^{l,m}_K(\Omega) := \Big\{ f \in \mathrm{C}^{l,m}(\Omega) : \ \operatorname{supp} f \subseteq K \Big\}$$

is a Banach space for finite $l, m$, and a Fréchet space for at least one of them infinite.

$$\mathrm{C}^{l,m}_c(\Omega) := \bigcup_{n \in \mathbf{N}} \mathrm{C}^{l,m}_{K_n}(\Omega)$$

with the topology of strict inductive limit is a complete locally convex topological vector space.

# Anisotropic distributions

The space of anisotropic distributions is the dual of $\mathrm{C}_c^{l,m}(\Omega)$

$$\mathcal{D}'_{l,m}(\Omega) := (\mathrm{C}_c^{l,m}(\Omega))' \,.$$

In fact

$$T \in \mathcal{D}'_{l,m}(\Omega) \iff \begin{cases} T \in \mathcal{D}'(\Omega)\,, \text{ and} \\ (\forall K \Subset \Omega)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^\infty(\Omega)) \quad |\langle T, \varphi \rangle| \leqslant C p_K^{l,m}(\varphi)\,, \end{cases}$$

The definition can easily be extended to *differential manifolds without boundary of dimension* $d$:
a locally Euclidean (of the fixed dimension $d$, i.e. locally diffeomorphic to $\mathbf{R}^d$) second countable Hausdorff topological space on which an equivalence class of $\mathrm{C}^\infty$ smooth atlases is given.

# Kernel theorem on manifolds without boundary

**Teorem.** *Let $X$ and $Y$ be differential manifolds, of dimension $d$ and $r$, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$. Then the following statements hold:*

i) *If $K \in \mathcal{D}'_{l,m}(X \times Y)$, then for each $\varphi \in \mathrm{C}^l_c(X)$ the linear form $K_\varphi$, defined by $\psi \mapsto \langle K, \varphi \otimes \psi \rangle$, is a distribution of order not more than $m$ on $Y$. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ with its strict inductive limit topology to $\mathcal{D}'_m(Y)$ with weak $*$ topology, is linear and continuous.*

ii) *Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator, in the pair of topologies as in (i) above. Then there exists a unique distribution of anisotropic order $K \in \mathcal{D}'_{l,r(m+2)}(X \times Y)$ such that for any $\varphi \in \mathrm{C}^l_c(X)$ and $\psi \in \mathrm{C}^{r(m+2)}_c(Y)$ one has*

$$\langle K, \varphi \otimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle \ .$$

∎

# Anisotropic distributions on manifolds with boundary

The definition of *differential manifold with boundary* differs from the notion of a differential manifold without boundary only in that the former is diffeomorphic either to $\mathbf{R}^d$ or to the closed half-space $\operatorname{Cl}\mathbf{R}^d_+ = \{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d : x_d \geqslant 0\}$.

For simplicity, we shall consider only $X = \Omega \subseteq \mathbf{R}^d$ open, and $Y = \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

The space of distributions on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ of order $l \in \mathbf{N} \cup \{\infty\}$ we define by

$$\mathcal{D}'_l(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) = \left(\mathrm{C}^l(\mathrm{K}_{0,\infty}(\mathbf{R}^d))\right)',$$

where the case $l = \infty$ we shall also denote by $\mathcal{D}'(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
[This corresponds to supported distributions of R. Melrose.]

The space of anisotropic distributions on $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$ of order $(l, m) \in (\mathbf{N} \cup \{\infty\})^2$ is defined by

$$\mathcal{D}'_{l,m}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)) = \left(\mathrm{C}^{l,m}_c(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))\right)'.$$

# Kernel theorem on $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$

Note that it is sufficient to introduce distributions on $\Omega \times \mathrm{S}^d_{[0,r_1]}$ since by applying the pushforward $(\mathcal{J}^{-1})_*$ we have a one-to-one correspondence with distributions on $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

**Corollary.** *Let $\Omega \subseteq \mathbf{R}^d$ be open and $l, m \in \mathbf{N}_0 \cup \{\infty\}$. Furthermore, let $A : \mathrm{C}^l_c(\Omega) \to \mathcal{D}'_m(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ be a continuous linear operator, taking $\mathrm{C}^l_c(\Omega)$ with its inductive limit topology and $\mathcal{D}'_m(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ with weak $*$ topology. Then there exists a unique distribution of anisotropic order $K \in \mathcal{D}'_{l,d(m+2)}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that for any $\varphi \in \mathrm{C}^l_c(X)$ and $\psi \in \mathrm{C}^{d(m+2)}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ one has*

$$\langle K, \varphi \otimes \psi \rangle = \langle A\varphi, \psi \rangle \ .$$

∎

# One-scale H-measures

$\Omega \subseteq \mathbf{R}^d$ open, $p \in \langle 1, \infty \rangle$, $\frac{1}{p} + \frac{1}{p'} = 1$

**Teorem**

If $u_n \rightharpoonup 0$ in $\mathrm{L}^2_{\rm loc}(\Omega)$, $v_n \rightharpoonup 0$ in $\mathrm{L}^2_{\rm loc}(\Omega)$ and $\omega_n \to 0^+$, then there exist $(u_{n'})$, $(v_{n'})$ and $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)\mathbf{R}^d)$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d)\mathbf{R}^d)$

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 v_{n'}}(\boldsymbol{\xi})} \psi(\omega_{n'} \boldsymbol{\xi}) \, d\boldsymbol{\xi} = \langle \mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \ .$$

The measure $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ is called the one-scale H-measure with characteristic length $(\omega_{n'})$ associated to the (sub)sequences $(u_{n'})$ and $(v_{n'})$.

$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$

Determine $E$ such that

— $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is continuous

— The First commutation lemma is valid

# Existence of one-scale H-distributions

**Teorem.** *Let $\Omega \subseteq \mathbf{R}^d$ be open. If $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$ and $(v_n)$ is bounded in $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$ (for some $p \in (1,\infty)$ and $q \geqslant p'$, where $1/p + 1/p' = 1$), and if $\omega_n \to 0^+$, then there exist subsequences $(u_{n'})$, $(v_{n'})$, and a complex valued (supported) distribution $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'_{0,\kappa}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$, where $\kappa := d(\lfloor \frac{d}{2} \rfloor + 3)$, such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, one has:*

$$
\begin{aligned}
\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})(\mathbf{x}) \overline{(\varphi_2 v_{n'})(\mathbf{x})}\, d\mathbf{x} &= \lim_{n'\to\infty} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x}) \overline{\mathcal{A}_{\bar{\psi}_{n'}}(\varphi_2 v_{n'})(\mathbf{x})}\, d\mathbf{x} \\
&= \left\langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \otimes \psi \right\rangle,
\end{aligned} \tag{1}
$$

*where $\psi_n = \psi(\omega_n \cdot)$. The distribution $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ we call the* one-scale H-distribution (with the characteristic length $(\omega_{n'})$) *associated to (sub)sequences $(u_{n'})$ and $(v_{n'})$.*
*Moreover, for $p = 2$ the one-scale H-distribution above is the one-scale H-measures with characteristic length $(\omega_{n'})$ associated to (sub)sequences $(u_{n'})$ and $(v_{n'})$.*

∎

# Immediate properties of one-scale H-distributions

Changing the order of sequences; $(v_n)$ and $(u_n)$ determine the distribution

$$\left\langle \bar{\nu}_{\mathrm{K}_{0,\infty}}, \Psi \right\rangle = \overline{\left\langle \nu_{\mathrm{K}_{0,\infty}}, \bar{\Psi} \right\rangle} .$$

Supports: if $u_n, v_n$ are supported in closed sets $F_1, F_2 \subseteq \Omega$, then any one-scale distribution they determine is supported in $(F_1 \cap F_2) \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

**Lema.** *Let $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$, for some $p \in (1,\infty)$. Then the following statements are equivalent:*

(a) *$u_n \to 0$ (strongly) in $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$.*

(b) *For every bounded sequence $(v_n)$ in $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega)$ and every $\omega_n \to 0^+$, $(u_n)$ and $(v_n)$ form an $(\omega_n)$-pure pair and the corresponding one-scale H-distribution is zero.*

(c) *For $v_n = |u_n|^{p-2} u_n$ and some $\omega_n \to 0^+$, $(u_n)$ and $(v_n)$ form an $(\omega_n)$-pure pair and the corresponding one-scale H-distribution is zero.*

∎

**Theorem.** Let $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ satisfy

$$\sum_{|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \,,$$

where $(\varepsilon_n)$ is a sequence of positive real numbers, $\mathbf{A}^{\boldsymbol{\alpha}}_n \in \mathrm{C}(\Omega;\mathrm{M}_{\mathrm{q}\times\mathrm{r}}(\mathbf{C}))$, such that for any $\boldsymbol{\alpha}\in\mathbf{N}_0^d$ the sequence $\mathbf{A}^{\boldsymbol{\alpha}}_n \to \mathbf{A}^{\boldsymbol{\alpha}}$ in the space $\mathrm{C}(\Omega;\mathrm{M}_{\mathrm{q}\times\mathrm{r}}(\mathbf{C}))$ (in other words, $\mathbf{A}^{\boldsymbol{\alpha}}_n$ converges locally uniformly to $\mathbf{A}^{\boldsymbol{\alpha}}$), while $(\mathsf{f}_n)$ is a sequence of functions in $\mathrm{W}^{-m,p}_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ satisfying $(\varepsilon_n)$-local compactness condition

$$(\forall\varphi\in\mathrm{C}^\infty_c(\Omega)) \qquad \mathcal{A}_{\frac{1}{1+|\varepsilon_n\boldsymbol{\xi}|^m}}(\varphi\mathsf{f}_n) \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^p(\mathbf{R}^d;\mathbf{C}^r)\,.$$

Moreover, let $(\mathsf{v}_n)$ be a bounded sequence in $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ and let $\omega_n \to 0^+$ be a sequence of positive reals such that $c := \lim_n \frac{\omega_n}{\varepsilon_n}$ exists (in $[0,\infty]$).

# Localisation principle for one-scale H-distributions (cont.)

Then any one-scale H-distribution $\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}}^{(\omega_n)}$ associated to (sub)sequences (of) $(\mathsf{u}_n)$ and $(\mathsf{v}_n)$ with characteristic length $(\omega_n)$ satisfies:

$$\mathbf{p}_c(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}\,,$$

where, with respect to the value of $c$, we have

i) $c = 0$ :

$$\mathbf{p}_0(\mathbf{x},\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}|=m} (2\pi i)^m \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{1+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,,$$

ii) $c \in (0,\infty)$ :

$$\mathbf{p}_c(\mathbf{x},\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}|\leqslant m} \Big(\frac{2\pi i}{c}\Big)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{1+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,,$$

iii) $c = \infty$ :

$$\mathbf{p}_\infty(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{1+|\boldsymbol{\xi}|^m}\mathbf{A}^{0}(\mathbf{x})\,.$$

# Localisation principle for one-scale H-measures ($c = 1$)

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,.$$

**Theorem.** *Under previous assumptions, one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* $(\varepsilon_n)$ *corresponding to* $(\mathsf{u}_n)$ *satisfies*

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}\,,$$

*where*

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

∎

# Small-amplitude homogenisation of elastic plate

# Averaging of strongly inhomogeneious materials

First works at the end of 19th century (mathematical physics)
Poisson, Maxwell, Rayleigh

Examples of strongly inhomogeneous media:
– fibre reinforced materials (reinforced glass, concrete, fibreglass, ...)
– layered materials (plywood)
– gas concrete
– porous media (interesting e.g. for oil extraction)
– leaf

Inhomogeneous structures are usually better than homogeneous (they better optimise in order to achieve some property).

The notion of *effective property:*
the simplest structure of inhomogeneities — periodic structure (true for crystals, man-made materials, ...)
apply the averaging procedure, which will produce the effective coefficients (constants in this case!)
we replace a strongly inhomogeneious material by a homogeneous one — thus the name *homogenisation* (Ivo Babuška, 1974)

# A one-dimensional example

Consider heat conduction in a bar $[0,1]$, made of two materials $A$ and $B$, with conductivities $0 < \alpha < \beta < \infty$, in proportion $\vartheta : (1-\vartheta)$ ($\vartheta \in \langle 0,1\rangle$).
We can take

$$a_1(x) := \begin{cases} \alpha & \text{for } x \in [0,\vartheta\rangle \\ \beta & \text{for } x \in [\vartheta,1\rangle \end{cases} ,$$

extend it periodically to $\mathbf{R}$, and then define $a_n(x) := a(nx)$.

We have (for larger $n$) rapidly changing coefficients $a_n$. If we assume there are no internal heat sources, then the heat flow obeys the Fourier law, with data at $0$ and $1$:

$$\begin{cases} -\left(a_n(x)u_n'(x)\right)' = f(x) \\ u_n(0) = u_n(1) = 0 \ . \end{cases}$$

The numerical computation of $u_n$ requires very fine mesh, and the solution exhibits rapid oscillation. However, its behaviour above this $1/n$ scale is much better.

Can we find some effective or averaged coefficients such that the solution of the same equation with those coefficients will be, in a sense, the limit of solutions $u_n$?

# A one-dimensional example (cont.)

Assume only that $\alpha \leqslant a_n \leqslant \beta$ (i.e. $a_n \in \mathrm{L}^\infty(\langle 0,1\rangle)$), and $f \in \mathrm{L}^2(\langle 0,1\rangle)$ (with no periodicity).
The equation can be written in the variational form:

$$(\forall\, v \in \mathrm{H}_0^1(\langle 0,1\rangle)) \qquad \int_0^1 a_n u_n' v' = \int_0^1 f v\ .$$

LHS: equivalent to the scalar product on $\mathrm{H}_0^1(\langle 0,1\rangle)$,
RHS: a bounded linear functional,
so (for any fixed $n$) by the Riesz representation theorem there is a unique $u_n \in \mathrm{H}_0^1(\langle 0,1\rangle)$ representing the right hand side. Also, $u_n'$ is bounded in $\mathrm{L}^2(\langle 0,1\rangle)$, and therefore has a weak accumulation point (say, $u_\infty$).
On the other hand, $p_n := a_n u_n' \in \mathrm{L}^2(\langle 0,1\rangle)$, while $p_n' = (a_n u_n')' = -f \in \mathrm{L}^2(\langle 0,1\rangle)$, and $p_n$ is bounded in $\mathrm{H}_0^1(\langle 0,1\rangle)$.

We can pass to a subsequence once more $p_n \rightharpoonup p$ in $\mathrm{H}^1(\langle 0,1\rangle)$, which by the Rellich compact embedding gives $p_n \longrightarrow p$ in $\mathrm{L}^2(\langle 0,1\rangle)$.

N.B. we do not explicitly write the subsequences.

# A one-dimensional example (cont.)

Writting

$$u_n' = \frac{1}{a_n} p_n \; ,$$

we can pass to the limit in the product (as $\frac{1}{\alpha} \geqslant \frac{1}{a_n} \geqslant \frac{1}{\beta}$ and there is a further subsequence such that $\frac{1}{a_n} \overset{*}{\rightharpoonup} \frac{1}{a_\infty}$), or after taking the derivative:

$$-\left( \frac{u_\infty}{\frac{1}{a_\infty}} \right)' = -p' = f \; .$$

Thus the effective coefficients are $a_\infty$, as the limit of solutions satisfies the same equation as $u_n$, but with $a_\infty$ instead of $a_n$.

In the periodic case, the limit $1/a_\infty = \fint (1/a_n)$, and it does not depend on the choice of subsequences. Any limit $u_\infty$ has to satisfy the equation, which has the unique solution—thus the whole sequence $u_n$ converges.

In the general case we only know the result for an accummulation point.

# Assumptions for Kirchhoff-Love plates

- the plate is thin, but not very thin
  (rougly, the thickness is 1–20% of the leading dimension)
- the plate thickness might vary only slowly
  (so that the 3D stress effects are ignored)
- the plate is symmetric about mid-surface
- applied transverse loads are distributed over plate surface areas (no concentrated loads)
- there is no significant extension of the mid-surface

There are no transverse shear deformations.
The variation of vertical displacement in the direction of thickness can be neglected.
The planes perpendicular to the mid-surface will remain plane and perpendicular to the deformed mid-surface.

# Kirchhoff-Love plate equation

The above leads to a linear elliptic problem, with homogeneous Dirichlet boundary conditions:

$$\left\{ \begin{array}{l} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}_0^2(\Omega)\,, \end{array} \right.$$

where:

- $\Omega \subseteq \mathbf{R}^d$ is a bounded domain ($d = 2$ ... for the plate)
- $f \in \mathrm{H}^{-2}(\Omega)$ is the external load
- $u \in \mathrm{H}_0^2(\Omega)$ is the vertical displacement of the plate
- **M** describes (non-homogeneous) properties of the material plate is made of. At a point it is a linear operator from symmetric matrices to symmetric matrices, and we take **M** from the set:

$$\mathfrak{M}_2(\alpha, \beta; \Omega) := \Big\{ \mathbf{N} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym}) \quad \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geqslant \alpha \mathbf{S} : \mathbf{S} \ (\mathrm{ae}\ \mathbf{x}) \ \& \ \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geqslant \frac{1}{\beta}\mathbf{S} : \mathbf{S} \ (\mathrm{ae}\ \mathbf{x}) \Big\}$$

This ensures the boundedness and coercivity, so we have the existence and uniqueness of solutions via the Lax-Milgram lemma in a standard way.

# Homogenisation: H-convergence

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in \mathrm{H}^{-2}(\Omega)$ the sequence of solutions $u_n$ of problems

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

coverges weakly to a limit $u$ in $\mathrm{H}^2_0(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $\mathrm{L}^2(\Omega; \mathrm{Sym})$.

This convergence comes indeed from a weak topology on $X = \bigcup \mathfrak{M}_2(1/n, n; \Omega)$, where we consider the maps $\mathbf{M} \mapsto u$, with weak topology on $\mathrm{H}^2_0(\Omega)$, for any fixed $f \in \mathrm{H}^{-2}(\Omega)$, as well as $\mathbf{M} \mapsto \mathbf{M}\nabla\nabla u$, with weak topology on $\mathrm{L}^2(\Omega; \mathrm{Sym})$.

for second order elliptic equations:
Tartar & Murat, 1977

general form for higher-order elliptic equations:
Žikov, Kozlov, Oleinik, Ngoan, 1979

for plates: N.A. & N. Balenović, 1999–2000
revisited: K. Burazin, J. Jankov (& M. Vrdoljak), 2018–21

# Properties: Compactness

**Theorem.** *Let* $(\mathbf{M}^n)$ *be a sequence in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*. Then there is a subsequence* $(\mathbf{M}^{n_k})$ *and a tensor function* $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ *such that* $(\mathbf{M}^{n_k})$ *H-converges to* $\mathbf{M}$*.* ∎

**Theorem.** *(compactness by compensation) Let the following convergences be valid:*

$$
\begin{aligned}
w^n &\rightharpoonup w^\infty \quad \text{in} \quad \mathrm{H}^2_{\mathrm{loc}}(\Omega)\,, \\
\mathbf{D}^n &\rightharpoonup \mathbf{D}^\infty \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})\,,
\end{aligned}
$$

*with an additional assumption that the sequence* $(\operatorname{div}\operatorname{div}\mathbf{D}^n)$ *is contained in a precompact (for the strong topology) set of the space* $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$*. Then we have*

$$
\nabla\nabla w^n : \mathbf{D}^n \xrightharpoonup{*} \nabla\nabla w^\infty : \mathbf{D}^\infty
$$

*in the space of Radon measures.* ∎

to Dependence on parameters

# Locality and irrelevance of boundary conditions

**Theorem.** *(locality of H-convergence) Let* $(\mathbf{M}^n)$ *and* $(\mathbf{O}^n)$ *be two sequences of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, which H-converge to* $\mathbf{M}$ *and* $\mathbf{O}$*, respectively. Let* $\omega$ *be an open subset compactly embedded in* $\Omega$*. If* $\mathbf{M}^n(\mathbf{x}) = \mathbf{O}^n(\mathbf{x})$ *in* $\omega$*, then* $\mathbf{M}(\mathbf{x}) = \mathbf{O}(\mathbf{x})$ *in* $\omega$*.* ∎

**Theorem.** *(irrelevance of boundary conditions) Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converges to* $\mathbf{M}$*. For any sequence* $(z_n)$ *such that*

$$\begin{aligned} z_n &\longrightarrow z \quad &&\text{in } \mathrm{H}^2_{\mathrm{loc}}(\Omega) \\ \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla z_n) = f_n &\longrightarrow f \quad &&\text{in } \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega), \end{aligned}$$

*the weak convergence* $\mathbf{M}^n \nabla\nabla z_n \rightharpoonup \mathbf{M}\nabla\nabla z$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$ *holds.* ∎

# Convergence of energies

**Theorem.** *Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converges to* $\mathbf{M}$*. For any* $f \in \mathrm{H}^{-2}(\Omega)$*, the sequence* $(u_n)$ *of solutions of*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \end{cases}$$

*satisfies* $\mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \rightharpoonup \mathbf{M}\nabla\nabla u : \nabla\nabla u$ *weakly-*$*$ *in the space of Radon measures and* $\int_\Omega \mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \, d\mathbf{x} \to \int_\Omega \mathbf{M}\nabla\nabla u : \nabla\nabla u \, d\mathbf{x}$*, where* $u$ *is the solution of the homogenised equation*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}_0^2(\Omega)\,. \end{cases}$$

∎

# Ordering property for symmetric tensors ...

**Theorem.** *Let* $(\mathbf{M}^n)$ *and* $(\mathbf{O}^n)$ *be two sequences of symmetric tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converge to the homogenised tensors* $\mathbf{M}$ *and* $\mathbf{O}$*, respectively. Furthermore, assume that, for any* $n$*,*

$$(\forall \boldsymbol{\xi} \in \mathrm{Sym}) \qquad \mathbf{M}^n \boldsymbol{\xi} : \boldsymbol{\xi} \leqslant \mathbf{O}^n \boldsymbol{\xi} : \boldsymbol{\xi} \; .$$

*Then the homogenised limits are also ordered:*

$$(\forall \boldsymbol{\xi} \in \mathrm{Sym}) \qquad \mathbf{M} \boldsymbol{\xi} : \boldsymbol{\xi} \leqslant \mathbf{O} \boldsymbol{\xi} : \boldsymbol{\xi} \; .$$

∎

**Theorem.** *Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that either converges strongly to a limit tensor* $\mathbf{M}$ *in* $\mathrm{L}^1(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, or converges to* $\mathbf{M}$ *almost everywhere in* $\Omega$*. Then,* $\mathbf{M}^n$ *also H-converges to* $\mathbf{M}$*.* ∎

**Theorem.** *Let* $F = \{f_n : n \in \mathbf{N}\}$ *be a countable dense family in* $\mathrm{H}^{-2}(\Omega)$, $\mathbf{M}$ *and* $\mathbf{O}$ *tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$, *and* $(u_n)$, $(v_n)$ *sequences of solutions to*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u_n) = f_n \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

*and*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{O}\nabla\nabla v_n) = f_n \\ v_n \in \mathrm{H}^2_0(\Omega) \end{cases} .$$

*Then,*

$$d(\mathbf{M}, \mathbf{O}) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|u_n - v_n\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{M}\nabla\nabla u_n - \mathbf{O}\nabla\nabla v_n\|_{\mathrm{H}^{-1}(\Omega;\mathrm{Sym})}}{\|f_n\|_{\mathrm{H}^{-2}(\Omega)}}$$

*is a metric on* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *and H-convergence is equivalent to the convergence with respect to* $d$.

∎

# Correctors

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $\mathbf{M}$, and $(w_n^{ij})_{1\leqslant i,j\leqslant d}$ a family of test functions satisfying

$$\begin{aligned} w_n^{ij} &\rightharpoonup \frac{1}{2}x_i x_j \quad &\text{in} \quad &\mathrm{H}^2(\Omega) \\ \mathbf{M}^n \nabla\nabla w_n^{ij} &\rightharpoonup \cdots &\text{in} \quad &\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathsf{Sym}) \\ \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla w_n^{ij}) &\to \cdots &\text{in} \quad &\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega). \end{aligned}$$

The sequence of tensors $\mathbf{W}^n$ defined by $W^n_{ijkm} = [\nabla\nabla w_n^{km}]_{ij}$ is called the sequence of correctors.

It is unique, indeed:

**Theorem.** *Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor $\mathbf{M}$. A sequence of correctors $(\mathbf{W}^n)$ is unique in the sense that, if there exist two sequences of correctors $(\mathbf{W}^n)$ and $(\tilde{\mathbf{W}}^n)$, their difference $(\mathbf{W}^n - \tilde{\mathbf{W}}^n)$ converges strongly to zero in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.* ∎

# Corrector result

**Theorem.** *Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *which H-converges to* $\mathbf{M}$*. For* $f \in \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$*, let* $(u_n)$ *be the solution of*

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f & \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

*and let* $u$ *be the weak limit of* $(u_n)$ *in* $\mathrm{H}^2_0(\Omega)$*, i.e. the solution of the homogenised equation*

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M} \nabla\nabla u) = f & \text{in} \quad \Omega \\ u \in \mathrm{H}^2_0(\Omega)\,. \end{cases}$$

*Then* $\mathbf{R}_n := \nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u \to \mathbf{0}$ *strongly in* $\mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathrm{Sym})$*.*

∎

# Smoothness with respect to a parameter $p \in P$

**Theorem.** *Let* $\mathbf{M}^n : \Omega \times P \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ *be a sequence of tensors, such that* $\mathbf{M}^n(\cdot, p) \in \mathfrak{M}_2(\alpha, \beta; \Omega)$*, for* $p \in P$*. Assume that* $p \mapsto \mathbf{M}^n(\cdot, p)$ *is of class* $\mathrm{C}^k$ *from* $P$ *to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, with derivatives (up to order* $k$*) being equicontinuous on every compact set* $K \subseteq P$*:*

$$(\forall K \in \mathcal{K}(P))(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})(\forall i \leq k)$$

$$|p - q| < \delta \Rightarrow \|(\mathbf{M}^n)^{(i)}(\cdot, p) - (\mathbf{M}^n)^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))} < \varepsilon.$$

*Then there is a subsequence* $(\mathbf{M}^{n_k})$ *such that for every* $p \in P$

$$\mathbf{M}^{n_k}(\cdot, p) \xrightarrow{H} \mathbf{M}(\cdot, p) \quad \text{in} \quad \mathfrak{M}_2(\alpha, \beta; \Omega)$$

*and* $p \mapsto \mathbf{M}(\cdot, p)$ *is a* $\mathrm{C}^k$ *mapping from P to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*.* ∎

In particular, the above is valid for $k = \infty$ and $k = \omega$ (the analytic functions).

# Small-amplitude homogenisation

Consider a sequence of problems

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n(\cdot\,;\gamma)\nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}_0^2(\Omega)\,, \end{cases}$$

where we assume that the coefficients are a small perturbation of a given continuous tensor function $\mathbf{A}_0$, for small $\gamma$

$$\mathbf{M}^n(\cdot\,;\gamma) := \mathbf{A}_0 + \gamma\mathbf{B}^n + \gamma^2\mathbf{C}^n + o(\gamma^2)\,,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{O}$ in $\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$. For small $\gamma$ we, in fact, we can assume that the function is analytic in $\gamma$.
Then (after passing to a subsequence if needed)

$$\mathbf{M}^n(\cdot\,;\gamma) \xrightarrow{H} \mathbf{M}(\cdot\,;\gamma) = \mathbf{A}_0 + \gamma\mathbf{B}_0 + \gamma^2\mathbf{C}_0 + o(\gamma^2)\,;$$

the limit being measurable in $\mathbf{x}$, and analytic in $\gamma$.

The goal is to obtain the explicit formula for the leading terms $\mathbf{B}_0$ and $\mathbf{C}_0$ in the expansion of the homogenisation limit.

## Small-amplitude homogenisation procedure

Take $u \in \mathrm{H}^2_0(\Omega)$ and define $f_\gamma := \operatorname{div}\operatorname{div}(\mathbf{M}(\cdot\,;\gamma)\nabla\nabla u)$, depending analytically on $\gamma$. Using $f_\gamma$, let $u^n_\gamma$ be the solution (for each $n$ and $\gamma$) of

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n(\cdot\,;\gamma)\nabla\nabla u^n_\gamma) = f_\gamma \text{ in } \Omega \\ \qquad\qquad\qquad\qquad u^n_\gamma \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

which analytically depends on $\gamma$, hence one can write

$$u^n_\gamma := u^n_0 + \gamma u^n_1 + \gamma^2 u^n_2 + o(\gamma^2)\,.$$

As $\mathbf{M}^n(\cdot\,;\gamma) \xrightarrow{H} \mathbf{M}(\cdot\,;\gamma)$, we have weak convergences in $\mathrm{L}^2(\Omega;\mathrm{Sym})$:

$$(*) \qquad \begin{aligned} \mathbf{E}^n_\gamma &:= \nabla\nabla u^n_\gamma \rightharpoonup \nabla\nabla u \\ \mathbf{D}^n_\gamma &:= \mathbf{M}^n(\cdot\,;\gamma)\mathbf{E}^n_\gamma \rightharpoonup \mathbf{M}(\cdot\,;\gamma)\nabla\nabla u\,. \end{aligned}$$

$\mathbf{E}^n_\gamma$ and $\mathbf{D}^n_\gamma$ are analytic in $\gamma$ and consequently each can be expanded in the Taylor series:

$$\begin{aligned} \mathbf{E}^n_\gamma &= \mathbf{E}^n_0 + \gamma\mathbf{E}^n_1 + \gamma^2\mathbf{E}^n_2 + o(\gamma^2) \\ \mathbf{D}^n_\gamma &= \mathbf{D}^n_0 + \gamma\mathbf{D}^n_1 + \gamma^2\mathbf{D}^n_2 + o(\gamma^2)\,. \end{aligned}$$

For $\gamma = 0$, the uniqueness of solution implies $u^n_0 = u$. Moreover, this gives us

$$\mathbf{E}^n_0 = \nabla\nabla u \quad \text{and} \quad \mathbf{D}^n_0 = \mathbf{A}_0\nabla\nabla u\,.$$

## Small-amplitude homogenisation procedure (cont.)

After inserting the above expansions into $(*)$ and equating the terms with equal powers of $\gamma$, one can conclude that $\mathbf{E}_1^n, \mathbf{E}_2^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, and

$$\mathbf{D}_1^n = \mathbf{A}_0 \mathbf{E}_1^n + \mathbf{B}^n \nabla\nabla u .$$

Since $\mathbf{E}_1^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, while $\mathbf{B}^n \overset{*}{\rightharpoonup} \mathbf{O}$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$:

$$\mathbf{D}_1^n \rightharpoonup \mathbf{0} \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}) \; .$$

Similarly, by using

$$\mathbf{D}_\gamma^n = \mathbf{M}^n(\cdot\,; \gamma)\mathbf{E}_\gamma^n \rightharpoonup \mathbf{M}(\cdot\,; \gamma)\nabla\nabla u = (\mathbf{A}_0 + \gamma \mathbf{B}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2))\nabla\nabla u \; ,$$

after equating the terms standing by $\gamma^1$, we obtain that

$$\mathbf{D}_1^n \rightharpoonup \mathbf{B}_0 \nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}) \; .$$

The limits are equal, so $\mathbf{B}_0 \nabla\nabla u = \mathbf{0}$.
Since $u \in \mathrm{H}_0^2(\Omega)$ can be arbitrary, we conlude that $\mathbf{B}_0 = \mathbf{O}$.

# The corrector can be expressed by H-measure

Analogously, equating the terms standing by $\gamma^2$ gives:

$$\mathbf{D}_2^n = \mathbf{A}_0\mathbf{E}_2^n + \mathbf{B}^n\mathbf{E}_1^n + \mathbf{C}^n\nabla\nabla u \rightharpoonup \mathbf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym})\,.$$

On the other hand, as $\mathbf{E}_2^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$ and $\mathbf{C}^n \xrightharpoonup{*} \mathbf{O}$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, we have

$$\mathbf{D}_2^n \rightharpoonup \lim_n \mathbf{B}^n\mathbf{E}_1^n = \mathbf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym})\,.$$

Obviously, identifying the corrector of order $2$ in $\gamma$ requires the computation of the weak limit of $(\mathbf{B}^n\mathbf{E}_1^n)$, the product of two weakly convergent sequences.

And such limits can be expressed by using H-measures.

For a physical plate, we assume that $\Omega$ is a bounded region, so $\mathrm{L}^\infty$ weak $*$ topology is stronger than $\mathrm{L}^2$ weak, and we are indeed in the situation where both sequences converge weakly in $\mathrm{L}^2$ to zero.

Let $\tilde{\boldsymbol{\mu}}$ be the H-measure corresponding to the sequence $[\mathbf{B}^n \;\; \mathbf{E}_1^n]^T$:

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\sigma} \\ \boldsymbol{\rho} & \boldsymbol{\nu} \end{bmatrix},$$

which is defined as a $(d^4 + d^2) \times (d^4 + d^2)$ Hermitian nonnegative matrix Radon measure.

More precisely, block $\boldsymbol{\mu}$ is the H-measure associated to (a subsequence of) $(\mathbf{B}^n)$, while $\boldsymbol{\sigma} = \boldsymbol{\rho}^*$ is the H-measure corresponding to the product $\mathbf{B}^n \mathbf{E}_1^n$.. For simplicity, by $\mathbf{v}^n := [\mathbf{B}^n \;\; \mathbf{E}_1^n]^T$ we denote the $(d^4 + d^2) \times 1$ column matrix, but we still use the original four indices for $\mathbf{B}^n$ and two for $\mathbf{E}_1^n$, avoiding explicit writing of the appropriate bijection from $\{1, \ldots, d\}^4 \bigcup \{1, \ldots, d\}^2$ to $\{1, \ldots, d^4 + d^2\}$, as such notation will be needed again for interpretation of the limit. All indices have range in $\{1, \ldots, d\}$.

After computing this limit, we write it as $\mathbf{C}_0 \nabla\nabla u$, and thus identify $\mathbf{C}_0$. Our goal is to use the localisation principle for H-measures to express that limit, i.e. the measure $\boldsymbol{\sigma}$, from the H-measure $\boldsymbol{\mu}$. To this end we need to choose certain expressions relating $\mathbf{E}_1^n$ and $\mathbf{B}^n$.

# Computing the H-measure

Firstly, we insert the expansions for $\mathsf{M}^n(\cdot\,;\gamma), \mathsf{M}(\cdot\,;\gamma)$ and $u^n_\gamma$ into BVP

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathsf{M}^n(\cdot\,;\gamma)\nabla\nabla u^n_\gamma) = f_\gamma = \mathsf{div}\,\mathsf{div}\,(\mathsf{M}(\cdot\,;\gamma)\nabla\nabla u) \text{ in } \Omega \\ u^n_\gamma \in \mathrm{H}^2_0(\Omega)\ , \end{cases}$$

and after comparing expressions corresponding to the first power of $\gamma$, we obtain

$$\mathsf{div}\,\mathsf{div}\,(\mathsf{A}_0\mathbf{E}^n_1 + \mathsf{B}^n\nabla\nabla u) = \mathsf{div}\,\mathsf{div}\,(\mathsf{B}_0\nabla\nabla u).$$

Due to $\mathsf{B}_0 = \mathsf{O}$ we have

$$(+) \qquad \mathsf{div}\,\mathsf{div}\,(\mathsf{A}_0\mathbf{E}^n_1 + \mathsf{B}^n\nabla\nabla u) = 0\ ,$$

as well as Schwarz's symmetries:

$$(++) \qquad \partial_r\partial_s(\mathbf{E}^n_1)_{kl} - \partial_k\partial_l(\mathbf{E}^n_1)_{rs} = 0\ .$$

Additionally assume that $\nabla\nabla u$ is continuous, and apply the Localisation principle to relations $(+)$ and $(++)$.

## Localisation on $(+)$

For chosen $i, j \in \{1, \dots, d\}$, after defining matrix $\mathbf{A}^{ij} \in M_{1\times(d^4+d^2)}(\mathbf{R})$ by

$$\mathbf{A}^{ij} := \left[\mathbf{A}^{\mathsf{B}^{ij}}, \mathbf{A}^{\mathbf{E}_1^{ij}}\right] ,$$

where each $\mathbf{A}^{\mathsf{B}^{ij}}$ is a $1 \times d^4$ matrix with entries

$$\left[\mathbf{A}^{\mathsf{B}^{ij}}\right]_{vwkl} := \begin{cases} \partial_k \partial_l u, & \text{if } (v, w) = (i, j) \\ 0, & \text{otherwise ,} \end{cases}$$

and each $\mathbf{A}^{\mathbf{E}_1^{ij}}$ is a $1 \times d^2$ matrix with entries given by

$$\left[\mathbf{A}^{\mathbf{E}_1^{ij}}\right]_{kl} := [\mathsf{A}_0]_{ijkl}.$$

It is easy to check that the assumptions of Theorem are fulfilled for $h = 2$. Therefore

$$\left(\sum_{i,j=1}^{d} (2\pi i)^2 \, \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2} \, \mathbf{A}^{ij}(\mathbf{x})\right) \tilde{\boldsymbol{\mu}}^T = \mathbf{0}$$

and from here we can conclude that

$$\sum_{i,j,k,l=1}^{d} \xi_i \xi_j \boldsymbol{\mu}_{ijkl}^{pqrs} \partial_k \partial_l u + \sum_{i,j,k,l=1}^{d} \xi_i \xi_j \bar{\boldsymbol{\rho}}_{pqrs}^{kl} [\mathsf{A}_0]_{ijkl} = 0 \; .$$

# Localisation on $(++)$

For fixed $k, l, r, s \in \{1, \ldots, d\}$, $(k,l) \neq (r,s)$, define $\mathbf{A}^{ij} \in M_{1\times(d^4+d^2)}(\mathbf{R})$ by

$$\mathbf{A}^{ij} := \left[\mathbf{0}, \mathbf{A}^{\mathbf{E}_1^{ij}}\right] ,$$

where $\mathbf{A}^{\mathbf{E}_1^{ij}}$ is a $1 \times d^2$ matrix whose entries are given by

$$\left[\mathbf{A}^{\mathbf{E}_1^{ij}}\right]_{vw} = \begin{cases} 1, & \text{if } (i,j,v,w) = (r,s,k,l) \\ -1, & \text{if } (i,j,v,w) = (k,l,r,s) \\ 0, & \text{otherwise} \end{cases} .$$

Again, the Localisation principle with $h = 2$ gives us

$$\left( \sum_{i,j=1}^{d} (2\pi i)^2 \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2} \mathbf{A}^{ij}(\mathbf{x}) \right) \tilde{\boldsymbol{\mu}}^T = \mathbf{0} ,$$

which yields

$$\xi_r \xi_s \bar{\boldsymbol{\rho}}^{kl}_{pqrs} = \xi_k \xi_l \bar{\boldsymbol{\rho}}^{rs}_{pqrs} .$$

The above is trivially satisfied for $(k,l) = (r,s)$.

## Combining two relations

By multiplying the relation obtained from $(+)$ by $\xi_r\xi_s$ and summing over $r, s$

$$\sum_{i,j,k,l,r,s=1}^{d} \xi_i\xi_j\xi_r\xi_s \boldsymbol{\mu}_{ijkl}^{pqrs}\partial_k\partial_l u + \sum_{i,j,k,l,r,s=1}^{d} \xi_i\xi_j\xi_r\xi_s \bar{\boldsymbol{\rho}}_{pqrs}^{kl}[\mathbf{A}_0]_{ijkl} = 0.$$

By using the other relation, we can rewrite it in an equivalent form

$$\sum_{i,j,k,l,r,s=1}^{d} \xi_i\xi_j\xi_r\xi_s \boldsymbol{\mu}_{ijkl}^{pqrs}\partial_k\partial_l u + \sum_{i,j,k,l,r,s=1}^{d} \xi_i\xi_j\xi_k\xi_l \bar{\boldsymbol{\rho}}_{pqrs}^{rs}[\mathbf{A}_0]_{ijkl} = 0 \ ,$$

which, after division by

$$\sum_{i,j,k,l=1}^{d} [\mathbf{A}_0]_{ijkl}\xi_i\xi_j\xi_k\xi_l = \mathbf{A}_0(\boldsymbol{\xi}\otimes\boldsymbol{\xi}) : (\boldsymbol{\xi}\otimes\boldsymbol{\xi}) > 0$$

yields

$$\sum_{r,s=1}^{d} \bar{\boldsymbol{\rho}}_{pqrs}^{rs} = - \sum_{i,j,k,l,r,s=1}^{d} \frac{\xi_i\xi_j\xi_r\xi_s}{\mathbf{A}_0(\boldsymbol{\xi}\otimes\boldsymbol{\xi}) : (\boldsymbol{\xi}\otimes\boldsymbol{\xi})} \boldsymbol{\mu}_{ijkl}^{pqrs}\partial_k\partial_l u \ .$$

Recall that $\lim_n \mathbf{B}^n\mathbf{E}_1^n = \mathbf{C}_0\nabla\nabla u$ weakly in $L^2(\Omega)$, and thus also weak $*$ in the space of Radon measures.

## Hermitian character of H-measures

As $\boldsymbol{\sigma} = \boldsymbol{\rho}^*$ is the H-measure corresponding to the product $\mathbf{B}^n \mathbf{E}_1^n$, for an arbitrary $\varphi \in \mathrm{C}_c(\Omega)$, we have in components

$$\begin{aligned}
\int_\Omega \varphi(\mathbf{x}) \sum_{r,s=1}^{d} [\mathbf{C}_0(\mathbf{x})]_{pqrs} \partial_r \partial_s u(\mathbf{x})\, d\mathbf{x} &= \left\langle \sum_{r,s=1}^{d} \overline{[\mathbf{C}_0]_{pqrs} \partial_r \partial_s u}, \varphi \right\rangle \\
&= \left\langle \sum_{r,s=1}^{d} \bar{\boldsymbol{\sigma}}_{rs}^{pqrs}, \varphi \boxtimes 1 \right\rangle \\
&= \int_{\Omega \times S^{d-1}} \varphi(\mathbf{x}) d\left( \sum_{r,s=1}^{d} \boldsymbol{\sigma}_{rs}^{pqrs} \right)(\mathbf{x}, \boldsymbol{\xi}) \\
&= \int_{\Omega \times S^{d-1}} \varphi(\mathbf{x}) d\left( \sum_{r,s=1}^{d} \left(\bar{\boldsymbol{\rho}}^T\right)_{rs}^{pqrs} \right)(\mathbf{x}, \boldsymbol{\xi}) \\
&= \int_{\Omega \times S^{d-1}} \varphi(\mathbf{x}) d\left( \sum_{r,s=1}^{d} \bar{\boldsymbol{\rho}}_{pqrs}^{rs} \right)(\mathbf{x}, \boldsymbol{\xi}) \,.
\end{aligned}$$

## The result

Finally, inserting the expression for $\sum_{r,s=1}^{d} \bar{\boldsymbol{\rho}}_{pqrs}^{rs}$ from before

$$\sum_{r,s=1}^{d} \int_{\Omega} [\mathbf{C}_0]_{pqrs} \varphi \partial_r \partial_s u \, d\mathbf{x} = - \int_{\Omega \times S^{d-1}} \sum_{i,j,k,l,r,s=1}^{d} \frac{\xi_i \xi_j \xi_k \xi_l}{\mathbf{A}_0(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi})} \varphi \partial_r \partial_s u \, d\boldsymbol{\mu}_{ijrs}^{pqkl}(\mathbf{x}, \boldsymbol{\xi}).$$

By varying $u \in \mathrm{C}^2(\Omega)$ (e.g. choosing $\nabla\nabla u$ constant on the support of $\varphi$), one easily deduces the result which is stated in the following theorem.

**Theorem.** *The tensor* $\mathbf{M}(\cdot\,;\gamma)$ *admits the expansion*

$$\mathbf{M}(\cdot\,;\gamma) := \mathbf{A}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2) \,,$$

*where the second-order H-correction* $\mathbf{C}_0 \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ *satisfies*

$$\int_{\Omega} [\mathbf{C}_0]_{pqrs} \varphi \, d\mathbf{x} = - \sum_{i,j,k,l=1}^{d} \left\langle \boldsymbol{\mu}_{pqkl}^{ijrs}, \frac{\varphi \xi_i \xi_j \xi_k \xi_l}{\mathbf{A}_0(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi})} \right\rangle .$$

∎

# Sequences not converging to zero

If we take $\mathbf{B}^n \xrightharpoonup{*} \mathbf{B}^0$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ and $\mathbf{C}^n \xrightharpoonup{*} \mathbf{C}^0$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, we get

$$\mathbf{M}(\cdot\,;\gamma) := \mathbf{A}_0 + \gamma\mathbf{B}^0 + \gamma^2(\mathbf{C}^0 + \mathbf{C}_0) + o(\gamma^2),$$

where $\mathbf{C}_0$ is given in the Theorem.

## Periodic case

- Let $Y$ be the $d$-dimensional torus, $\mathbf{M} \in \mathrm{L}^\infty(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) \cap \mathfrak{M}_2(\alpha, \beta; Y)$
- Assume $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), \mathbf{x} \in \Omega \subseteq \mathbf{R}^d$ (projection of $\mathbf{R}^d$ to $Y$ assumed)
- $\mathrm{H}^2(Y)$ consists of 1-periodic functions, with the norm taken over the fundamental period
- $\mathrm{H}^2(Y)/\mathbf{R}$ is equipped with the norm $\|\nabla\nabla \cdot \|_{\mathrm{L}^2(Y)}$
- $\mathbf{E}_{ij}, 1 \leqslant i, j \leqslant d$ are $\mathrm{M}_{d\times d}$ matrices defined as

$$[\mathbf{E}_{ij}]_{kl} = \begin{cases} 1, & \text{if } i = j = k = l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem.** $(\mathbf{M}^n)$ *H-converges to a constant tensor* $\mathbf{M}^\infty \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ *defined as*

$$m^\infty_{klij} = \int_Y \mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x})) : (\mathbf{E}_{kl} + \nabla\nabla w_{kl}(\mathbf{x}))\, d\mathbf{x},$$

*where* $(w_{ij})$ *is the family of unique solutions in* $\mathrm{H}^2(Y)/\mathbf{R}$ *of*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x}))) = 0 \text{ in Y} \\ \mathbf{x} \to w_{ij}(\mathbf{x}) \quad \textit{is } Y\textit{-periodic.} \end{cases}$$

∎

## Small-amplitude assumptions

**Theorem.** *Let* $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym};\mathrm{Sym})$ *be a constant coercive tensor,* $\mathbf{B}^n(\mathbf{x}) := \mathbf{B}(n\mathbf{x})$, $\mathbf{x} \in \Omega$, *where* $\Omega \subseteq \mathbf{R}^d$ *is a bounded, open set, and* $\mathbf{B}$ *is a Y-periodic,* $L^\infty$ *tensor function, satisfying* $\int_Y \mathbf{B}(\mathbf{x})\,d\mathbf{x} = 0$. *Then*

$$\mathbf{M}^n_\gamma(\mathbf{x}) := \mathbf{A}_0 + \gamma\mathbf{B}^n(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

*H-converges (for any small* $\gamma$*) to a tensor* $\mathbf{M}_\gamma := \mathbf{A}_0 + \gamma^2\mathbf{C}_0 + o(\gamma^2)$ *, where*

$$\begin{aligned}\mathbf{C}_0\mathbf{E}_{mn} : \mathbf{E}_{rs} = (2\pi i)^2 \sum_{\mathbf{k}\in J} a^{mn}_{-\mathbf{k}}\mathbf{B}_\mathbf{k}(\mathbf{k}\otimes\mathbf{k}) : \mathbf{E}_{rs} + \\ + (2\pi i)^4 \sum_{\mathbf{k}\in J} a^{mn}_{\mathbf{k}} a^{rs}_{-\mathbf{k}}\mathbf{A}_0(\mathbf{k}\otimes\mathbf{k}) : \mathbf{k}\otimes\mathbf{k} + \\ + (2\pi i)^2 \sum_{\mathbf{k}\in J} a^{rs}_{-\mathbf{k}}\mathbf{B}_\mathbf{k}\mathbf{E}_{mn} : \mathbf{k}\otimes\mathbf{k}\,,\end{aligned}$$

*with* $m,n,r,s \in \{1,2,\cdots,d\}$, $J := \mathbf{Z}^d \setminus \{0\}$, *and*

$$a^{mn}_{\mathbf{k}} = -\frac{\mathbf{B}_\mathbf{k}\mathbf{E}_{mn}\mathbf{k}\cdot\mathbf{k}}{(2\pi i)^2\mathbf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}, \quad \mathbf{k} \in J,$$

*and* $\mathbf{B}_\mathbf{k}$ *are the Fourier coefficients of function* $\mathbf{B}$*.* ∎

# Result by applying H-measures

The corresponding H-measure of the sequence $(\mathbf{B}^n)$ can be explicitly computed

$$\boldsymbol{\mu}_{ijrs}^{pqkl} = \lambda(\mathbf{x}) \sum_{\mathbf{k}\in\mathbf{Z}^d} [\mathbf{B}_\mathbf{k}]_{pqkl} [\overline{\mathbf{B}}_\mathbf{k}]_{ijrs}\, \delta_{\frac{\mathbf{k}}{|\mathbf{k}|}}(\boldsymbol{\xi}),$$

where $\lambda$ denotes the Lebesgue measure on $\mathbf{R}^d$ and $\mathbf{B}_\mathbf{k}$, $\mathbf{k} \in \mathbf{Z}^d$, are Fourier coefficients of function $\mathbf{B}$. After inserting this expression in the formula in the Theorem, we can easily calculate $\mathbf{C}_0$ explicitly:

$$\mathbf{C}_0 = -\sum_{\mathbf{k}\in\mathbf{Z}^d} \frac{\mathbf{B}_\mathbf{k}(\mathbf{k}\otimes\mathbf{k}) \otimes \mathbf{B}_\mathbf{k}^T(\mathbf{k}\otimes\mathbf{k})}{\mathbf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})},$$

where the tensor product of two matrices $\mathbf{A}, \mathbf{B} \in M_d(\mathbf{C})$ is the fourth-order tensor with entries

$$[\mathbf{A}\otimes\mathbf{B}]_{ijkl} = a_{ij}\bar{b}_{kl}.$$

This coincides with the result obtained via explicit formula for the homogenisation limit of a periodic sequence of tensors describing material properties in the Kirchhoff-Love model.

Thank you for your attention!