Fundamental solutions of linear partial differential operators with constant coefficients

Sandeep Kumar Soni

Department of Mathematics, Faculty of Science, University of Zagreb

PMF-MO

Seminar for differential equations and numerical analysis

Zagreb, 6^{th} June 2022

Introduction

Theorem

Malgrange-Ehrenpreis theorem: Every linear partial differential operator (not identically vanishing) with constant coefficients possesses a fundamental solution in the space of distributions.

Introduction

Theorem

Malgrange-Ehrenpreis theorem: Every linear partial differential operator (not identically vanishing) with constant coefficients possesses a fundamental solution in the space of distributions.

Corollary

Every partial differential operator (not identically vanishing) with constant coefficients is locally solvable. And regularity properties of the solutions can be deduced by examining the fundamental solution.

Introduction

Theorem

Malgrange-Ehrenpreis theorem: Every linear partial differential operator (not identically vanishing) with constant coefficients possesses a fundamental solution in the space of distributions.

Corollary

Every partial differential operator (not identically vanishing) with constant coefficients is locally solvable. And regularity properties of the solutions can be deduced by examining the fundamental solution.

Proofs:
– Non-constructive proofs using Hahn-Banach theorem.
– Elementary proof based on L^2 theory.
– Constructive proofs.

Assumptions and notations

$\mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n), \mathcal{E}(\mathbb{R}^n), \mathcal{E}'(\mathbb{R}^n)$ are usual.

Assumptions and notations

$\mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n), \mathcal{E}(\mathbb{R}^n), \mathcal{E}'(\mathbb{R}^n)$ are usual.

$\partial^\alpha = \partial^{\alpha_1} ... \partial^{\alpha_n}$, and $|\alpha| = \alpha_1 + ... + \alpha_n$ for a multi-index $\alpha \in \mathbb{N}_0^n$.
$P(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$ is an operator of degree m.
$P_m(\partial) = \sum_{|\alpha| = m} c_\alpha \partial^\alpha$ is the principal part of $P(\partial)$.

Assumptions and notations

$\mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n), \mathcal{E}(\mathbb{R}^n), \mathcal{E}'(\mathbb{R}^n)$ are usual.

$\partial^\alpha = \partial^{\alpha_1}...\partial^{\alpha_n}$, and $|\alpha| = \alpha_1 + ... + \alpha_n$ for a multi-index $\alpha \in \mathbb{N}_0^n$.
$P(\partial) = \sum_{|\alpha|\leq m} c_\alpha \partial^\alpha$ is an operator of degree m.
$P_m(\partial) = \sum_{|\alpha|=m} c_\alpha \partial^\alpha$ is the principal part of $P(\partial)$.

Fourier transform:

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n),\ \mathcal{F}(\varphi)(x) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(\xi)\mathrm{d}\xi\ .$$

Assumptions and notations

$\mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n), \mathcal{E}(\mathbb{R}^n), \mathcal{E}'(\mathbb{R}^n)$ are usual.

$\partial^\alpha = \partial^{\alpha_1} ... \partial^{\alpha_n}$, and $|\alpha| = \alpha_1 + ... + \alpha_n$ for a multi-index $\alpha \in \mathbb{N}_0^n$.
$P(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$ is an operator of degree m.
$P_m(\partial) = \sum_{|\alpha| = m} c_\alpha \partial^\alpha$ is the principal part of $P(\partial)$.

Fourier transform:

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n),\ \mathcal{F}(\varphi)(x) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \varphi(\xi) \mathrm{d}\xi \ .$$

By duality or density, this yields the isomorphism

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\ \langle \mathcal{F}T, \varphi \rangle := \langle T, \mathcal{F}\varphi \rangle , \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \ .$$

Assumptions and notations

$\mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n), \mathcal{E}(\mathbb{R}^n), \mathcal{E}'(\mathbb{R}^n)$ are usual.

$\partial^\alpha = \partial^{\alpha_1} ... \partial^{\alpha_n}$, and $|\alpha| = \alpha_1 + ... + \alpha_n$ for a multi-index $\alpha \in \mathbb{N}_0^n$.
$P(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$ is an operator of degree m.
$P_m(\partial) = \sum_{|\alpha| = m} c_\alpha \partial^\alpha$ is the principal part of $P(\partial)$.

Fourier transform:

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n),\ \mathcal{F}(\varphi)(x) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \varphi(\xi) \mathrm{d}\xi \ .$$

By duality or density, this yields the isomorphism

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\ \langle \mathcal{F}T, \varphi \rangle := \langle T, \mathcal{F}\varphi \rangle, \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \ .$$

For $\zeta \in \mathbb{C}^n, T \in \mathcal{D}'(\mathbb{R}^n), S \in \mathcal{S}'(\mathbb{R}^n), U \in \mathcal{E}'(\mathbb{R}^n)$, the following hold in $\mathcal{D}'(\mathbb{R}^n)$:

$$P(\partial)(e^{\zeta \cdot x} T) = e^{\zeta \cdot x} (P(\partial + \zeta) T) \, ;$$
$$P(\partial) \mathcal{F}^{-1} S = \mathcal{F}_\xi^{-1} (P(i\xi) S) \, ;$$
$$(e^{\zeta \cdot x} U) * (e^{\zeta \cdot x} T) = e^{\zeta \cdot x} (U * T) \ .$$

Fundamental Solution: A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a fundamental solution of a differential operator $P(\partial) \in \mathbb{C}[\partial_1, ..., \partial_n]$ iff $P(\partial)E = \delta$.

Lemma

If $\lambda_0, ..., \lambda_m \in \mathbb{C}$ are pairwise different, then $a_j = \Pi_{k=0,k\neq j}^{m}(\lambda_j - \lambda_k)^{-1}$ is the unique solution of

$$\sum_{j=0}^{m} a_j \lambda_j^k = \begin{cases} 0, & \text{if } k = 0, ..., m-1, \\ 1, & \text{if } k = m\ . \end{cases}$$

Fundamental Solution: A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a fundamental solution of a differential operator $P(\partial) \in \mathbb{C}[\partial_1, ..., \partial_n]$ iff $P(\partial)E = \delta$.

Lemma

If $\lambda_0, ..., \lambda_m \in \mathbb{C}$ are pairwise different, then $a_j = \Pi_{k=0,k\neq j}^m (\lambda_j - \lambda_k)^{-1}$ is the unique solution of

$$\sum_{j=0}^{m} a_j \lambda_j^k = \begin{cases} 0, & \text{if } k = 0, ..., m-1, \\ 1, & \text{if } k = m \ . \end{cases}$$

Proof: Vandermonde's determinant is not 0, implies the uniqueness.
For $p(z) = \Pi_{j=0}^m (z - \lambda_j)$, $p'(\lambda_j) = \Pi_{k=0,k\neq j}^m (\lambda_j - \lambda_k) = a_j^{-1}$, by Residue theorem,

$$\sum_{j=0}^{m} a_j \lambda_j^k = \sum_{j=0}^{m} \frac{\lambda^k}{p'(\lambda_j)} = \frac{1}{2\pi i} \lim_{N\to\infty} \int_{|z|=N} \frac{z^k}{p(z)} \mathrm{d}z = \begin{cases} 0, & \text{if } k = 0, ..., m-1, \\ 1, & \text{if } k = m \ . \end{cases}$$

Theorem

Let $P(\xi) = \sum_{|\alpha|\leq m} c_\alpha \xi^\alpha$ be a not identically vanishing polynomial in $\mathbb{R}^n$ of degree m. If $\eta \in \mathbb{R}^n$ with $P_m(\eta) \neq 0$, the real numbers $\lambda_0, ..., \lambda_m$ are pairwise different, and $a_j = \Pi_{k=0,k\neq j}^{m}(\lambda_j - \lambda_k)^{-1}$, then

$$E = \frac{1}{\overline{P_m(2\eta)}} \sum_{j=0}^{m} a_j e^{\lambda_j \eta \cdot x} \mathcal{F}_\xi^{-1} \left(\frac{\overline{P(i\xi + \lambda_j \eta)}}{P(i\xi + \lambda_j \eta)} \right)$$

is a fundamental solution of $P(\partial)$, i.e. $P(\partial)E = \delta$.

The Malgrange-Ehrenpreis theorem

Theorem

Let $P(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha$ *be a not identically vanishing polynomial in* $\mathbb{R}^n$ *of degree* m*. If* $\eta \in \mathbb{R}^n$ *with* $P_m(\eta) \neq 0$*, the real numbers* $\lambda_0, ..., \lambda_m$ *are pairwise different, and* $a_j = \Pi_{k=0, k \neq j}^m (\lambda_j - \lambda_k)^{-1}$*, then*

$$E = \frac{1}{\overline{P_m(2\eta)}} \sum_{j=0}^{m} a_j e^{\lambda_j \eta \cdot x} \mathcal{F}_\xi^{-1} \left(\frac{\overline{P(i\xi + \lambda_j \eta)}}{P(i\xi + \lambda_j \eta)} \right)$$

is a fundamental solution of $P(\partial)$*, i.e.* $P(\partial)E = \delta$ *.*

Proof: (1) The expression within the brackets, is indeed, a tempered distribution and so E is well defined.

Theorem

Let $P(\xi) = \sum_{|\alpha| \le m} c_\alpha \xi^\alpha$ *be a not identically vanishing polynomial in* $\mathbb{R}^n$ *of degree* m. *If* $\eta \in \mathbb{R}^n$ *with* $P_m(\eta) \neq 0$, *the real numbers* $\lambda_0, ..., \lambda_m$ *are pairwise different, and* $a_j = \Pi_{k=0, k\neq j}^m (\lambda_j - \lambda_k)^{-1}$, *then*

$$E = \frac{1}{\overline{P_m(2\eta)}} \sum_{j=0}^m a_j e^{\lambda_j \eta \cdot x} \mathcal{F}_\xi^{-1} \left(\frac{\overline{P(i\xi + \lambda_j \eta)}}{P(i\xi + \lambda_j \eta)} \right)$$

is a fundamental solution of $P(\partial)$, *i.e.* $P(\partial)E = \delta$.

Proof: (1) The expression within the brackets, is indeed, a tempered distribution and so E is well defined.
For $\lambda \in \mathbb{R}$ fixed, $N = \{\xi \in \mathbb{R}^n : P(i\xi + \lambda\eta) = 0\}$ has Lebesgue measure 0. By a linear change of coordinates, we can assume that $P_m(1, 0, ..., 0) \neq 0$, and since $N_{\xi'} := \{\xi_1 \in \mathbb{R} : P(i(\xi_1, \xi') + \lambda\eta) = 0\}$ are finite for $\xi' \in \mathbb{R}^{n-1}$, we get by Fubini's theorem that $\int_N \mathrm{d}\xi = \int_{\mathbb{R}^{n-1}} \left(\int_{N_{\xi'}} \mathrm{d}\xi_1 \right) \mathrm{d}\xi' = 0$. Which means

$$S(\xi) = \frac{\overline{P(i\xi + \lambda\eta)}}{P(i\xi + \lambda\eta)} \in L^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \, .$$

(2) Verifying that $P(\partial)E = \delta$.

(2) Verifying that $P(\partial)E = \delta$.

For $S \in \mathcal{S}'(\mathbb{R}^n)$, $\zeta \in \mathbb{C}^n$, we have

$$P(\partial)(e^{\zeta\cdot x}\mathcal{F}^{-1}S) = e^{\zeta\cdot x}P(\partial+\zeta)\mathcal{F}^{-1}S = e^{\zeta x}\mathcal{F}_{\xi}^{-1}(P(i\xi+\zeta)S) \ .$$

The Malgrange-Ehrenpreis theorem

(2) Verifying that $P(\partial)E = \delta$.

For $S \in \mathcal{S}'(\mathbb{R}^n)$, $\zeta \in \mathbb{C}^n$, we have

$$P(\partial)(e^{\zeta\cdot x}\mathcal{F}^{-1}S) = e^{\zeta\cdot x}P(\partial+\zeta)\mathcal{F}^{-1}S = e^{\zeta x}\mathcal{F}_{\xi}^{-1}(P(i\xi+\zeta)S)\ .$$

So,

$$P(\partial)\left(e^{\lambda\eta\cdot x}\mathcal{F}^{-1}\left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)}\right)\right) = e^{\lambda\eta\cdot x}\mathcal{F}^{-1}\left(\overline{P(i\xi+\lambda\eta)}\right) = e^{\lambda\eta\cdot x}\overline{P(-\partial+\lambda\eta)}\delta\ .$$

Hence,

$$\begin{aligned} P(\partial)\left(e^{\lambda\eta\cdot x}\mathcal{F}^{-1}\left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)}\right)\right) &= e^{\lambda\eta\cdot x}\overline{P(-\partial+\lambda\eta)}\delta = \overline{P(-\partial+2\lambda\eta)}(e^{\lambda\eta\cdot x}\delta) \\ &= \overline{P(-\partial+2\lambda\eta)}\delta \quad \left(e^{\lambda\eta\cdot x}\delta = \delta\right) \\ &= \overline{\left(\lambda^m P_m(2\eta) + \sum_{k=0}^{m-1}\lambda^k Q_k(\partial)\right)}\delta \quad \text{(Taylor)} \end{aligned}$$

For $T_k := \overline{Q_k(\partial)}\delta \in \mathcal{E}'(\mathbb{R}^n)$, we have

$$P(\partial)\left(e^{\lambda\eta\cdot x}\mathcal{F}^{-1}\left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)}\right)\right) = \lambda^m\overline{P_m(2\eta)}\delta + \sum_{k=0}^{m-1}\lambda^k T_k\ .$$

So by linearity and previous lemma,

$$\begin{aligned}P(\partial)\left(\sum_{j=0}^{m} a_j e^{\lambda_j\eta\cdot x}\mathcal{F}^{-1}\left(\frac{\overline{P(i\xi+\lambda_j\eta)}}{P(i\xi+\lambda_j\eta)}\right)\right) &= \sum_{j=0}^{m} a_j\lambda_j^m\overline{P_m(2\eta)}\delta + \sum_{k=0}^{m-1}\sum_{j=0}^{m} a_j\lambda_j^k T_k \\ &= \overline{P_m(2\eta)}\delta + 0\end{aligned}$$

Thus,

$$P(\partial)E = \delta\ .$$

Benifits

1. With fundamental solution in hand, by convolution, we are able to find the solution of inhomogeneous linear partial differential equations.

1. With fundamental solution in hand, by convolution, we are able to find the solution of inhomogeneous linear partial differential equations. Not only for $f \in \mathcal{D}(\mathbb{R}^n)$, infact, for $f \in \mathcal{E}'(\mathbb{R}^n)$, we have

$$P(\partial)(E * f) = P(\partial)E * f = \delta * f = f \ .$$

Benifits

1. With fundamental solution in hand, by convolution, we are able to find the solution of inhomogeneous linear partial differential equations. Not only for $f \in \mathcal{D}(\mathbb{R}^n)$, infact, for $f \in \mathcal{E}'(\mathbb{R}^n)$, we have

$$P(\partial)(E * f) = P(\partial)E * f = \delta * f = f \ .$$

2. For each $T \in \mathcal{E}'(\mathbb{R}^n) \setminus \{0\}$ with finite support, there exists a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$. i.e. $T * E = \delta$.

Benifits

1. With fundamental solution in hand, by convolution, we are able to find the solution of inhomogeneous linear partial differential equations. Not only for $f \in \mathcal{D}(\mathbb{R}^n)$, infact, for $f \in \mathcal{E}'(\mathbb{R}^n)$, we have

$$P(\partial)(E * f) = P(\partial)E * f = \delta * f = f \ .$$

2. For each $T \in \mathcal{E}'(\mathbb{R}^n) \setminus \{0\}$ with finite support, there exists a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$. i.e. $T * E = \delta$.
3. More recently(2017), a generalisation of this theorem has been done for fractional PDEs by Dumitru Baleanu and Arran Fernandez.

...thank you for your attention :)

Peter Wagner: *A new constructive proof of the Malgrange-Ehrenpreis Theorem*, The American Mathematical Monthly 116:5 (2009) 457-462.
https://doi.org/10.1080/00029890.2009.11920961