Asymptotic behaviour of solutions to the wave equation with variable coefficients

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Problem

We consider the following problem

$$\begin{cases} (\rho(t,x)u'(t,x))' - \operatorname{div}(\mathbf{A}(t,x)\nabla u(t,x)) = f(t,x) & \text{in } (0,T)\times\Omega, \\ u(0) = u^0 \in H_0^1(\Omega), \quad \rho u'(0) = u^1 \in L^2(\Omega) & \text{in } \Omega, \end{cases} \tag{1}$$

with $f \in \mathcal{M}([0,T];L^2(\Omega))$, $\rho \in BV(0,T;L^\infty(\Omega))$, $\mathbf{A} \in BV(0,T;\mathbf{M}_{\text{div}}(\Omega))$. Let us assume there exists $\alpha > 0$ such that it holds

$$\rho(t,x) \geq \alpha \quad \text{a.e. on } (0,T)\times\Omega, \tag{2}$$

$$\mathbf{A}(t,x)\xi\cdot\xi \geq \alpha|\xi|^2 \quad \text{a.e. on } (0,T)\times\Omega,\ \xi\in\mathbb{R}^d. \tag{3}$$

Existence and uniqueness of solution

Theorem. *There exists unique solution $u \in L^\infty(0,T;H_0^1(\Omega))$ with $u' \in L^\infty(0,T;L^2(\Omega))$ of (1): Moreover, we have the following estimate*

$$\|u(t)\|^2_{H_0^1(\Omega)} + \|u'(t)\|^2_{L^2(\Omega)} \lesssim \|u^0\|^2_{H_0^1(\Omega)} + \|u^1\|^2_{L^2(\Omega)} + \|f\|^2_{\mathcal{M}([0,T];L^2(\Omega))}, \qquad \text{a.e. } t\in[0,T],$$

where the constant depends continously on α, $\|\rho\|_{BV(0,T;L^\infty(\Omega))}$ and $\|\mathbf{A}\|_{BV(0,T;\mathbf{M}_{\text{div}}(\Omega))}$.

Outline of the proof:

- Approximating coefficients in the main problem with those with more regularity in time using property (c) of BV functions.
- Galerkin approach: Separate symmetric from antisymmetric part of $\mathbf{A}$; the symmetric part is treated in the usual way, while the condition on $\operatorname{div}\mathbf{A}$ is used to treat antisymmetric part of $\mathbf{A}$.
- Uniqueness follows from the existence of the transpose problem, which is of the same type since $\mathbf{A}\in\mathbf{M}_{\text{div}}(\Omega)$ implies $\mathbf{A}^T\in\mathbf{M}_{\text{div}}(\Omega)$.

Homogenization

Consider a sequence of problems (1) where $\rho, \mathbf{A}, f, u^0, u^1$ are replaced with $\rho_n, \mathbf{A}_n, f_n, u_n^0, u_n^1$. Assume

$$\rho_n \text{ bounded in } BV(0,T;L^\infty(\Omega)), \quad \mathbf{A}_n \text{ bounded in } BV(0,T;L^\infty(\Omega;M_d^{\text{sym}}))$$

and that (2) and (3) hold for all n.
One can then extract subsequences and find $\rho\in L^\infty((0,T)\times\Omega)$, $\mathbf{A}\in L^\infty(0,T;L^\infty(\Omega;M_d^{\text{sym}})$ such that it holds

$$\rho_n \overset{*}{\rightharpoonup} \rho \text{ in } L^\infty((0,T)\times\Omega), \qquad \mathbf{A}_n(t,\cdot) \overset{H}{\rightharpoonup} \mathbf{A}(t,\cdot) \text{ up to a countable subset of } (0,T), \tag{4}$$

so it is not a restriction to assume (4) in the sequel.

Theorem. *With assumptions on $\rho_n, \mathbf{A}_n$ as above, assume additionally that there exist $f\in\mathcal{M}([0,T];L^2(\Omega))$, $u^0\in H_0^1(\Omega)$ and $u^1\in L^2(\Omega)$ such that*

$$f_n \overset{*}{\rightharpoonup} f \text{ in } \mathcal{M}([0,T];L^2(\Omega)), \quad u_n^0 \rightharpoonup u^0 \text{ in } H_0^1(\Omega), \quad u_n^1 \rightharpoonup u^1 \text{ in } L^2(\Omega).$$

We then have that the unique solution u_n of (1) satisfies

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)), \quad u_n' \overset{*}{\rightharpoonup} u' \text{ in } L^\infty(0,T;L^2(\Omega)),$$

where u is the unique solution of the problem of the same structure.

What can be said about non-symmetric A_n?

The H-limit of $\mathbf{A}_n\in BV(0,T;\mathbf{M}_{\text{div}}(\Omega))$ lies in $BV(0,T;L^\infty(\Omega;\mathcal{M}_d))$, but at this point it is not clear it is also in $BV(0,T;\mathbf{M}_{\text{div}}(\Omega))$. With that additional assumption, similar result follows. This assumption is always satisfied in the periodic case, where the H-limit is known to be a constant function, as well as in the case of simple laminates.

References

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[2] L. Tartar, The General Theory of Homogenization.
[3] S. Brahim-Otsmane, G.A. Francfort, F. Murat, Correctors for the homogenization of the wave and heat equations,

BV spaces

Let X be a Banach space and $f:[0,T]\to X$. We say that f is of *bounded variation* if it holds

$$\sup_{0=t_0<\dots<t_n=T}\sum_{i=1}^n \|f(t_i)-f(t_{i-1})\|_X < \infty,$$

and we denote by $BV(0,T;X)$ the space of all such functions. It is a Banach space when endowed with the norm

$$\|f\|_{BV(0,T;X)} := \|f\|_X + V(f;[0,T]).$$

BV functions with values in a Banach space are similar in behaviour to those with scalar values, where it is known that BV functions are those whose weak derivatives are bounded measures. Therefore, some results for Banach-valued case can be naturally generalized.

Some of the key structural properties of BV space being used in proving the problem.

(a) Functions in $BV(0,T;X)$ are continuous up to a countable set of $(0,T)$, and have one-sided limit in all points of $(0,T)$.

(b) For each function in $BV(0,T;X)$ there is a unique bounded measure $\mu\in\mathcal{M}([0,T])$ such that

$$\|f(t)-f(s)\|_X \leq \mu(\langle s,t]), \quad s<t.$$

(c) For each $f\in BV(0,T;X)$ there is a sequence $f_n\in W^{1,1}(0,T;X)$ such that

$$\|f_n - f\|_{L^1(0,T;X)} \to 0,$$

$$\|f_n'\|_{L^1(0,T;X)} \to V(f;[0,T]).$$

Space $\mathbf{M}_{\text{div}}(\Omega)$

For $\mathbf{A}:\Omega\to M_d(\mathbb{R})$ we can define its *divergence* as a vector defined by

$$\operatorname{div}\mathbf{A}\cdot\xi := \operatorname{div}(\mathbf{A}^T\xi), \quad \xi\in\mathbb{R}^d.$$

Equivalently, if we denote by $S_1,\dots,S_d$ the columns of $\mathbf{A}$, we have

$$\operatorname{div}\mathbf{A} = \begin{bmatrix}\operatorname{div} S_1\\ \vdots \\ \operatorname{div} S_d\end{bmatrix}.$$

We will denote the space of all $\mathbf{A}\in L^\infty(\Omega;M_d(\mathbb{R}))$ such that $\operatorname{div}\mathbf{A}\in L^\infty(\Omega;\mathbb{R}^d)$ and $\operatorname{div}\mathbf{A}^T\in L^\infty(\Omega;\mathbb{R}^d)$ with $\mathbf{M}_{\text{div}}(\Omega)$. It is an intersection of two *graph spaces* and therefore it is a Banach space with the norm

$$\begin{aligned}\|\mathbf{A}\|_{\mathbf{M}_{\text{div}}(\Omega)} &:= \|\mathbf{A}\|_{L^\infty(\Omega;M_d(\mathbb{R}))} \\ &+ \|\operatorname{div}\mathbf{A}\|_{L^\infty(\Omega;\mathbb{R}^d)} \\ &+ \|\operatorname{div}\mathbf{A}^T\|_{L^\infty(\Omega;\mathbb{R}^d)}\end{aligned}$$

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