H-distributions on Hörmander spaces

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The history of H-measures and H-distributions

Hörmander spaces

Preliminary results

Construction of a variant of H-distributions

H-measures

[Tartar, Gérard (1990/91)]

If scalar sequences $u_n, v_n \rightharpoonup 0$ in $L^2(\mathbf{R}^d)$, then there exist subsequences $(u_{n'}), (v_{n'})$ and a complex Radon measure μ on $\mathbf{R}^d \times S^{d-1}$ such that, for every $\varphi_1, \varphi_2 \in C_0(\mathbf{R}^d)$ and every $\psi \in C(S^{d-1})$,

$$\lim_{n'} \langle \mathcal{A}_\psi(\varphi_1 u_{n'}) | \varphi_2 v_{n'} \rangle := \lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} dx = \langle \mu, \varphi_1 \overline{\varphi_2} \psi \rangle,$$

where $\mathcal{A}_\psi u := \mathcal{F}^{-1}(\psi \hat{u})$.

H-distributions

[Antonić, Mitrović (2011)]

If $u_n \rightharpoonup 0$ in $L^p(\mathbf{R}^d)$, $1 < p < \infty$ and $v_n \overset{*}{\rightharpoonup} 0$ in $L^q(\mathbf{R}^d)$, $q \geq \max\{p', 2\}$, then there exist subsequences $(u_{n'}), (v_{n'})$ and a complex valued distribution $\mu \in \mathcal{D}'(\mathbf{R}^d \times S^{d-1})$ of order no more than $\kappa = \lfloor d/2 \rfloor + 1$ in ξ, such that, for every $\varphi_1, \varphi_2 \in C_c^\infty(\mathbf{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$ one has:

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} dx &= \lim_{n'} \int_{\mathbf{R}^d} \varphi_1 u_{n'} \overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_{n'})} dx \\ &= \langle \mu, \varphi_1 \overline{\varphi_2} \psi \rangle.\end{aligned}$$

Plan for today

$B_{p,k}$ ... $k\hat{u} \in L^p(\mathbf{R}^d)$, $u \in \mathcal{S}'$, $k(\xi + \eta) \le (1 + C|\xi|)^N k(\eta)$

B_s^p ... $k(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} = \langle \xi \rangle^s$

$u_n \rightharpoonup 0$ in B_s^p

$v_n \rightharpoonup 0$ in $B_{-s+m}^{p'}$

$\mu \in \Big(\mathcal{S}(\mathbf{R}^d) \hat{\otimes} (s_{\infty,N+1}^m)_0 \Big)'$

$\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d),\ \psi \in (s_{\infty,N+1}^m)_0$

When $B_{p,k}$ is an algebra?

$$u \cdot v = ? \qquad\qquad u \cdot v \stackrel{?}{=} \mathcal{F}^{-1}(\hat{u} * \hat{v})$$

$$\langle u, v\rangle,\ u \in B_{p,k},\ v \in B_{p',\frac{1}{k}}$$

$$u, v \in B_{p,k} \stackrel{?}{\Longrightarrow} u \cdot v \in B_{p,k}$$

I) [Messina, Rodino (2001)]

$$K(\xi,\eta) = \frac{k(\xi)}{k(\xi-\eta)k(\eta)}, \quad \sup_{\xi} \int |K(\xi,\eta)|^{p'} d\eta < \infty$$

II) [Pi, Wong (1992)]

$$B_s^p,\ s > \frac{d}{p'}$$

Symbol classes

$S^m_{\rho,\delta}$ ... $|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\beta|}$

$S^m_{1,0}$ ... $|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$, shorter S^m

$\dot{S}^m_q$ ... $|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta} \langle x \rangle^{q-|\alpha|} \langle \xi \rangle^m$

$\dot{S}^m_{q,N}$ ... if above is valid for $|\alpha|, |\beta| \le N \in \mathbf{N}_0$

The last one is a Banach space with the norm

$$|\sigma|_{\dot{S}^m_{q,N}} = \max_{|\alpha|,|\beta|\le N} \sup_{x,\xi\in\mathbf{R}^d} \frac{|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)|}{\langle x \rangle^{q-|\alpha|} \langle \xi \rangle^m}$$

The corresponding pseudodifferential operator T_σ:

$$T_\sigma \varphi(x) = \int_{\mathbf{R}^d} e^{ix\xi} \sigma(x,\xi) \hat{\varphi}(\xi) đ\xi$$

Continuity of pseudodifferential operators

$S^0_{\rho,\delta}$ $\quad \rho \geq \delta \ \ldots \ L^2$ continuity

$\rho = 1,\ \delta \in [0,1\rangle \ \ldots \ L^p$ continuity

$\ldots$ mixed-norm L^p (2019)

$\dot{S}^m_0 \implies T_\sigma : B^p_s \longrightarrow B^p_{s-m}$ bounded (Iancu 2000)

Theorem 1. *Let* $1 < p < \infty,\ m, s \in \mathbf{R}$*, let* N *be an integer such that* $N > |s-m| + 7d + 8$*, and let the operator* T *be defined by*

$$T(\sigma, u) = T_\sigma u,\ \sigma \in \dot{S}^m_{0,N},\ u \in B^p_s.$$

Then T *is a continuous bilinear operator from* $\dot{S}^m_{0,N} \times B^p_s$ *to* B^p_{s-m} *and there is a constant* $C_{N,s-m} > 0$ *such that*

$$\|T_\sigma u\|_{B^p_{s-m}} \leq C_{N,s-m} |\sigma|_{\dot{S}^m_{0,N}} \|u\|_{B^p_s}.$$

∎

Idea of the proof

It is enough to consider the case $m = s = 0$ because in the general case:

$$T_{\sigma_2} = T_{\langle\cdot\rangle^{s-m}} T_\sigma T_{\langle\cdot\rangle^{-s}} = T_{\langle\cdot\rangle^{s-m}} T_{\sigma\langle\cdot\rangle^{-s}}.$$

$$\widehat{T_\sigma u} = \tilde{T}^*_{\bar{\tilde{\sigma}}} \hat{u}, \qquad \begin{array}{l} \tilde{\sigma}(x,\xi) = \sigma(-x,\xi) \\ \tilde{T}_\sigma u(\xi) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x,\xi) \hat{u}(x) đx \end{array}$$

$$\implies \|\widehat{T_\sigma u}\|_p = \|\tilde{T}^*_{\bar{\tilde{\sigma}}} \hat{u}\|_p \leq C_M |\bar{\tilde{\sigma}}^*|_{S^0_M} \|\hat{u}\|_p$$

It remains to prove: $|\bar{\tilde{\sigma}}^*|_{S^0_M} \leq C_M |\sigma|_{\dot{S}^0_{0,N}}$

We denote $\sigma_1(\xi,x) = \bar{\tilde{\sigma}}^*(x,\xi) = \iint e^{-iy\eta} \sigma(-x-y,\xi+\eta) dy đ\eta$
and estimate

Idea of the proof (cont.)

$$
\begin{aligned}
\frac{|\partial_\xi^\beta \partial_x^\alpha \sigma_1(\xi,x)|}{\langle x\rangle^{-|\alpha|}} &= \langle x\rangle^{|\alpha|} \Big| \iint e^{-iy\eta} \partial_\xi^\beta \partial_x^\alpha \sigma(-x-y,\xi+\eta) dy đ\eta \Big| \\
&= \langle x\rangle^{|\alpha|} \Big| \iint \frac{e^{-iy\eta}}{\langle y\rangle^{2k}} \langle D_\eta\rangle^{2k} \left(\frac{\langle D_y\rangle^{2l} \partial_\xi^\beta \partial_x^\alpha \sigma(-x-y,\xi+\eta)}{\langle \eta\rangle^{2l}} \right) dy đ\eta \Big| \\
&\le \langle x\rangle^{|\alpha|} \cdot C_{2k} \iint \frac{\sum_{|\gamma|\le 2k} |\partial_\eta^\gamma \langle D_y\rangle^{2l} \partial_\xi^\beta \partial_x^\alpha \sigma(-x-y,\xi+\eta)|}{\langle y\rangle^{2k} \langle \eta\rangle^{2l}} dy đ\eta \\
&\le \langle x\rangle^{|\alpha|} \cdot C_{2k} \cdot |\sigma|_{\dot{S}^0_{0,M+\max\{2k,2l\}}} \iint \frac{\langle -x-y\rangle^{-|\alpha|-2l}}{\langle y\rangle^{2k} \langle \eta\rangle^{2l}} dy đ\eta
\end{aligned}
$$

$$
\langle D_\eta\rangle^{2k} = (1-\triangle_\eta)^k
$$

Fourier multipliers

$\mathcal{A}_\psi u = \mathcal{F}^{-1}(\psi \hat{u})$
We shall need $\psi \in (s^m_{\infty,N+1})_0$

$$s^m_{\infty,N} \;\ldots\; |\psi|_{s^m_{\infty,N}} := \max_{|\alpha|\leq N} \|\partial_\xi^\alpha \psi(\xi)\langle\xi\rangle^{-m+|\alpha|}\|_\infty < \infty$$

$$(s^m_{\infty,N})_0 \;\ldots\; \lim_{n\to\infty} \sup_{|\xi|\geq n} \frac{|\partial_\xi^\alpha \psi(\xi)|}{\langle\xi\rangle^{m-|\alpha|}} = 0, \qquad \text{for all } |\alpha| \leq N$$

$((s^m_{\infty,N+1})_0, |\cdot|_{s^m_{\infty,N}})$ is separable

Symbol of the commutator

$$C = [T_\psi, T_\varphi] = T_\psi T_\varphi - T_\varphi T_\psi$$

Lemma 1. *Let* $\psi \in s^m_{\infty,N}$, $\varphi \in \mathcal{S}(\mathbf{R}^d)$, $m \in \mathbf{R}$, $N \geq d+5$. *If* σ *denotes the symbol of the commutator* $C = [T_\psi, T_\varphi]$, *then* $\sigma \in \dot{S}^{m-1}_{-\epsilon,N'}$, *for any* $\epsilon \in \langle 0, 1 \rangle$ *and* $N' \leq \frac{N-d-5}{2}$.

■

$$\sigma(x,\xi) = \sum_{|\gamma|=1} \partial_\xi^\gamma \psi(\xi) D_x^\gamma \varphi(x) + 2 \sum_{|\gamma|=2} \frac{1}{\gamma!} \int_0^1 (1-\theta) I_3(x,\xi) d\theta,$$

where

$$I_3(x,\xi) = \iint e^{-iy\eta} \partial_\xi^\gamma \psi(\xi + \theta\eta) D_y^\gamma \varphi(x+y) dy\, \text{đ}\eta.$$

Compactness of the commutator

Theorem 2. *Let* $\psi \in s^m_{\infty,N}$, $\varphi \in \mathcal{S}(\mathbf{R}^d)$, $m, s \in \mathbf{R}$, $N \geq 15d + 2|s| + 25$. *Then the commutator* $C = [T_\psi, T_\varphi]$ *is a compact operator from* B^p_{m+s} *to* B^p_s *for* $1 < p < \infty$.

$\phi \in C_c^\infty(\mathbf{R}^d)$, $\phi(\xi) = 1$ for $|\xi| \leq 1$ and $\phi(\xi) = 0$ for $|\xi| \geq 2$
$\sigma_n(x,\xi) = \phi(\frac{x}{n})\sigma(x,\xi) = \phi_n(x)\sigma(x,\xi)$, σ - symbol of C

$$B^p_{m+s} \xrightarrow[\text{bounded}]{C} B^p_{s+1} \xrightarrow[\text{bounded}]{T_{\phi_n}} B^p_{s+1} \xrightarrow[\text{compact}]{I} B^p_s$$

We claim: $T_{\sigma_n} \longrightarrow C$ in the operator norm

Theorem 1 $\implies$ $\|(T_{\sigma_n} - C)u\|_{B^p_s} \leq C|\sigma_n - \sigma|_{\dot{S}^m_{0,N'}} \|u\|_{B^p_{m+s}}$

It is enough to prove: $|\sigma_n - \sigma|_{\dot{S}^m_{0,N'}} \to 0$, $n \to \infty$

Proof (cont.)

$$\frac{|\partial_x^\alpha \partial_\xi^\beta (\sigma_n - \sigma)|}{\langle x\rangle^{-|\alpha|}\langle \xi\rangle^m} = \frac{\left|\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\partial_x^{\alpha-\gamma}(\phi(\frac{x}{n})-1)\partial_x^\gamma\partial_\xi^\beta\sigma(x,\xi)\right|}{\langle x\rangle^{-|\alpha|}\langle \xi\rangle^m}$$

$$\leq C\left(\sup_{n\leq|x|\leq 2n}\sum_{\gamma<\alpha}\frac{\frac{1}{n^{|\alpha-\gamma|}}|\partial_x^\gamma\partial_\xi^\beta\sigma(x,\xi)|}{\langle x\rangle^{-|\alpha|}\langle \xi\rangle^m} + \sup_{|x|\geq n}\frac{|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)|}{\langle x\rangle^{-|\alpha|}\langle \xi\rangle^m}\right)$$

$$\leq C\left(\sup_{n\leq|x|\leq 2n}\sum_{\gamma<\alpha}\frac{\frac{1}{n^{|\alpha-\gamma|}}\langle x\rangle^{-\epsilon-|\gamma|}\langle \xi\rangle^{m-1}}{\langle x\rangle^{-|\alpha|}\langle \xi\rangle^m} + \sup_{|x|\geq n}\frac{\langle x\rangle^{-\epsilon-|\alpha|}\langle \xi\rangle^{m-1}}{\langle x\rangle^{-|\alpha|}\langle \xi\rangle^m}\right)$$

$$\leq C\left(\sup_{n\leq|x|\leq 2n}\sum_{\gamma<\alpha}\frac{1}{n^{|\alpha-\gamma|}}\langle x\rangle^{-\epsilon-|\gamma|+|\alpha|} + \sup_{|x|\geq n}\langle x\rangle^{-\epsilon}\right)$$

$$\leq Cn^{-\epsilon}$$

Theorem 3. *Let* $u_n \rightharpoonup 0$ *in* B^p_s, $v_n \rightharpoonup 0$ *in* $B^{p'}_{-s+m}$, $1 < p < \infty$, $m, s \in \mathbb{R}$, $N \geq 15d + 2|s| + 25$. *Then, up to subsequences, there exists a distribution* $\mu \in (\mathcal{S}(\mathbf{R}^d)\hat{\otimes}(s^m_{\infty,N+1})_0)'$ *such that for every* $\psi \in (s^m_{\infty,N+1})_0$ *and for every* $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)$ *it holds that*

$$\lim_{n\to\infty} \langle \varphi_1 u_n, T_\psi(\varphi_2 v_n)\rangle = \langle \mu, \varphi_1\varphi_2\psi\rangle.$$

∎

Proof

$$\mu_n(\varphi,\psi) = \langle u_n, T_\psi(\varphi v_n)\rangle,\ n \in \mathbf{N},\ \varphi \in \mathcal{S}(\mathbf{R}^d),\ \psi \in (s^m_{\infty,N+1})_0$$

$T_\sigma = T_\psi T_\varphi \overset{\text{Lemma 1}}{\Longrightarrow} \sigma \in \dot{S}^m_{0,N'}$

Theorem 1 $\Longrightarrow \|T_\psi(\varphi v_n)\|_{B^{p'}_{-s}} \leq C|\sigma|_{\dot{S}^m_{0,N'}} \|v_n\|_{B^{p'}_{-s+m}}$

Next, by estimating oscillatory integral (as earlier)

$$\sigma(x,\xi) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} e^{-iy\eta}\psi(\xi+\eta)\varphi(x+y)dy đ\eta$$

we get $|\sigma|_{\dot{S}^m_{0,N'}} \leq C|\psi|_{s^m_{\infty,N}}|\varphi|_{k_0}$, for some $k_0 \in \mathbf{N}$.

$\Longrightarrow\ \psi \mapsto \mu_n(\varphi,\psi)$ is linear and continuous, and $\varphi \mapsto \mu_n(\varphi,\psi)$ is linear and continuous

Proof (cont.)

$\varphi \in \mathcal{S}(\mathbf{R}^d)$ fixed, $(B_n\varphi)$ sequence in $((s^m_{\infty,N+1})_0)'$: $\langle B_n\varphi, \cdot\rangle = \mu_n(\varphi, \cdot)$

Banach-Alaoglu-Bourbaki $\implies$ there exists a subsequence $B_k\varphi \overset{*}{\rightharpoonup} B\varphi$

Diagonalisation argument $\implies$ we can define B on a countable dense set $D = \{\varphi_l : l \in \mathbf{N}\} \subset \mathcal{S}(\mathbf{R}^d)$:

$$\langle B\varphi_l, \psi\rangle = \lim_{k\to\infty} \langle B_{k,k}\varphi_l, \psi\rangle,\ \varphi_l \in D,\ \psi \in (s^m_{\infty,N+1})_0$$

Banach-Steinhaus $\implies$ we can define B on the whole $\mathcal{S}(\mathbf{R}^d)$:

$$\langle B\varphi, \psi\rangle = \lim_{k\to\infty} \langle B_{k,k}\varphi, \psi\rangle,\ \varphi \in D,\ \psi \in (s^m_{\infty,N+1})_0$$

Schwartz kernel theorem $\implies$ there exists $\mu \in (\mathcal{S}(\mathbf{R}^d)\hat{\otimes}(s^m_{\infty,N+1})_0)'$:

$$\langle \mu, \varphi\psi\rangle = \lim_{k\to\infty} \langle B_{k,k}\varphi, \psi\rangle = \lim_{k\to\infty} \langle u_{k,k}, T_\psi(\varphi v_{k,k})\rangle$$

Finally, $\varphi \in \mathcal{S}(\mathbf{R}^d)$ can be written as $\varphi = \varphi_1\varphi_2$ for some $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)$, and compactness of the commutator gives the claim.