Abstract Friedrichs operators and the graph space

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PMF-MO

Seminar for differential equations and numerical analysis

Zagreb, 24^{th} February 2022

Joint work with Marko Erceg

IP-2018-01-2449 (MiTPDE)

Classical Friedrichs operators

Assumptions:
$d, r \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary;
$\mathbf{A}_k \in W^{1,\infty}(\Omega; \mathrm{M}_r(\mathbb{C}))$, $k \in \{1, \ldots, d\}$, and $\mathbf{B} \in L^\infty(\Omega; \mathrm{M}_r(\mathbb{C})$ satisfying (a.e. on Ω):

(F1) $$\mathbf{A}_k = \mathbf{A}_k^* \, ;$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{B} + \mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \, .$$

Define $\mathcal{L}, \widetilde{\mathcal{L}} : L^2(\Omega)^r \to \mathcal{D}'(\Omega)^r$ by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u} \, , \qquad \widetilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \Big(\mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\Big)\mathsf{u} \, .$$

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Aim: impose boundary conditions such that for any $\mathsf{f} \in \mathrm{L}^2(\Omega)^r$ we have a unique solution of $\mathcal{L}\mathsf{u} = \mathsf{f}$.
Gain: many important (semi)linear equations of mathematical physics can be written in the form of classical Friedrichs operators.

K. O. Friedrichs: *Symmetric positive linear differential equations*, Commun. Pure Appl. Math. **11** (1958) 333–418.

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– Contributions: C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...
– treating the equations of mixed type, such as the Tricomi equation:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;$$

– unified treatment of equations and systems of different type;
– more recently: better numerical properties.

Shortcommings:
– no satisfactory well-posedness result,
– no intrinsic (unique) way to pose boundary conditions.

The classical theory in short

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$\rightsquigarrow$ development of the abstract theory

Abstract Friedrichs operators

$(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ complex Hilbert space ($\mathcal{H}' \equiv \mathcal{H}$), $\| \cdot \| := \sqrt{\langle \cdot \mid \cdot \rangle}$
$\mathcal{D} \subseteq \mathcal{H}$ dense subspace

Definition

Let $T, \widetilde{T} : \mathcal{D} \to \mathcal{H}$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

(T1) $(\forall \varphi, \psi \in \mathcal{D}) \qquad \langle T\varphi \mid \psi \rangle = \langle \varphi \mid \widetilde{T}\psi \rangle ;$

(T2) $(\exists c > 0)(\forall \varphi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\varphi\| \leqslant c\|\varphi\| ;$

(T3) $(\exists \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \qquad \langle (T + \widetilde{T})\varphi \mid \varphi \rangle \geqslant \mu_0 \|\varphi\|^2 .$

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems*, Comm. Partial Diff. Eq. **32** (2007) 317–341.

N. Antonić, K. Burazin: *Intrinsic boundary conditions for Friedrichs systems*, Comm. Partial Diff. Eq. **35** (2010) 1690–1715.

Classical is *abstract*

$\mathbf{A}_k \in W^{1,\infty}(\Omega; \mathrm{M}_r(\mathbb{C}))$ and $\mathbf{C} \in L^\infty(\Omega; \mathrm{M}_r(\mathbb{C}))$ satisfy (F1)–(F2):

(F1) $$\mathbf{A}_k = \mathbf{A}_k^* \, ;$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{B} + \mathbf{B}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant \mu_0 \mathbf{I} \, .$$

$\mathcal{D} := C_c^\infty(\Omega)^r$, $\mathcal{H} := L^2(\Omega)^r$, and

$$T\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u} \, , \qquad \widetilde{T}\mathsf{u} := -\sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \Big(\mathbf{B}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k\Big)\mathsf{u} \, .$$

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(T1) $\langle T\mathsf{u} \mid \mathsf{v} \rangle_{L^2} = \langle \mathsf{u} \mid -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k^* \mathsf{v}) + \big(\mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\big)\mathsf{v} \rangle_{L^2} \overset{\text{(F1)}}{=} \langle \mathsf{u} \mid \widetilde{T}\mathsf{v} \rangle_{L^2} \, .$

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Since $(T + \widetilde{T})\mathsf{u} = \Big(\mathbf{B} + \mathbf{B}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k\Big)\mathsf{u}$,

(T2) $\|(T + \widetilde{T})\mathsf{u}\|_{L^2} \leqslant \Big(2\|\mathbf{B}\|_{L^\infty} + \sum_{k=1}^d \|\mathbf{A}_k\|_{W^{1,\infty}}\Big)\|\mathsf{u}\|_{L^2}\,,$

(T3) $\langle (T + \widetilde{T})\mathsf{u} \mid \mathsf{u} \rangle_{L^2} \overset{\text{(F2)}}{\geqslant} \mu_0 \|\mathsf{u}\|_{L^2}^2\,.$

Well-posedness result

Goal: For $(T, \widetilde{T})$ satisfying (T1)–(T3) find $\mathcal{V} \supseteq \mathcal{D}$ ($\widetilde{\mathcal{V}} \supseteq \mathcal{D}$) such that T ($\widetilde{T}$) extended to $\mathcal{V}$ ($\widetilde{\mathcal{V}}$) is a linear bijection.

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$$\exists \textit{ maximal operators}: \quad \begin{aligned} &T_1 : \mathcal{W} \subseteq \mathcal{H} \to \mathcal{H}\,, \quad T \subseteq T_1\,, \\ &\widetilde{T}_1 : \mathcal{W} \subseteq \mathcal{H} \to \mathcal{H}\,, \quad \widetilde{T} \subseteq \widetilde{T}_1\,. \end{aligned} \qquad (\operatorname{dom} T_1 = \operatorname{dom} \widetilde{T}_1 =: \mathcal{W})$$

Boundary map (form): $D : \mathcal{W} \to \mathcal{W}'$,

$$[u \mid v] := {}_{\mathcal{W}'}\langle Du, v \rangle_{\mathcal{W}} := \langle\, T_1 u \mid v \,\rangle - \langle\, u \mid \widetilde{T}_1 v \,\rangle\,. \qquad ([u \mid v] = \overline{[v \mid u]})$$

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Boundary map (form): $D : \mathcal{W} \to \mathcal{W}'$,

$$[u \,|\, v] := {}_{\mathcal{W}'}\langle Du, v \rangle_{\mathcal{W}} := \langle T_1 u \,|\, v \rangle - \langle u \,|\, \widetilde{T}_1 v \rangle . \qquad ([u \,|\, v] = \overline{[v \,|\, u]})$$

For $\mathcal{V}, \widetilde{\mathcal{V}} \subseteq \mathcal{W}$ we introduce two conditions:

(V1)
$$(\forall\, u \in \mathcal{V}) \qquad [u \,|\, u] \geqslant 0$$
$$(\forall\, v \in \widetilde{\mathcal{V}}) \qquad [v \,|\, v] \leqslant 0$$

(V2)
$$\mathcal{V} = \{u \in \mathcal{W} : (\forall\, v \in \widetilde{\mathcal{V}}) \quad [v \,|\, u] = 0\}$$
$$\widetilde{\mathcal{V}} = \{v \in \mathcal{W} : (\forall\, u \in \mathcal{V}) \quad [u \,|\, v] = 0\} \qquad (\implies \mathcal{D} \subseteq \mathcal{V} \cap \widetilde{\mathcal{V}})$$

Well-posedness result

Goal: For $(T,\widetilde{T})$ satisfying (T1)–(T3) find $\mathcal{V}\supseteq\mathcal{D}$ ($\widetilde{\mathcal{V}}\supseteq\mathcal{D}$) such that T ($\widetilde{T}$) extended to $\mathcal{V}$ ($\widetilde{\mathcal{V}}$) is a linear bijection.

$\exists$ *maximal operators* : $T_1 : \mathcal{W}\subseteq\mathcal{H}\to\mathcal{H}\,,\quad T\subseteq T_1\,,$ $\widetilde{T}_1 : \mathcal{W}\subseteq\mathcal{H}\to\mathcal{H}\,,\quad \widetilde{T}\subseteq\widetilde{T}_1\,.$ $\quad(\mathrm{dom}\,T_1=\mathrm{dom}\,\widetilde{T}_1=:\mathcal{W})$

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(V1)
$$(\forall\, u\in\mathcal{V})\qquad [u\,|\,u]\geqslant 0$$
$$(\forall\, v\in\widetilde{\mathcal{V}})\qquad [v\,|\,v]\leqslant 0$$

(V2)
$$\mathcal{V}=\{u\in\mathcal{W} : (\forall\, v\in\widetilde{\mathcal{V}})\quad [v\,|\,u]=0\}$$
$$\widetilde{\mathcal{V}}=\{v\in\mathcal{W} : (\forall\, u\in\mathcal{V})\quad [u\,|\,v]=0\}$$
$$(\implies \mathcal{D}\subseteq\mathcal{V}\cap\widetilde{\mathcal{V}})$$

Theorem (Ern, Guermond, Caplain, 2007)

(T1)–(T3) + (V1)–(V2) $\implies$ $T_1|_{\mathcal{V}},\widetilde{T}_1|_{\widetilde{\mathcal{V}}}$ *bijective realisations* .

Example 1 (Scalar elliptic PDE)

$\Omega \subseteq \mathbb{R}^d$, $\mu > 0$ and $f \in \mathrm{L}^2(\Omega)$ given.

$$-\triangle u + \mu u = f \iff -\operatorname{div} \nabla u + \mu u = f \iff \begin{cases} \nabla u + \mathsf{p} = 0 \\ \operatorname{div} \mathsf{p} + \mu u = f \end{cases}$$

$$\iff T\mathsf{v} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{v}) + \mathbf{C}\mathsf{v} = \mathsf{g}\,,$$

where $\mathsf{v} := [\mathsf{p}\ u]^\top$, $\mathsf{g} := [0\ f]^\top$, $(\mathbf{A}_k)_{ij} := \delta_{i,k}\delta_{j,d+1} + \delta_{i,d+1}\delta_{j,k}$, $\mathbf{C} := \operatorname{diag}\{1, \ldots, 1, \mu\}$.
Assumtions (F1) and (F2) are satisfied.

$L = \mathrm{L}^2(\Omega)^{d+1}$, $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$

- $V = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)$ ... Dirichelt boundary condition ($u = 0$ on Γ)
- $V = \mathrm{L}^2_{\mathrm{div},0}(\Omega) \times \mathrm{H}^1(\Omega)$ ... Neumann boundary condition ($\mathsf{p} \cdot \nu = \nabla u \cdot \nu = 0$ on Γ)

Characterisation of joint pair of abstract Friedrichs operators

Theorem

$$(T1)-(T3) \iff \begin{cases} T \subseteq \widetilde{T}^* \quad \& \quad \widetilde{T} \subseteq T^*; \\ \overline{T+\widetilde{T}} \textit{ bounded self-adjoint in } \mathcal{H} \textit{ with strictly positive bottom}; \\ \operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} \quad \& \quad \operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* . \end{cases}$$

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Condition (T3) is used in this theorem only to get that $\overline{T+\widetilde{T}}$ has strictly positive bottom. More precisely, a pair $(T,\widetilde{T})$ satisfies conditions (T1)–(T2) if and only if $T\subseteq\widetilde{T}^*$, $\widetilde{T}\subseteq T^*$, and $\overline{T+\widetilde{T}}$ is an everywhere defined, bounded, self-adjoint operator on $\mathcal{H}$. Since many statements hold even in this case, we shall explicitly emphasise in which particular situations condition (T3) is necessary.

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Dual pairs : Operators A, B on $\mathcal{H}$ with the property that $A \subseteq B^*$ and $B \subseteq A^*$ are often referred to as *dual pairs*.

Thus, operators forming a joint pair of abstract Friedrichs operators are dual pairs (this follows merely from condition (T1)).

Let $(T, \widetilde{T})$ be a joint pair of abstract Friedrichs operators. By (T1) it is evident that T and $\widetilde{T}$ are closable. Since $T + \widetilde{T}$ is a bounded operator, graph norms $\|\cdot\|_T$ and $\|\cdot\|_{\widetilde{T}}$ are equivalent.

(1)
$$\begin{aligned} \operatorname{dom}\overline{T} &= \operatorname{dom}\overline{\widetilde{T}} =: \mathcal{W}_0 \,, \\ \operatorname{dom}T^* &= \operatorname{dom}\widetilde{T}^* =: \mathcal{W} \,, \end{aligned}$$

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and $(\overline{T + \widetilde{T}})|_{\mathcal{W}} = \widetilde{T}^* + T^*$. This implies that $(\overline{T}, \overline{\widetilde{T}})$ is also a pair of abstract Friedrichs operators. Now we simplify our notation by introducing

$$T_0 \;:=\; \overline{T} \;, \quad \widetilde{T}_0 \;:=\; \overline{\widetilde{T}} \;, \quad T_1 \;:=\; \widetilde{T}^* \;, \quad \widetilde{T}_1 \;:=\; T^* \;.$$

Therefore, we have

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$$T_0 \;\subseteq\; T_1 \quad \text{and} \quad \widetilde{T}_0 \;\subseteq\; \widetilde{T}_1 \;.$$

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When equipped with the graph norm (one of two equivalent norms $\|\cdot\|_{T_1}$ and $\|\cdot\|_{\widetilde{T}_1}$), the space $\mathcal{W}$ becomes a Banach space, thus we shall call it the *graph space*. $\mathcal{W}_0$ is a closed subspace of the graph space $\mathcal{W}$, while it is dense in $\mathcal{H}$ (since it contains $\mathcal{D}$).

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When equipped with the graph norm (one of two equivalent norms $\|\cdot\|_{T_1}$ and $\|\cdot\|_{\widetilde{T}_1}$), the space $\mathcal{W}$ becomes a Banach space, thus we shall call it the *graph space*. $\mathcal{W}_0$ is a closed subspace of the graph space $\mathcal{W}$, while it is dense in $\mathcal{H}$ (since it contains $\mathcal{D}$). As an illustration, for $\mathcal{H} = L^2(\Omega)$ and a certain choice of operators it could be that $\mathcal{W}$ and $\mathcal{W}_0$ are Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$, respectively.

Characterisation of joint pair of abstract Friedrichs operators

Remark: Since T_1 and $\widetilde{T}_1$ are closed, their kernels $\ker T_1$ and $\ker \widetilde{T}_1$ are closed both in $\mathcal{H}$ and $\mathcal{W}$. Indeed, for any convergent sequence (u_n) in, say, $\ker T_1$ with the limit $u \in \mathcal{H}$, we have $u_n \xrightarrow{\mathcal{H}} u$ and $T_1 u_n = 0$. This implies $u \in \operatorname{dom} T_1 = \mathcal{W}$ and $T_1 u = 0$, i.e. $u \in \ker T_1$. Thus, we also have $u_n \xrightarrow{\mathcal{W}} u$.

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Lemma

Let a pair of operators $(T, \widetilde{T})$ on $\mathcal{H}$ satisfies (T1)–(T2)*. Then the boundary operator D is continuous and satisfies*

i) $(\forall u, v \in \mathcal{W}) \quad {}_{\mathcal{W}'}\langle Du, v \rangle_{\mathcal{W}} = \overline{{}_{\mathcal{W}'}\langle Dv, u \rangle_{\mathcal{W}}}\,,$

ii) $\ker D = \mathcal{W}_0\,,$

iii) $\operatorname{ran} D = \mathcal{W}_0^0\,,$

where ^{0} *stands for the annihilator.*

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Theorem

If $(T, \widetilde{T})$ satisfies (T1)–(T2), then

$$(V2) \quad \Longleftrightarrow \quad \begin{cases} \mathcal{D} \subseteq \mathcal{V}, \widetilde{\mathcal{V}} \subseteq \mathcal{W} \\ (\widetilde{T}^*|_{\mathcal{V}})^* = T^*|_{\widetilde{\mathcal{V}}} \\ (T^*|_{\widetilde{\mathcal{V}}})^* = \widetilde{T}^*|_{\mathcal{V}}\,. \end{cases}$$

We seek for bijective closed operators $S \equiv \widetilde{T}^*|_{\mathcal{V}}$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^*\,,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$. If $(\mathrm{dom}\,S, \mathrm{dom}\,S^*)$ satisfies $(V1)$ we call (S, S^*) an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.

Bijective realisations with signed boundary map

We seek for bijective closed operators $S \equiv \widetilde{T}^*|_{\mathcal{V}}$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^*,$$

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Theorem (Antonić, Michelangeli, Erceg , 2017)

Let $(T, \widetilde{T})$ satisfies (T1)–(T3).

(i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.*

(ii)

$\ker \widetilde{T}^* \neq \{0\}$ & $\ker T^* \neq \{0\}$ $\Longrightarrow$ *uncountably many adjoint pairs of bijective realisations with signed boundary map*

$\ker \widetilde{T}^* = \{0\}$ *or* $\ker T^* = \{0\}$ $\Longrightarrow$ *only one adjoint pair of bijective realisations with signed boundary map*

Classification

For $(T, \widetilde{T})$ satisfying (T1)–(T3) we have

$$\overline{T} \subseteq \widetilde{T}^* \qquad \text{and} \qquad \overline{\widetilde{T}} \subseteq T^* \,,$$

while by the previous theorem there exists closed T_{r} such that

- $\overline{T} \subseteq T_{\mathrm{r}} \subseteq \widetilde{T}^*$ ($\iff$ $\overline{\widetilde{T}} \subseteq T_{\mathrm{r}}^* \subseteq T^*$),
- $T_{\mathrm{r}} : \operatorname{dom} T_{\mathrm{r}} \to \mathcal{H}$ bijection,
- $(T_{\mathrm{r}})^{-1} : \mathcal{H} \to \operatorname{dom} T_{\mathrm{r}}$ bounded.

Thus, we can apply a universal classification (classification of dual (adjoint) pairs).

We used Grubb's universal classification

G. Grubb: *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa **22** (1968) 425–513.

Result: complete classification of all adjoint pairs of bijective realisations with signed boundary map.
To do: apply this result to general classical Friedrichs operators from the beginning (*nice class of non-self-adjoint differential operators of interest*)

Example 2 (First order ode on an interval)

$L := \mathrm{L}^2(0,1)$, $\mathcal{D} := \mathrm{C}_c^\infty(0,1)$
$T, \widetilde{T} : \mathcal{D} \to L$,

$$T\varphi := \frac{\mathrm{d}}{\mathrm{d}x}\varphi + \varphi \qquad \text{and} \qquad \widetilde{T}\varphi := -\frac{\mathrm{d}}{\mathrm{d}x}\varphi + \varphi\,.$$

We have

$$\operatorname{dom}\overline{T} = \operatorname{dom}\overline{\widetilde{T}} = \mathrm{H}_0^1(0,1) =: W_0$$

$$\operatorname{dom}T^* = \operatorname{dom}\widetilde{T}^* = \mathrm{H}^1(0,1) =: W\,.$$

As $D[u,v] = u(1)\overline{v(1)} - u(0)\overline{v(0)}$, for

$$V := \widetilde{V} := \{u \in \mathrm{H}^1(0,1) : u(0) = u(1)\}$$

we have that $T_\mathrm{r} := \widetilde{T}^*|_V$, $T_\mathrm{r}^* = T^*|_V$ form an adjoint pair of bijective realisations with signed boundary map.
Classification: all adjoint pairs of bijective realisations with signed boundary map

$$\{(T_{\alpha,\beta}, T_{\alpha,\beta}^*) : \alpha \leqslant -e^{-1}\,,\ \beta \in \mathbb{R}\} \cup \{(T_\mathrm{r}, T_\mathrm{r}^*)\}$$

$$\operatorname{dom}T_{\alpha,\beta}^{(*)} = \Big\{u \in \mathrm{H}^1(0,1) : \Big(2e^{-1} - (+)\alpha(1+e) - \mathrm{i}\beta(1+e)\Big)u(1) = \Big(2 + \alpha(1+e) - (+)\mathrm{i}\beta(1+e)\Big)u(0)\Big\}$$

(P1) **Grubb's decomposition :**

$$\operatorname{dom} T_1 = \operatorname{dom} T_r \dot{+} \ker T_1 \ ,$$
$$\operatorname{dom} \widetilde{T}_1 = \operatorname{dom} T_r^* \dot{+} \ker \widetilde{T}_1 \ .$$

Some preliminary results

(P1) **Grubb's decomposition :**

$$\operatorname{dom} T_1 = \operatorname{dom} T_r \dot{+} \ker T_1 \ ,$$
$$\operatorname{dom} \widetilde{T}_1 = \operatorname{dom} T_r^* \dot{+} \ker \widetilde{T}_1 \ .$$

(P2) $(\mathcal{W}, [\cdot|\cdot])$ is indefinite inner product space and

$$\mathcal{W}_0 \subseteq \mathcal{V} \subseteq \mathcal{W} \text{ is closed in } \mathcal{W} \iff \mathcal{V} = \mathcal{V}^{[\perp][\perp]}.$$

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(P3) If $\mathcal{V}, \widetilde{\mathcal{V}} \subset \mathcal{W}$ and $(\mathcal{V}, \widetilde{\mathcal{V}})$ satisfies the condition (V1) then

$$(\forall u \in \mathcal{V}) \quad |\langle T_1 u | u \rangle| \geq \mu_0 \|u\|^2 \,,$$
$$(\forall v \in \widetilde{\mathcal{V}}) \quad |\langle \widetilde{T}_1 v | v \rangle| \geq \mu_0 \|v\|^2 \ .$$

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(P4)

$$\mathcal{H} = \operatorname{ran} T_0 \oplus \ker \widetilde{T}_1 = \operatorname{ran} \widetilde{T}_0 \oplus \ker T_1 \ .$$

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(P2) $(\mathcal{W}, [\cdot|\cdot])$ is indefinite inner product space and

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(P4)

$$\mathcal{H} = \operatorname{ran} T_0 \oplus \ker \widetilde{T}_1 = \operatorname{ran} \widetilde{T}_0 \oplus \ker T_1 \ .$$

(P5) $(\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1, \mathcal{W}_0 \dot{+} \ker T_1)$ satisfies (V1) condition.

Decomposition of the graph space

Theorem

$(T_0, \widetilde{T}_0)$ is a joint pair of closed abstract Friedrichs operators then

$$\mathcal{W} = \mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1 .$$

Corollary

$\left(T_1|_{\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1}, \widetilde{T}_1|_{\mathcal{W}_0 \dot{+} \ker T_1}\right)$ is a pair of mutually adjoint pair of bijective realisations relative to $(T, \tilde{T})$.

Corollary

$\left(T_1|_{\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1}, \widetilde{T}_1|_{\mathcal{W}_0 \dot{+} \ker T_1}\right)$ is a pair of mutually adjoint pair of bijective realisations relative to $(T, \tilde{T})$.

Proof: From (P5) it is sufficient to prove only

$$\mathcal{W}_0 + \ker T_1 = (\mathcal{W}_0 + \ker \widetilde{T}_1)^{[\perp]} \qquad \text{and} \qquad \mathcal{W}_0 + \ker \widetilde{T}_1 = (\mathcal{W}_0 + \ker T_1)^{[\perp]} .$$

Let $u_0, v_0 \in \mathcal{W}_0$, $\nu \in \ker T_1$ and $\tilde{\nu} \in \ker \widetilde{T}_1$ be arbitrary. Then

$$[\, v_0 + \tilde{\nu} \mid u_0 + \nu \,] = \overline{[\, \nu \mid \tilde{\nu} \,]} = \overline{\langle T_1 \nu \mid \tilde{\nu} \rangle} - \overline{\langle \nu \mid \widetilde{T}_1 \tilde{\nu} \rangle} = 0 \, .$$

Thus, $\mathcal{W}_0 + \ker T_1 \subseteq (\mathcal{W}_0 + \ker \widetilde{T}_1)^{[\perp]}$.

Corollary

$\left(T_1|_{\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1}, \widetilde{T}_1|_{\mathcal{W}_0 \dot{+} \ker T_1}\right)$ *is a pair of mutually adjoint pair of bijective realisations relative to* $(T, \tilde{T})$.

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Let $u_0, v_0 \in \mathcal{W}_0$, $\nu \in \ker T_1$ and $\tilde{\nu} \in \ker \widetilde{T}_1$ be arbitrary. Then

$$[\, v_0 + \tilde{\nu} \mid u_0 + \nu \,] = \overline{[\, \nu \mid \tilde{\nu} \,]} = \overline{\langle\, T_1 \nu \mid \tilde{\nu} \,\rangle} - \overline{\langle\, \nu \mid \widetilde{T}_1 \tilde{\nu} \,\rangle} = 0\,.$$

Thus, $\mathcal{W}_0 + \ker T_1 \subseteq (\mathcal{W}_0 + \ker \widetilde{T}_1)^{[\perp]}$.
Let $u \in (\mathcal{W}_0 + \ker \widetilde{T}_1)^{[\perp]}$. By the above theorem there exist $u_0 \in \mathcal{W}_0$, $\nu \in \ker T_1$ and $\tilde{\nu} \in \ker \widetilde{T}_1$ such that $u = u_0 + \nu + \tilde{\nu}$. For any $v_0 \in \mathcal{W}_0$ and $\tilde{\nu}_1 \in \ker \widetilde{T}_1$ we have

$$0 = [\, v_0 + \tilde{\nu}_1 \mid u \,] = [\, v_0 + \tilde{\nu}_1 \mid u_0 + \nu + \tilde{\nu} \,] = [\, \tilde{\nu}_1 \mid \nu \,] + [\, \tilde{\nu}_1 \mid \tilde{\nu} \,] = [\, \tilde{\nu}_1 \mid \tilde{\nu} \,]\,,$$

where we have used $\ker T_1 \subseteq (\ker \widetilde{T}_1)^{[\perp]}$.
Putting $\tilde{\nu}_1 = \tilde{\nu}$ we get

$$0 = [\, \tilde{\nu} \mid \tilde{\nu} \,] = \langle\, T_1 \tilde{\nu} \mid \tilde{\nu} \,\rangle = \langle\, (T_1 + \widetilde{T}_1)\tilde{\nu} \mid \tilde{\nu} \,\rangle \geq 2\mu_0 \|\tilde{\nu}\|^2 \,,$$

where the last inequality is due to condition (T3). Hence, necessarily $\tilde{\nu} = 0$, which implies $u = u_0 + \nu \subseteq \mathcal{W}_0 + \ker T_1$.
The second equation is analogous to the first.
Hence the proof is complete.

Decomposition of the graph space

Lemma

$\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1$ *is direct and closed in* $\mathcal{W}$.
In particular , $\mathcal{W}_0 \dot{+} \ker T_1$ *and* $\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1$ *are direct and closed in* $\mathcal{W}$.

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In particular , $\mathcal{W}_0 \dot{+} \ker T_1$ and $\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1$ are direct and closed in $\mathcal{W}$.

Proof: The second part is just a simple consequence, so let us focus only on the first part. Using (P5) and (P3) we have that the operators $T_1|_{\mathcal{W}_0 + \ker \widetilde{T}_1}$ and $\widetilde{T}_1|_{\mathcal{W}_0 + \ker T_1}$ are $\mathcal{H}$-coercive, hence injective.

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let $u_0 \in \mathcal{W}_0$, $\nu \in \ker T_1$ and $\tilde{\nu} \in \ker \widetilde{T}_1$ be such that $u_0 + \nu + \tilde{\nu} = 0$. Then

$$0 = |T_1(u_0 + \nu + \tilde{\nu})| = |T_1(u_0 + \tilde{\nu})| \geq \mu_0 \|u_0 + \tilde{\nu}\| \,,$$

implying $u_0 + \tilde{\nu} = 0$. Acting by $\widetilde{T}_1$ we get

$$0 = |\widetilde{T}_1(u_0 + \tilde{\nu})| = |\widetilde{T}_1(u_0)| \geq \mu_0 \|u_0\| \,.$$

Thus, $u_0 = 0$, which implies $\tilde{\nu} = 0$, and then finally $\nu = 0$. Which proves that the sum is direct.

Decomposition of the graph space

Let $u_n = u_n^0 + \nu_n + \tilde{\nu}_n \in \mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1$ ($u_n^0 \in \mathcal{W}_0$, $\nu_n \in \ker T_1$, $\tilde{\nu}_n \in \ker \widetilde{T}_1$) converges to $u \in \mathcal{W}$ in graph norm.
$T_1(u_n^0 + \nu_n + \tilde{\nu}_n) = T_1(u_n^0 + \tilde{\nu}_n)$ is a Cauchy sequence in $\mathcal{H}$ and $T_1|_{\mathcal{W}_0 + \ker \widetilde{T}_1}$ is $\mathcal{H}$-coercive, $(u_n^0 + \tilde{\nu}_n)$ is a Cauchy sequence in $\mathcal{H}$ as well, hence converges to some $w \in \mathcal{H}$ and $\nu := u - w \in \mathcal{H}$.

Decomposition of the graph space

Let $u_n = u_n^0 + \nu_n + \tilde{\nu}_n \in \mathcal{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$ ($u_n^0 \in \mathcal{W}_0$, $\nu_n \in \ker T_1$, $\tilde{\nu}_n \in \ker \widetilde{T}_1$) converges to $u \in \mathcal{W}$ in graph norm.
$T_1(u_n^0 + \nu_n + \tilde{\nu}_n) = T_1(u_n^0 + \tilde{\nu}_n)$ is a Cauchy sequence in $\mathcal{H}$ and $T_1|_{\mathcal{W}_0 + \ker \widetilde{T}_1}$ is $\mathcal{H}$-coercive, $(u_n^0 + \tilde{\nu}_n)$ is a Cauchy sequence in $\mathcal{H}$ as well, hence converges to some $w \in \mathcal{H}$ and $\nu := u - w \in \mathcal{H}$.

$$\begin{aligned} \|\nu_n - \nu\| &= \|(u_n^0 + \nu_n + \tilde{\nu}_n) - u - (u_n^0 + \tilde{\nu}_n - w)\| \\ &\leq \|u_n^0 + \nu_n + \tilde{\nu}_n - u\| + \|u_n^0 + \tilde{\nu}_n - w\|, \end{aligned}$$

gives $\lim_n \nu_n = \nu$. T_1 is closed implies $\ker T_1$ is closed in both $\mathcal{H}$ and $\mathcal{W}$, we get $\nu \in \ker T_1$ and $\nu_n \xrightarrow{\mathcal{W}} \nu$.

Let $u_n = u_n^0 + \nu_n + \tilde{\nu}_n \in \mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1$ ($u_n^0 \in \mathcal{W}_0$, $\nu_n \in \ker T_1$, $\tilde{\nu}_n \in \ker \widetilde{T}_1$) converges to $u \in \mathcal{W}$ in graph norm.
$T_1(u_n^0 + \nu_n + \tilde{\nu}_n) = T_1(u_n^0 + \tilde{\nu}_n)$ is a Cauchy sequence in $\mathcal{H}$ and $T_1|_{\mathcal{W}_0 + \ker \widetilde{T}_1}$ is $\mathcal{H}$-coercive, $(u_n^0 + \tilde{\nu}_n)$ is a Cauchy sequence in $\mathcal{H}$ as well, hence converges to some $w \in \mathcal{H}$ and $\nu := u - w \in \mathcal{H}$.

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gives $\lim_n \nu_n = \nu$. T_1 is closed implies $\ker T_1$ is closed in both $\mathcal{H}$ and $\mathcal{W}$, we get $\nu \in \ker T_1$ and $\nu_n \xrightarrow{\mathcal{W}} \nu$.
So far $u_n^0 + \tilde{\nu}_n \xrightarrow{\mathcal{W}} u - \nu$, implying that $\widetilde{T}_1(u_n^0 + \tilde{\nu}_n) = \widetilde{T}_0(u_n^0)$ is a Cauchy sequence in $\mathcal{H}$. Since $\widetilde{T}_0$ is also $\mathcal{H}$-coercive , (u_n^0) is also a Cauchy and hence convergent sequence in $\mathcal{H}$. $\widetilde{T}_0$ is closed implies that (u_n^0) converges to some $u_0 \in \mathcal{W}_0$ (in the graph norm).

Decomposition of the graph space

Let $u_n = u_n^0 + \nu_n + \tilde{\nu}_n \in \mathcal{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$ ($u_n^0 \in \mathcal{W}_0$, $\nu_n \in \ker T_1$, $\tilde{\nu}_n \in \ker \widetilde{T}_1$) converges to $u \in \mathcal{W}$ in graph norm.
$T_1(u_n^0 + \nu_n + \tilde{\nu}_n) = T_1(u_n^0 + \tilde{\nu}_n)$ is a Cauchy sequence in $\mathcal{H}$ and $T_1|_{\mathcal{W}_0 + \ker \widetilde{T}_1}$ is $\mathcal{H}$-coercive, $(u_n^0 + \tilde{\nu}_n)$ is a Cauchy sequence in $\mathcal{H}$ as well, hence converges to some $w \in \mathcal{H}$ and $\nu := u - w \in \mathcal{H}$.

$$\begin{aligned}\|\nu_n - \nu\| &= \|(u_n^0 + \nu_n + \tilde{\nu}_n) - u - (u_n^0 + \tilde{\nu}_n - w)\| \\ &\leq \|u_n^0 + \nu_n + \tilde{\nu}_n - u\| + \|u_n^0 + \tilde{\nu}_n - w\|,\end{aligned}$$

gives $\lim_n \nu_n = \nu$. T_1 is closed implies $\ker T_1$ is closed in both $\mathcal{H}$ and $\mathcal{W}$, we get $\nu \in \ker T_1$ and $\nu_n \xrightarrow{\mathcal{W}} \nu$.
So far $u_n^0 + \tilde{\nu}_n \xrightarrow{\mathcal{W}} u - \nu$, implying that $\widetilde{T}_1(u_n^0 + \tilde{\nu}_n) = \widetilde{T}_0(u_n^0)$ is a Cauchy sequence in $\mathcal{H}$. Since $\widetilde{T}_0$ is also $\mathcal{H}$-coercive , (u_n^0) is also a Cauchy and hence convergent sequence in $\mathcal{H}$. $\widetilde{T}_0$ is closed implies that (u_n^0) converges to some $u_0 \in \mathcal{W}_0$ (in the graph norm).
$\tilde{\nu} := u - u_0 - \nu$. Analogously as for (ν_n), we get that $\tilde{\nu}_n \xrightarrow{\mathcal{W}} \tilde{\nu} \in \ker \widetilde{T}_1$. Thus, $u_n^0 + \nu_n + \tilde{\nu}_n \xrightarrow{\mathcal{W}} u_0 + \nu + \tilde{\nu}$. Uniqueness of the limit finally implies $u = u_0 + \nu + \tilde{\nu} \in \mathcal{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$.
Which completes the proof.

Decomposition involving a reference operator

Lemma

For any bijective realisation T_{r},

$$\mathcal{W} = \mathcal{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \dotplus \ker T_1 = \mathcal{W}_0 \dotplus (T_{\mathrm{r}}^*)^{-1}(\ker \widetilde{T}_1) \dotplus \ker T_1 \ .$$

Lemma

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Proof: From (P1) we have

$$\mathcal{W} = \mathrm{dom}\, T_{\mathrm{r}} \dotplus \ker T_1 \ .$$

Sufficient to prove

$$\mathrm{dom}\, T_{\mathrm{r}} \ = \ \mathcal{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \ .$$

Here $T_0 \subset T_{\mathrm{r}}$ and $T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \subset \mathrm{dom}\, T_{\mathrm{r}}$. So, $\mathcal{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \subset \mathrm{dom}\, T_{\mathrm{r}}$.

Decomposition involving a reference operator

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For any bijective realisation T_{r}*,*

$$\mathcal{W} = \mathcal{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \dotplus \ker T_1 = \mathcal{W}_0 \dotplus (T_{\mathrm{r}}^*)^{-1}(\ker \widetilde{T}_1) \dotplus \ker T_1 \ .$$

Proof: From (P1) we have

$$\mathcal{W} = \mathrm{dom}\, T_{\mathrm{r}} \dotplus \ker T_1 \ .$$

Sufficient to prove

$$\mathrm{dom}\, T_{\mathrm{r}} \ = \ \mathcal{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \ .$$

Here $T_0 \subset T_{\mathrm{r}}$ and $T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \subset \mathrm{dom}\, T_{\mathrm{r}}$. So, $\mathcal{W}_0 \dotplus T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \subset \mathrm{dom}\, T_{\mathrm{r}}$.
Let $u \in \mathrm{dom}\, T_{\mathrm{r}}$. Since $T_{\mathrm{r}} u \in \mathcal{H}$ by (P4) , for some $u_0 \in \mathcal{W}_0$, $\tilde{\nu} \in \ker \widetilde{T}_1$.

$$u = T_0 u_0 + \tilde{\nu} = T_{\mathrm{r}} u_0 + \tilde{\nu} \ .$$

$T1|_{\mathrm{dom}\, T_{\mathrm{r}}} = T_{\mathrm{r}}$ is a bijection, we have

$$u = T_{\mathrm{r}}^{-1} T_{\mathrm{r}} u = T_r^{-1}(T_{\mathrm{r}} u_0 + \tilde{\nu}) = u_0 + T_{\mathrm{r}}^{-1}(\tilde{\nu}) \implies u \in \mathcal{W}_0 + T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \ .$$

Decomposition involving a reference operator

Let $u_0 \in \mathcal{W}_0$ and $\tilde{\nu} \in \ker \widetilde{T}_1$ such that $u_0 + T_{\mathrm{r}}^{-1}(\tilde{\nu}) = 0$. Then

$$T_{\mathrm{r}}(u_0 + \tilde{\nu}) = 0 \implies \tilde{\nu} = -T_{\mathrm{r}}(u_0) = -T_0(u_0)$$

Which means $\tilde{\nu} \in \ker \widetilde{T}_1 \cap \operatorname{ran} T_0 = \{0\}$. So, $\tilde{\nu} = 0$ and by injectivity of T_0 $u_0 = 0$ as well. Hence the decomposition is direct.

Let $u_0 \in \mathcal{W}_0$ and $\tilde{\nu} \in \ker \widetilde{T}_1$ such that $u_0 + T_{\mathrm{r}}^{-1}(\tilde{\nu}) = 0$. Then

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Which means $\tilde{\nu} \in \ker \widetilde{T}_1 \cap \operatorname{ran} T_0 = \{0\}$. So, $\tilde{\nu} = 0$ and by injectivity of T_0 $u_0 = 0$ as well. Hence the decomposition is direct.
We get the second equality by replacing the role of T_0 by $\widetilde{T}_0$, because of symmetry condition (T1).
Which completes the proof.

Decomposition of the graph space

Lemma

$$\mathcal{W} = \left(\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1\right)^{[\perp][\perp]}.$$

Lemma

$\mathcal{W} = \left(\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1\right)^{[\perp][\perp]}.$

Proof: Since $\mathcal{W}_0^{[\perp]} = \mathcal{W}$, it is sufficient to prove

$$\left(\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1\right)^{[\perp]} = \mathcal{W}_0 .$$

$\ker D = \mathcal{W}_0 \implies \mathcal{W}_0 \subseteq \left(\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1\right)^{[\perp]}.$

Let $u \in \left(\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1\right)^{[\perp]} \subseteq \mathcal{W}$, so by previous lemma

$$\exists!\; u_0 + T_{\mathrm{r}}^{-1}(\tilde{\nu}) + \nu = u \in \mathcal{W}_0 \dot{+} T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \dot{+} \ker T_1 = \mathcal{W} .$$

Let $v_0 + \nu_1 + \tilde{\nu}_1 \in \mathcal{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$, then

(3)
$$
\begin{aligned}
0 = [\, u \mid v_0 + \nu_1 + \tilde{\nu}_1 \,] &= [\, u_0 + T_{\mathrm{r}}^{-1}(\tilde{\nu}) + \nu \mid v_0 + \nu_1 + \tilde{\nu}_1 \,] \\
&= [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) + \nu \mid \nu_1 + \tilde{\nu}_1 \,] \\
&= [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \nu_1 \,] + [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \tilde{\nu}_1 \,] + [\, \nu \mid \nu_1 \,] + [\, \nu \mid \tilde{\nu}_1 \,] \\
&= [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \nu_1 \,] + [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \tilde{\nu}_1 \,] + [\, \nu \mid \nu_1 \,] \,,
\end{aligned}
$$

for $\nu_1 = 0$ and $\tilde{\nu}_1 = \tilde{\nu}$ we get

$$
0 = [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \tilde{\nu} \,] = \langle\, \tilde{\nu} \mid \tilde{\nu} \,\rangle = \|\tilde{\nu}\|^2 \implies \tilde{\nu} = 0 \,.
$$

Let $v_0 + \nu_1 + \tilde{\nu}_1 \in \mathcal{W}_0 \dotplus \ker T_1 \dotplus \ker \widetilde{T}_1$, then

(3)
$$\begin{aligned} 0 = [\, u \mid v_0 + \nu_1 + \tilde{\nu}_1 \,] &= [\, u_0 + T_{\mathrm{r}}^{-1}(\tilde{\nu}) + \nu \mid v_0 + \nu_1 + \tilde{\nu}_1 \,] \\ &= [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) + \nu \mid \nu_1 + \tilde{\nu}_1 \,] \\ &= [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \nu_1 \,] + [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \tilde{\nu}_1 \,] + [\, \nu \mid \nu_1 \,] + [\, \nu \mid \tilde{\nu}_1 \,] \\ &= [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \nu_1 \,] + [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \tilde{\nu}_1 \,] + [\, \nu \mid \nu_1 \,] \,, \end{aligned}$$

for $\nu_1 = 0$ and $\tilde{\nu}_1 = \tilde{\nu}$ we get

$$0 = [\, T_{\mathrm{r}}^{-1}(\tilde{\nu}) \mid \tilde{\nu} \,] = \langle\, \tilde{\nu} \mid \tilde{\nu} \,\rangle = \|\tilde{\nu}\|^2 \implies \tilde{\nu} = 0 \,.$$

From (3) again, taking $\nu_1 = \nu$ and using $\widetilde{T}_1|_{\mathcal{W}_0 + \ker T_1}$ is $\mathcal{H}$-coercive ((P5) and (P3)) we get

$$0 = |[\, \nu \mid \nu \,]| = |\langle\, \widetilde{T}_1 \nu \mid \nu \,\rangle| \geq \mu_0 \|\nu\|^2 \implies \nu = 0 \,.$$

So, $u = u_0 \in \mathcal{W}_0$. Which completes the proof.

Proof of the decomposition

Proof.

We have $\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1$ is direct and closed in $\mathcal{W}$ and by (P2) we have

$$\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1 = \left(\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1\right)^{[\perp][\perp]}$$

Which is $\mathcal{W}$ by previous lemma.

□

...thank you for your attention :)

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