Friedrichs operators as dual pairs

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Classical Friedrichs operators

Assumptions:
$d, r \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary;
$\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega)^{r\times r}$, $k \in \{1, \dots, d\}$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega)^{r\times r}$ satisfying (a.e. on Ω):

(F1) $$\mathbf{A}_k = \mathbf{A}_k^* \,;$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant \mu_0 \mathbf{I} \,.$$

Define $\mathcal{L}, \widetilde{\mathcal{L}} : \mathrm{L}^2(\Omega)^r \to \mathcal{D}'(\Omega)^r$ by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} \,, \qquad \widetilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \Big(\mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\Big)\mathsf{u} \,.$$

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Aim: impose boundary conditions such that for any $\mathsf{f} \in \mathrm{L}^2(\Omega)^r$ we have a unique solution of $\mathcal{L}\mathsf{u} = \mathsf{f}$.
Gain: many important (semi)linear equations of mathematical physics can be written in the form of classical Friedrichs operators.

The classical theory in short

K. O. Friedrichs: *Symmetric positive linear differential equations*, Commun. Pure Appl. Math. **11** (1958) 333–418.

Unified treatment of linear hyperbolic systems like Maxwell's, Dirac's, or higher order equations (e.g. the wave equation).

– Contributions: C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, . . .
– treating the equations of mixed type, such as the Tricomi equation:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;$$

– unified treatment of equations and systems of different type;
– more recently: better numerical properties.

Shortcommings:
– no satisfactory well-posedness result,
– no intrinsic (unique) way to pose boundary conditions.

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$\rightsquigarrow$ development of the abstract theory

Abstract Friedrichs operators

$(L, \langle \cdot \mid \cdot \rangle)$ complex Hilbert space ($L' \equiv L$), $\| \cdot \| := \sqrt{\langle \cdot \mid \cdot \rangle}$
$\mathcal{D} \subseteq L$ dense subspace

Definition

Let $T, \widetilde{T} : \mathcal{D} \to L$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

(T1) $$(\forall\, \phi, \psi \in \mathcal{D}) \qquad \langle\, T\phi \mid \psi \,\rangle = \langle\, \phi \mid \widetilde{T}\psi \,\rangle\,;$$

(T2) $$(\exists\, c > 0)(\forall\, \phi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\phi\| \leqslant c\|\phi\|\,;$$

(T3) $$(\exists\, \mu_0 > 0)(\forall\, \phi \in \mathcal{D}) \qquad \langle\, (T + \widetilde{T})\phi \mid \phi \,\rangle \geqslant \mu_0 \|\phi\|^2\,.$$

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems*, Comm. Partial Diff. Eq. **32** (2007) 317–341.

N. Antonić, K. Burazin: *Intrinsic boundary conditions for Friedrichs systems*, Comm. Partial Diff. Eq. **35** (2010) 1690–1715.

Classical is *abstract*

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$\mathcal{D} := \mathrm{C}_c^\infty(\Omega)^r$, $L := \mathrm{L}^2(\Omega)^r$, and

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(T1) $\langle\, T\mathsf{u} \mid \mathsf{v}\,\rangle_{\mathrm{L}^2} = \langle\, \mathsf{u} \mid -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k^*\mathsf{v}) + \big(\mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\big)\mathsf{v}\,\rangle_{\mathrm{L}^2} \overset{\text{(F1)}}{=} \langle\, \mathsf{u} \mid \widetilde{T}\mathsf{v}\,\rangle_{\mathrm{L}^2}\,.$

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Since $(T+\widetilde{T})\mathsf{u} = \Big(\mathbf{C} + \mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\Big)\mathsf{u}$,

(T2) $\|(T+\widetilde{T})\mathsf{u}\|_{\mathrm{L}^2} \leqslant \Big(2\|\mathbf{C}\|_{\mathrm{L}^\infty} + \sum_{k=1}^{d} \|\mathbf{A}_k\|_{\mathrm{W}^{1,\infty}}\Big)\|\mathsf{u}\|_{\mathrm{L}^2}\,,$

(T3) $\langle\, (T+\widetilde{T})\mathsf{u} \mid \mathsf{u}\,\rangle_{\mathrm{L}^2} \overset{\rm (F2)}{\geqslant} \mu_0 \|\mathsf{u}\|_{\mathrm{L}^2}^2\,.$

Well-posedness result

Goal: For $(T, \widetilde{T})$ satisfying (T1)–(T3) find $V \supseteq \mathcal{D}$ ($\widetilde{V} \supseteq \mathcal{D}$) such that T ($\widetilde{T}$) extended to V ($\widetilde{V}$) is a linear bijection.

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$\exists$ *maximal operators* :

$$\begin{aligned} T_1 &: W \subseteq L \to L \,, \quad T \subseteq T_1 \,, \\ \widetilde{T}_1 &: W \subseteq L \to L \,, \quad \widetilde{T} \subseteq \widetilde{T}_1 \,. \end{aligned} \qquad (\operatorname{dom} T_1 = \operatorname{dom} \widetilde{T}_1 =: W)$$

Boundary map (form):

$$\begin{aligned} & D : W \times W \to \mathbb{C} \,, \\ & D[u,v] := \langle\, T_1 u \mid v \,\rangle - \langle\, u \mid \widetilde{T}_1 v \,\rangle \,. \end{aligned} \qquad (D[u,v] = \overline{D[v,u]})$$

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For $V, \widetilde{V} \subseteq W$ we introduce two conditions:

(V1)
$$(\forall\, u \in V) \qquad D[u,u] \geqslant 0$$
$$(\forall\, v \in \widetilde{V}) \qquad D[v,v] \leqslant 0$$

(V2)
$$V = \{u \in W : (\forall\, v \in \widetilde{V}) \quad D[v,u] = 0\}$$
$$\widetilde{V} = \{v \in W : (\forall\, u \in V) \quad D[u,v] = 0\}$$
$$(\implies \mathcal{D} \subseteq V \cap \widetilde{V})$$

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Theorem (Ern, Guermond, Caplain, 2007)

(T1)–(T3) + (V1)–(V2) $\implies$ $T_1|_V, \widetilde{T}_1|_{\widetilde{V}}$ *bijective realisations*

Example 1 (Scalar elliptic PDE)

$\Omega \subseteq \mathbb{R}^d$, $\mu > 0$ and $f \in \mathrm{L}^2(\Omega)$ given.

$$-\triangle u + \mu u = f \iff -\operatorname{div} \nabla u + \mu u = f \iff \begin{cases} \nabla u + \mathsf{p} = 0 \\ \operatorname{div} \mathsf{p} + \mu u = f \end{cases}$$

$$\iff T\mathsf{v} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{v}) + \mathbf{C}\mathsf{v} = \mathsf{g}\,,$$

where $\mathsf{v} := [\mathsf{p}\ u]^\top$, $\mathsf{g} := [0\ f]^\top$, $(\mathbf{A}_k)_{ij} := \delta_{i,k}\delta_{j,d+1} + \delta_{i,d+1}\delta_{j,k}$, $\mathbf{C} := \mathrm{diag}\{1, \ldots, 1, \mu\}$.
Assumtions (F1) and (F2) are satisfied.

$L = \mathrm{L}^2(\Omega)^{d+1}$, $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$

- $V = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)$ ... Dirichelt boundary condition ($u = 0$ on Γ)
- $V = \mathrm{L}^2_{\mathrm{div},0}(\Omega) \times \mathrm{H}^1(\Omega)$ ... Neumann boundary condition ($\mathsf{p} \cdot \nu = \nabla u \cdot \nu = 0$ on Γ)

Theorem

$$(T1)-(T3) \iff \begin{cases} T \subseteq \widetilde{T}^* \quad \& \quad \widetilde{T} \subseteq T^*; \\ \overline{T+\widetilde{T}} \text{ bounded self-adjoint in } L \text{ with strictly positive bottom}; \\ \operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} \quad \& \quad \operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* . \end{cases}$$

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$T_1 = \widetilde{T}^*$ *and* $\widetilde{T}_1 = T^*$.

$$T \subseteq \widetilde{T}^*|_V \subseteq \widetilde{T}^* \qquad \& \qquad \widetilde{T} \subseteq T^*|_{\widetilde{V}} \subseteq T^* .$$

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Theorem

If $(T, \widetilde{T})$ satisfies (T1)–(T2), then

$$(V2) \iff \begin{cases} \mathcal{D} \subseteq V, \widetilde{V} \subseteq W \\ (\widetilde{T}^*|_V)^* = T^*|_{\widetilde{V}} \\ (T^*|_{\widetilde{V}})^* = \widetilde{T}^*|_V . \end{cases}$$

Bijective realisations with signed boundary map

We are seeking for bijective closed operators $S \equiv \widetilde{T}^*|_V$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^* \,,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$. If $(\operatorname{dom} S, \operatorname{dom} S^*)$ satisfies $(V1)$ we call (S, S^*) an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.

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Questions:

1) Existence of such V
2) Infinity of such V
3) Classification of such V

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Theorem

Let $(T, \widetilde{T})$ satisfies (T1)–(T3).

(i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.*

(ii)

$\ker \widetilde{T}^* \neq \{0\}$ & $\ker T^* \neq \{0\}$	$\Longrightarrow$	*uncountably many adjoint pairs of bijective realisations with signed boundary map*
$\ker \widetilde{T}^* = \{0\}$ *or* $\ker T^* = \{0\}$	$\Longrightarrow$	*only one adjoint pair of bijective realisations with signed boundary map*

Classification

For $(T, \widetilde{T})$ satisfying (T1)–(T3) we have

$$\overline{T} \subseteq \widetilde{T}^* \qquad \text{and} \qquad \overline{\widetilde{T}} \subseteq T^* ,$$

while by the previous theorem there exists closed T_{r} such that

- $\overline{T} \subseteq T_{\mathrm{r}} \subseteq \widetilde{T}^*$ ($\iff \overline{\widetilde{T}} \subseteq T_{\mathrm{r}}^* \subseteq T^*$),
- $T_{\mathrm{r}} : \mathrm{dom}\, T_{\mathrm{r}} \to L$ bijection,
- $(T_{\mathrm{r}})^{-1} : L \to \mathrm{dom}\, T_{\mathrm{r}}$ bounded.

Thus, we can apply a universal classification (classification of dual (adjoint) pairs).

We used Grubb's universal classification

G. Grubb: *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa **22** (1968) 425–513.

Result: complete classification of all adjoint pairs of bijective realisations with signed boundary map.
To do: apply this result to general classical Friedrichs operators form the beginning (*nice class of non-self-adjoint differential operators of interest*)

Example 2 (First order ode on an interval)

$L := \mathrm{L}^2(0,1)$, $\mathcal{D} := \mathrm{C}_c^\infty(0,1)$
$T, \widetilde{T} : \mathcal{D} \to L$,

$$T\phi := \frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi \qquad \text{and} \qquad \widetilde{T}\phi := -\frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi\,.$$

We have

$$\operatorname{dom}\overline{T} = \operatorname{dom}\overline{\widetilde{T}} = \mathrm{H}_0^1(0,1) =: W_0$$
$$\operatorname{dom}T^* = \operatorname{dom}\widetilde{T}^* = \mathrm{H}^1(0,1) =: W\,.$$

As $D[u,v] = u(1)\overline{v(1)} - u(0)\overline{v(0)}$, for

$$V := \widetilde{V} := \{u \in \mathrm{H}^1(0,1) : u(0) = u(1)\}$$

we have that $T_{\mathrm{r}} := \widetilde{T}^*|_V$, $T_{\mathrm{r}}^* = T^*|_V$ form an adjoint pair of bijective realisations with signed boundary map.
<u>Classification:</u> all adjoint pairs of bijective realisations with signed boundary map

$$\{(T_{\alpha,\beta}, T_{\alpha,\beta}^*) : \alpha \leqslant -e^{-1}\,,\ \beta \in \mathbb{R}\} \cup \{(T_{\mathrm{r}}, T_{\mathrm{r}}^*)\}$$

$$\operatorname{dom}T_{\alpha,\beta}^{(*)} = \Big\{u \in \mathrm{H}^1(0,1) : \Big(2e^{-1} - (+)\alpha(1+e) - \mathrm{i}\beta(1+e)\Big)u(1) = \Big(2 + \alpha(1+e) - (+)\mathrm{i}\beta(1+e)\Big)u(0)\Big\}$$

...thank you for your attention :)

N. Antonić, M.E., A. Michelangeli: *Friedrichs systems in a Hilbert space framework: solvability and multiplicity*, J. Differential Equations 263 (2017) 8264-8294.

M.E., A. Michelangeli: *On contact interactions realised as Friedrichs systems*, Complex Analysis and Operator Theory (2018) https://doi.org/10.1007/s11785-018-0787-4