One-scale H-distributions

Marko Erceg

Department of Mathematics, Faculty of Science
University of Zagreb

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Joint work with Nenad Antonić

$\Omega \subseteq \mathbf{R}^d$ open

Theorem

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ *and* $\omega_n \to 0^+$, *then there exist* $(u_{n'})$, $(v_{n'})$ *and* $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 v_{n'}}(\boldsymbol{\xi})} \psi(\omega_{n'}\boldsymbol{\xi})\, d\boldsymbol{\xi} = \langle \mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,.$$

The measure $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ *is called the one-scale H-measure with characteristic length* $(\omega_{n'})$ *associated to the (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

Luc Tartar: *The general theory of homogenization: A personalized introduction, Springer* (2009)
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The distribution $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ *is called the one-scale H-distribution with characteristic length* $(\omega_{n'})$ *associated to the (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

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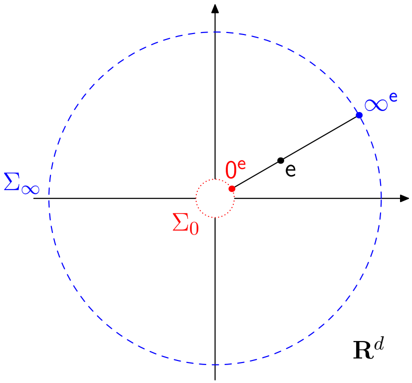
Determine E such that

- $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is continuous
- The First commutation lemma is valid

$\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ is a compactification of $\mathbf{R}^d_*$ homeomorphic to a spherical layer (i.e. an annulus in $\mathbf{R}^2$):



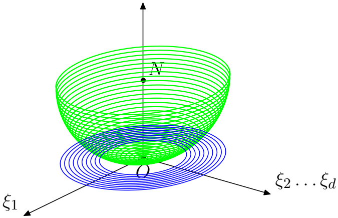
We shall need a differential structure on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

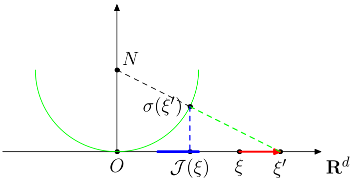
For fixed $r_0 > 0$ let us define $r_1 = \frac{r_0}{\sqrt{r_0^2+1}}$, and denote by

$$A[0, r_1, 1] := \left\{\boldsymbol{\zeta} \in \mathbf{R}^d : r_1 \leqslant |\boldsymbol{\zeta}| \leqslant 1\right\}$$

closed d-dimensional spherical layer equipped with the standard topology (inherited from $\mathbf{R}^d$). In addition let us define $A(0, r_1, 1) := \operatorname{Int} A[0, r_1, 1]$, and by $A_0[0, r_1, 1] := \mathrm{S}^{d-1}(0; r_1)$ and $A_\infty[0, r_1, 1] := \mathrm{S}^{d-1}$ we denote boundary spheres.

We want to construct a homeomorphism $\mathcal{J} : \mathbf{R}^d_* \longrightarrow A(0, r_1, 1)$.





From the previous construction we get that $\mathcal{J} : \mathbf{R}^d_* \longrightarrow A(0, r_1, 1)$ is given by

$$\mathcal{J}(\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}}{\sqrt{|\boldsymbol{\xi}|^2 + \left(\frac{|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|+r_0}\right)^2}} = \frac{|\boldsymbol{\xi}| + r_0}{|\boldsymbol{\xi}| K(\boldsymbol{\xi})}\boldsymbol{\xi}\,,$$

where $K(\boldsymbol{\xi}) = K(|\boldsymbol{\xi}|) := \sqrt{1 + (|\boldsymbol{\xi}| + r_0)^2}$.
$\boldsymbol{\xi}$ and $\mathcal{J}(\boldsymbol{\xi})$ lie on the same line:

$$\frac{\mathcal{J}(\boldsymbol{\xi})}{|\mathcal{J}(\boldsymbol{\xi})|} = \frac{\frac{|\boldsymbol{\xi}|+r_0}{|\boldsymbol{\xi}|K(\boldsymbol{\xi})}\boldsymbol{\xi}}{\frac{|\boldsymbol{\xi}|+r_0}{|\boldsymbol{\xi}|K(\boldsymbol{\xi})}|\boldsymbol{\xi}|} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\,.$$

$\mathcal{J}$ is homeomorphism and its inverse $\mathcal{J}^{-1} : A(0, r_1, 1) \longrightarrow \mathbf{R}^d_*$ is given by

$$\mathcal{J}^{-1}(\boldsymbol{\zeta}) = \frac{|\boldsymbol{\zeta}| - r_0\sqrt{1 - |\boldsymbol{\zeta}|^2}}{|\boldsymbol{\zeta}|\sqrt{1 - |\boldsymbol{\zeta}|^2}}\boldsymbol{\zeta} = \boldsymbol{\zeta}(1 - |\boldsymbol{\zeta}|^2)^{-\frac{1}{2}} - r_0\boldsymbol{\zeta}|\boldsymbol{\zeta}|^{-1}\,,$$

resulting that $(A[0, r_1, 1], \mathcal{J})$ is a compactification of $\mathbf{R}^d_*$.

Now we define

$$\Sigma_0 := \{0^{\mathsf{e}} : \mathsf{e} \in \mathrm{S}^{d-1}\} \qquad \text{and} \qquad \Sigma_\infty := \{\infty^{\mathsf{e}} : \mathsf{e} \in \mathrm{S}^{d-1}\}\,,$$

and $\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d_* \cup \Sigma_0 \cup \Sigma_\infty$.
Let us extend $\mathcal{J}$ to the whole $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ by $\mathcal{J}(0^{\mathsf{e}}) := r_1\mathsf{e}$ and $\mathcal{J}(\infty^{\mathsf{e}}) = \mathsf{e}$, which gives $\mathcal{J}^{\rightarrow}(\Sigma_0) = A_0[0, r_1, 1]$ and $\mathcal{J}^{\rightarrow}(\Sigma_\infty) = A_\infty[0, r_1, 1]$.
$d_*(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) := |\mathcal{J}(\boldsymbol{\xi}_1) - \mathcal{J}(\boldsymbol{\xi}_2)|$ is a metric on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$, so $(\mathrm{K}_{0,\infty}(\mathbf{R}^d), d_*)$ is a metric space isomorphic to $A[0, r_1, 1]$.

$$\lim_{|\boldsymbol{\xi}|\to 0}\left|\mathcal{J}(\boldsymbol{\xi}) - \mathcal{J}(0^{\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}})\right| = 0\;, \quad \lim_{|\boldsymbol{\xi}|\to \infty}\left|\mathcal{J}(\boldsymbol{\xi}) - \mathcal{J}(\infty^{\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}})\right| = 0\,,$$

$$\lim_{|\boldsymbol{\zeta}|\to r_1}|\mathcal{J}^{-1}(\boldsymbol{\zeta})| = 0\;, \quad \lim_{|\boldsymbol{\zeta}|\to 1}|\mathcal{J}^{-1}(\boldsymbol{\zeta})| = +\infty\,.$$

Lemma

For $\psi : \mathrm{K}_{0,\infty}(\mathbf{R}^d) \longrightarrow \mathbf{C}$ *the following is equivalent:*

a) $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$,

b) $(\exists \tilde{\psi} \in \mathrm{C}(A[0, r_1, 1]))\ \psi = \tilde{\psi} \circ \mathcal{J}$,

c) $\psi|_{\mathbf{R}^d_*} \in \mathrm{C}(\mathbf{R}^d_*)$, *and*

$$\lim_{|\boldsymbol{\xi}|\to 0} |\psi(\boldsymbol{\xi}) - \psi(0^{\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}})| = 0 \qquad \textit{and} \qquad \lim_{|\boldsymbol{\xi}|\to \infty} |\psi(\boldsymbol{\xi}) - \psi(\infty^{\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}})| = 0\,.$$

For $\psi \in \mathrm{C}(\mathbf{R}^d_*)$ we have $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ iff there exist $\psi_0, \psi_\infty \in \mathrm{C}(\mathrm{S}^{d-1})$ such that

$$\psi(\boldsymbol{\xi}) - \psi_0\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \to 0\,, \quad |\boldsymbol{\xi}| \to 0\,,$$

$$\psi(\boldsymbol{\xi}) - \psi_\infty\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \to 0\,, \quad |\boldsymbol{\xi}| \to \infty\,.$$

In particular, $\psi - \psi_0(\frac{\cdot}{|\cdot|}) \in \mathrm{C}_{ub}(\mathbf{R}^d)$ (uniformly continuous bounded functions).

For $\kappa \in \mathbf{N}_0 \cup \{\infty\}$ let us define

$$\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) := \left\{\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) : \psi^* := \psi \circ \mathcal{J}^{-1} \in \mathrm{C}^\kappa(A[0, r_1, 1])\right\}.$$

It is not hard to check that $\mathrm{C}^0(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ coincide.
For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we define $\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))} := \|\psi^*\|_{\mathrm{C}^\kappa(A[0,r_1,1])}$.

$$\mathrm{C}^\kappa(A[0, r_1, 1]) \quad \text{Banach algebra} \quad \Longrightarrow \quad \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) \quad \text{Banach algebra}$$

$$\begin{aligned} A[0, r_1, 1] \quad \text{compact} \quad &\Longrightarrow \quad \mathrm{C}^\kappa(A[0, r_1, 1]) \quad \text{separable} \\ &\Longrightarrow \quad \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) \quad \text{separable} \end{aligned}$$

Is $\mathcal{A}_\psi = (\psi \hat{\cdot})^\vee : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ continuous?

Theorem (Hörmander-Mihlin)

If for $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ *there exists* $C > 0$ *such that*

$$(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d_*)(\forall\, \boldsymbol{\alpha} \in \mathbf{N}^d_0,\ |\boldsymbol{\alpha}| \leqslant \kappa) \qquad |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})| \leqslant \frac{C}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}}\,,$$

where $\kappa = \lfloor \frac{d}{2} \rfloor + 1$, *then* ψ *is a Fourier multiplier. Moreover, we have*

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_d \max\Big\{p, \frac{1}{p-1}\Big\} C\,.$$

We shall use *Faá di Bruno formula*: for sufficiently smooth functions $\mathrm{g} : \mathbf{R}^d \longrightarrow \mathbf{R}^r$ and $f : \mathbf{R}^r \longrightarrow \mathbf{R}$ we have

$$\partial^{\boldsymbol{\alpha}}(f \circ \mathrm{g})(\boldsymbol{\xi}) = |\boldsymbol{\alpha}|! \sum_{1 \leqslant |\boldsymbol{\beta}| \leqslant |\boldsymbol{\alpha}|\,,\ \boldsymbol{\beta} \in \mathbf{N}^r_0} C(\boldsymbol{\beta}, \boldsymbol{\alpha})\,,$$

where

$$C(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \frac{(\partial^{\boldsymbol{\beta}} f)(\mathrm{g}(\boldsymbol{\xi}))}{\boldsymbol{\beta}!} \sum_{\substack{\sum_{i=1}^r \boldsymbol{\alpha}_i = \boldsymbol{\alpha}\,, \\ \boldsymbol{\alpha}_i \in \mathbf{N}^d_0}} \prod_{j=1}^r \sum_{\substack{\sum_{i=1}^{\beta_j} \boldsymbol{\gamma}_i = \boldsymbol{\alpha}_j\,, \\ \boldsymbol{\gamma}_i \in \mathbf{N}^d_0 \setminus \{0\}}} \prod_{s=1}^{\beta_j} \frac{\partial^{\boldsymbol{\gamma}_s} g_j(\boldsymbol{\xi})}{\boldsymbol{\gamma}_s!}\,.$$

Lemma

For every $j \in 1..d$ *and* $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ *we have*

$$\partial^{\boldsymbol{\alpha}}(\mathcal{J}_j)(\boldsymbol{\xi}) = P_{\boldsymbol{\alpha}}\Big(\boldsymbol{\xi}, \frac{1}{|\boldsymbol{\xi}|}\Big) K(\boldsymbol{\xi})^{-1-2|\boldsymbol{\alpha}|}\,, \quad \boldsymbol{\xi} \in \mathbf{R}_*^d\,,$$

where $P_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \eta)$ *is a polynomial of degree less or equal to* $|\boldsymbol{\alpha}| + 1$ *in* $\boldsymbol{\xi}$ *and* $2|\boldsymbol{\alpha}| + 1$ *in* η*, in addition that in the expression* $\lambda^{|\boldsymbol{\alpha}|} P_{\boldsymbol{\alpha}}\Big(\lambda, \ldots, \lambda, \frac{1}{\lambda}\Big)$ *we do not have terms of the negative order. Precisely, polynomial* $P_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \eta)$ *has only terms of the form* $C\boldsymbol{\xi}^{\boldsymbol{\beta}}\eta^k$ *where* $|\boldsymbol{\beta}| + |\boldsymbol{\alpha}| \geqslant k$*.*

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$$|\partial^{\boldsymbol{\alpha}}(\mathcal{J}_j)(\boldsymbol{\xi})| \leqslant \frac{C_{\boldsymbol{\alpha},d}}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}}\,, \quad \boldsymbol{\xi} \in \mathbf{R}_*^d\,.$$

Theorem

Let $\kappa \in \mathbf{N}_0$*. For every* $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *and* $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ *such that* $|\boldsymbol{\alpha}| \leqslant \kappa$ *we have*

$$|\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})| \leqslant C_{\kappa,d}\frac{\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}}\,, \quad \boldsymbol{\xi} \in \mathbf{R}_*^d\,.$$

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Therefore, for $\kappa \geqslant \lfloor \frac{d}{2} \rfloor + 1$ and $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we have

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_{d,p} \|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))} .$$

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Lemma

i) $\mathcal{S}(\mathbf{R}^d) \hookrightarrow \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *and*
ii) $\{\psi \circ \boldsymbol{\pi} \ : \ \psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})\} \hookrightarrow \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

Commutation lemma

$B_\varphi u := \varphi u$, $\mathcal{A}_\psi u := (\psi\hat{u})^\vee$.

Lemma

Let $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\kappa \geqslant \lfloor \frac{d}{2} \rfloor + 1$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, *and denote* $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. *Then the commutator can be expressed as a sum*

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K\,,$$

where for any $p \in \langle 1, \infty \rangle$ *we have that* K *is a compact operator on* $\mathrm{L}^p(\mathbf{R}^d)$, *while* $\tilde{C}_n \longrightarrow 0$ *in the operator norm on* $\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))$.

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Dem.

$$[B_\varphi, \mathcal{A}_{\psi_n}] = \underbrace{[B_\varphi, \mathcal{A}_{\psi_n - \psi_0 \circ \boldsymbol{\pi}}]}_{\tilde{C}_n} + \underbrace{[B_\varphi, \mathcal{A}_{\psi_0 \circ \boldsymbol{\pi}}]}_{K} ,$$

where $\boldsymbol{\pi}(\boldsymbol{\xi}) := \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$ and

$$\psi(\boldsymbol{\xi}) - (\psi_0 \circ \boldsymbol{\pi})(\boldsymbol{\xi}) \longrightarrow 0 \, , \quad |\boldsymbol{\xi}| \to 0 \, .$$

Let $r \in \langle 1, \infty \rangle$ and $\theta \in \langle 0, 1 \rangle$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r}$.

Proof of Comm. Lemma: $\tilde{C}_n := [B_\varphi, \mathcal{A}_{\psi_n - \psi_0 \circ \boldsymbol{\pi}}]$

$$\psi_n - \psi_0 \circ \boldsymbol{\pi} \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) \quad \Longrightarrow \quad \tilde{C}_n \text{ bounded on } \mathrm{L}^r(\mathbf{R}^d)$$

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$$\psi_n - \psi_0 \circ \boldsymbol{\pi} \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) \quad \Longrightarrow \quad \tilde{C}_n \text{ bounded on } \mathrm{L}^r(\mathbf{R}^d)$$

Lemma (Tartar, 2009)

Let $\psi \in \mathrm{C}_{ub}(\mathbf{R}^d)$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, and denote $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. Then the commutator $C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = B_\varphi \mathcal{A}_{\psi_n} - \mathcal{A}_{\psi_n} B_\varphi$ tends to zero in the operator norm on $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$.

Proof of Comm. Lemma: $\tilde{C}_n := [B_\varphi, \mathcal{A}_{\psi_n - \psi_0 \circ \boldsymbol{\pi}}]$

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By the Riesz-Thorin interpolation theorem we have

$$\|\tilde{C}_n\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant \|\tilde{C}_n\|^{\theta}_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))} \|\tilde{C}_n\|^{1-\theta}_{\mathcal{L}(\mathrm{L}^r(\mathbf{R}^d))},$$

implying $\tilde{C}_n \longrightarrow 0$ in the operator norm on $\mathrm{L}^p(\mathbf{R}^d)$.

Proof of Comm. Lemma: $K := [B_\varphi, \mathcal{A}_{\psi_0 \circ \boldsymbol{\pi}}]$

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Lemma (Tartar, 1990)

For $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *and* $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$ *the commutator* $C := [B_\varphi, \mathcal{A}_\psi]$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$.

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Lemma (Antonić, Mišur, Mitrović, 2016)

Let A be compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1,\infty\rangle \setminus \{2\}$. Then A is also compact on $\mathrm{L}^p(\mathbf{R}^d)$, for any p between 2 and r (i.e. such that $1/p = \theta/2 + (1-\theta)/r$, for some $\theta \in \langle 0,1\rangle$).

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r} \quad\Longrightarrow\quad K \text{ compact on } \mathrm{L}^p(\mathbf{R}^d)$$

Theorem

If $u_n \longrightarrow 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$ *and* (v_n) *is bounded in* $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$, *for some* $p \in \langle 1, \infty \rangle$ *and* $q \geqslant p'$, *and* $\omega_n \to 0^+$, *then there exist subsequences* $(u_{n'})$, $(v_{n'})$ *and a complex distribution of finite order* $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *where* $\kappa = \lfloor \frac{d}{2} \rfloor + 1$, *we have*

$$
\begin{aligned}
\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} \, d\mathbf{x} &= \lim_{n'} \int_{\mathbf{R}^d} \varphi_1 u_{n'} \overline{\mathcal{A}_{\bar{\psi}_{n'}}(\varphi_2 v_{n'})} \, d\mathbf{x} \\
&= \left\langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \right\rangle ,
\end{aligned}
$$

where $\psi_n := \psi(\omega_n \cdot)$. *The distribution* $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ *we call one-scale H-distribution (with characteristic length* $(\omega_{n'})$*) associated to (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

Theorem

If $u_n \longrightarrow 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$ and (v_n) is bounded in $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$, for some $p \in \langle 1, \infty\rangle$ and $q \geqslant p'$, and $\omega_n \to 0^+$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a complex distribution of finite order $\nu_{\mathrm{K}_{0,\infty}}^{(\omega_{n'})} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, where $\kappa = \lfloor \frac{d}{2} \rfloor + 1$, we have

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}}\, d\mathbf{x} = \lim_{n'} \int_{\mathbf{R}^d} \varphi_1 u_{n'} \overline{\mathcal{A}_{\bar{\psi}_{n'}}(\varphi_2 v_{n'})}\, d\mathbf{x}$$
$$= \left\langle \nu_{\mathrm{K}_{0,\infty}}^{(\omega_{n'})}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \right\rangle ,$$

where $\psi_n := \psi(\omega_n \cdot)$. The distribution $\nu_{\mathrm{K}_{0,\infty}}^{(\omega_{n'})}$ we call one-scale H-distribution (with characteristic length $(\omega_{n'})$) associated to (sub)sequences $(u_{n'})$ and $(v_{n'})$.

$$\int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}}\, d\mathbf{x} = \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \right\rangle .$$

K_m compacts such that $K_m \subseteq \operatorname{Int} K_{m+1}$ and $\bigcup_m K_m = \Omega$.

For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ such that $\operatorname{supp}\varphi_1, \operatorname{supp}\varphi_2 \subseteq K_m$, we have

$$| \langle \varphi_2 v_n, \mathcal{A}_{\psi_n}(\varphi_1 u_n) \rangle | \leqslant C_{m,d} \|\varphi_1\|_{\mathrm{L}^\infty(K_m)} \|\varphi_2\|_{\mathrm{L}^\infty(K_m)} \|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))} \,.$$

The existence of one-scale H-distributions: proof 1/2

For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ such that $\operatorname{supp}\varphi_1, \operatorname{supp}\varphi_2 \subseteq K_m$, we have

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By the Cantor diagonal procedure (we have separability) ... we get trilinear form L:

$$L(\varphi_1, \varphi_2, \psi) = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \right\rangle .$$

For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ such that $\operatorname{supp}\varphi_1, \operatorname{supp}\varphi_2 \subseteq K_m$, we have

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By the Cantor diagonal procedure (we have separability) ... we get trilinear form L:

$$L(\varphi_1, \varphi_2, \psi) = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\right\rangle .$$

L depends only on the product $\varphi_1\bar\varphi_2$: $\zeta_i \in \mathrm{C}_c(\Omega)$ such that $\zeta_i \equiv 1$ on $\operatorname{supp}\varphi_i$, $i = 1, 2$,

$$\begin{aligned}
\lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\right\rangle &= \lim_{n'} \left\langle \varphi_2 v_{n'}, \varphi_1 \mathcal{A}_{\psi_{n'}}(\zeta_1 u_n)\right\rangle \\
&= \lim_{n'} \left\langle \bar\varphi_1 \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\zeta_1 u_n)\right\rangle \\
&= \lim_{n'} \left\langle \zeta_1\zeta_2 v_{n'}, \varphi_1\bar\varphi_2 \mathcal{A}_{\psi_{n'}}(\zeta_1 u_n)\right\rangle \\
&= \lim_{n'} \left\langle \zeta_1\zeta_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1\bar\varphi_2 u_n)\right\rangle ,
\end{aligned}$$

$$\Longrightarrow \quad L(\varphi_1, \varphi_2, \psi) = L(\varphi_1\bar\varphi_2, \zeta_1\zeta_2, \psi) .$$

The existence of one-scale H-distributions: proof 2/2

For $\varphi \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we define

$$\mathcal{L}(\varphi, \psi) := L(\varphi, \zeta, \psi)\,,$$

where $\zeta \equiv 1$ on $\operatorname{supp} \varphi$.
$\mathcal{L}$ is continuous bilinear form on $\mathrm{C}_c(\Omega) \times \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

The existence of one-scale H-distributions: proof 2/2

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Theorem

Let $\Omega \subseteq \mathbf{R}^d$ be open, and let B be a continuous bilinear form on $\mathrm{C}^\infty_c(\Omega) \times \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$. Then there exists a unique distribution $\nu \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that

$$(\forall\, f \in \mathrm{C}^\infty_c(\Omega))(\forall\, g \in \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))) \quad B(f,g) = \langle \nu, f \boxtimes g\rangle\ .$$

Moreover, if B is continuous on $\mathrm{C}^k_c(\Omega) \times \mathrm{C}^l(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ for some $k,l \in \mathbf{N}_0$, ν is of a finite order $q \leqslant k+l+2d+1$.

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Therefore, we have that there exists $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'_{\kappa+2d+1}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that

$$\begin{aligned}\left\langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \psi\right\rangle =& \mathcal{L}(\varphi_1\bar\varphi_2, \psi)\\ =& L(\varphi_1\bar\varphi_2, \zeta_1\zeta_2, \psi)\\ =& L(\varphi_1, \varphi_2, \psi) = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\right\rangle\end{aligned}$$

$$\mathrm{H}^{s,p}(\mathbf{R}^d) := \Big\{ u \in \mathcal{S}' : \mathcal{A}_{(1+|\boldsymbol{\xi}|^2)^{\frac{s}{2}}} u \in \mathrm{L}^p(\mathbf{R}^d) \Big\}$$

$$\mathrm{H}^{s,p}_{\mathrm{loc}}(\Omega) := \Big\{ u \in \mathcal{D}' : (\forall\, \varphi \in \mathrm{C}^\infty_c(\Omega))\ \varphi u \in \mathrm{H}^{s,p}(\mathbf{R}^d) \Big\}$$

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Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $p \in \langle 1, \infty \rangle$, and

$$\sum_{0 \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,, \tag{$*$}$$

where

- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}^\infty(\Omega; \mathrm{M}_{\mathrm{q}\times\mathrm{r}}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m,p}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \mathcal{A}_{(1+|\varepsilon_n \boldsymbol{\xi}|^2)^{-\frac{m}{2}}} (\varphi \mathsf{f}_n) \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^q) \,. \tag{$**$}$$

Localisation principle: assumptions

$$\mathrm{H}^{s,p}(\mathbf{R}^d) := \Big\{ u \in \mathcal{S}' : \mathcal{A}_{(1+|\boldsymbol{\xi}|^2)^{\frac{s}{2}}} u \in \mathrm{L}^p(\mathbf{R}^d) \Big\}$$

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$$(1 + |\boldsymbol{\xi}|^2)^{-\frac{m}{2}} \text{ is a Fourier multiplier} \quad \Longrightarrow \quad \Big(\mathsf{f}_n \xrightarrow{\mathrm{L}^p_{\mathrm{loc}}} 0 \quad \Longrightarrow \quad (**) \Big)$$

$$\left| \partial^{\boldsymbol{\alpha}} \left(\left(\frac{1 + |\varepsilon_n \boldsymbol{\xi}|^2}{1 + |\boldsymbol{\xi}|^2} \right)^{\frac{m}{2}} \right) \right| \leqslant \frac{2^\kappa}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}} \quad \Longrightarrow \quad \Big((**) \quad \Longrightarrow \quad \mathsf{f}_n \xrightarrow{\mathrm{H}^{-m,p}_{\mathrm{loc}}} 0 \Big)$$

Theorem

Under previous assumptions let (v_n) *be a bounded sequence in* $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$*. Then one-scale H-distribution* $\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}}$ *associated to (sub)sequences* (v_n) *and* (u_n) *with characteristic length* (ε_n) *satisfies:*

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) = \sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m}(2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{(1+|\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,,$$

while q *is order of* $\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}}$.

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Dem. Multiplying $(*)$ by $\varphi\in\mathrm{C}^\infty_c(\Omega)$ and using the Leibniz rule we get

$$\sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m}\sum_{0\leqslant\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}}(-1)^{|\boldsymbol{\beta}|}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\varepsilon_n^{|\boldsymbol{\alpha}|}\partial_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\Big((\partial_{\boldsymbol{\beta}}\varphi)\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n\Big)=\varphi\mathsf{f}_n\,.$$

Lemma

Let (ε_n) be a sequence in $\mathbf{R}^+$ bounded from above and let (f_n) be a sequence of vector valued functions such that for some $k \in 0..m$ it converges strongly to zero in $\mathrm{H}^{-k,p}(\mathbf{R}^d;\mathbf{C}^q)$. Then $(\varepsilon_n^k \mathsf{f}_n)$ satisfies $(**)$.

$$\boldsymbol{\beta} \neq 0 \quad \Longrightarrow \quad \varepsilon_n^{|\boldsymbol{\alpha}|}\partial_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\Big((\partial_{\boldsymbol{\beta}}\varphi)\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n\Big) \text{ satisfies } (**)$$

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Thus, we have

$$\sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\varphi\mathsf{u}_n) = \tilde{\mathsf{f}}_n\,,$$

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Localisation principle: proof 1/2

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Lemma

For $m \in \mathbf{N}$ and $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ such that $m \geqslant 2q + |\boldsymbol{\alpha}| + 2$ we have $\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{(1+|\boldsymbol{\xi}|^2)^{\frac{m}{2}}} \in \mathrm{C}^q(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

$$(\forall\,|\boldsymbol{\alpha}| \leqslant m) \quad \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{(1+|\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}} \in \mathrm{C}^q(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$$

Localisation principle: proof 2/2

Applying $\mathcal{A}_{\psi_n^{m+2q+2,0}}$ we get

$$\sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m} \mathcal{A}_{(2\pi i)^{|\boldsymbol{\alpha}|}\psi_n^{m+2q+2,\boldsymbol{\alpha}}}(\varphi \mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^p(\mathbf{R}^d;\mathbf{C}^q)\,,$$

where $\psi_n^{m+2q+2,\boldsymbol{\alpha}} := \frac{(\varepsilon_n \boldsymbol{\xi})^{\boldsymbol{\alpha}}}{(1+|\varepsilon_n\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}}$.

Localisation principle: proof 2/2

Applying $\mathcal{A}_{\psi_n^{m+2q+2,0}}$ we get

$$\sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m} \mathcal{A}_{(2\pi i)^{|\boldsymbol{\alpha}|}\psi_n^{m+2q+2,\boldsymbol{\alpha}}}(\varphi \mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^p(\mathbf{R}^d;\mathbf{C}^q)\,,$$

where $\psi_n^{m+2q+2,\boldsymbol{\alpha}} := \frac{(\varepsilon_n \boldsymbol{\xi})^{\boldsymbol{\alpha}}}{(1+|\varepsilon_n\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}}$.

After applying $\mathcal{A}_{\psi(\varepsilon_n\cdot)}$, for $\psi \in \mathrm{C}^q(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, to the above sum, forming a tensor product with $\varphi_1 \mathsf{v}_n$, for $\varphi_1 \in \mathrm{C}_c^\infty(\Omega)$, and taking the complex conjugation, for the (i,j) component of the above sum we get

$$\begin{aligned}
0 &= \sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m} \sum_{s=1}^{d} \overline{\lim_n \int_{\mathbf{R}^d} \mathcal{A}_{(2\pi i)^{|\boldsymbol{\alpha}|}\psi_n \psi_n^{m+2q+2,\boldsymbol{\alpha}}}(\varphi A^{\boldsymbol{\alpha}}_{js} u_n^s)\overline{\varphi_1 v_n^k}\, d\mathbf{x}} \\
&= \sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m} \sum_{s=1}^{d} \Big\langle (2\pi i)^{|\boldsymbol{\alpha}|}\psi^{m+2q+2,\boldsymbol{\alpha}} A^{\boldsymbol{\alpha}}_{js} \nu^{ks}_{\mathrm{K}_{0,\infty}}, \bar{\varphi}\varphi_1 \boxtimes \bar{\psi} \Big\rangle \\
&= \Bigg\langle \sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{(1+|\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}} [\mathbf{A}^{\boldsymbol{\alpha}} \boldsymbol{\nu}^\top_{\mathrm{K}_{0,\infty}}]_{jk}, \bar{\varphi}\varphi_1 \boxtimes \bar{\psi} \Bigg\rangle\,.
\end{aligned}$$