Transport-collapse scheme for scalar conservation laws – initial-boundary value problem

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WeConMApp

Scalar conservation laws – initial and boundary conditions

Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain and $\mathbb{R}^+ = [0, \infty)$. We consider

$$\partial_t u + \mathrm{div}_{\mathbf{x}} f(u) = 0, \quad (t, \mathbf{x}) \in \mathbb{R}^+ \times \Omega, \tag{1}$$

$$u|_{t=0} = u_0(\mathbf{x}), \tag{2}$$

$$u|_{\mathbb{R}^+ \times \partial\Omega} = u_B(t, \mathbf{x}). \tag{3}$$

where $f \in C^2(\mathbb{R}; \mathbb{R}^d)$. If not stated otherwise, we assume that $u_0 \in L^1(\mathbb{R}^d)$, $u_B \in L^1_{loc}(\mathbb{R}^+ \times \partial\Omega)$. We also assume that

$$a \leq u_0, u_B \leq b \text{ for some constants } a \leq b. \tag{4}$$

Kruzhkov admissibility conditions – Cauchy problem

For every $\lambda \in \mathbb{R}$ it holds

$$\partial_t |u - \lambda| + \mathrm{div}_{\mathbf{x}}[\mathrm{sgn}(u - \lambda)(f(u) - f(\lambda))] \leq 0 \tag{5}$$

in the sense of distributions on $\mathcal{D}'(\mathbb{R}^d_+)$, and it holds $esslim_{t \to 0} \int_\Omega |u(t, \mathbf{x}) - u_0(\mathbf{x})| d\mathbf{x} = 0$.

The kinetic formulation

Roughly speaking, if we find derivative of the Kruzhkov admissibility conditions with respect to λ, we get the following statement.

The function $u \in C([0,\infty); L^1(\mathbb{R}^d)) \cap L^\infty_{loc}((0,\infty); L^\infty(\mathbb{R}^d))$ is the entropy admissible solution to (1), (2) if and only if there exists a non-negative Radon measure $m(t, \mathbf{x}, \lambda)$ such that $m((0,T)\times\mathbb{R}^{d+1}) < \infty$ for all $T > 0$ and such that the function $\chi(\lambda, u) = \begin{cases} 1, & 0 \le \lambda \le u \\ -1, & u \le \lambda \le 0 \\ 0, & else \end{cases}$, represents the distributional solution to

$$\partial_t \chi + \mathrm{div}_{\mathbf{x}}(f'(\lambda)\chi) = \partial_\lambda m(t, \mathbf{x}, \lambda), \quad (t, \mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{6}$$

$$\chi(\lambda, u(t=0, \mathbf{x})) = \chi(\lambda, u_0(\mathbf{x})). \tag{7}$$

Initial-boundary value problem – standard definition

A function $u \in L^\infty(\Omega)$ is said to be the weak entropy solution to (1), (2), (3) if there exists a constant $L \in \mathbb{R}$ such that for every $k \in \mathbb{R}$ and every non-negative $\varphi \in C_c(\mathbb{R}^d_+; \mathbb{R}^+)$, $\mathbb{R}^d_+ = \mathbb{R}^+ \times \mathbb{R}^d$, it holds

$$\int_{\mathbb{R}^d_+} \left(|u-k|_+ \partial_t \varphi + \mathrm{sgn}_+(u-k)(f(u)-f(k))\,\nabla_{\mathbf{x}}\varphi\right) d\mathbf{x}dt \qquad (8)$$
$$+ \int_{\mathbb{R}^d} |u_0 - k|_+ \varphi(0,\cdot) d\mathbf{x} + L \int_{\mathbb{R}^+ \times \partial\Omega} \varphi\, |u_B - k|_+ d\gamma(\mathbf{x}) dt \geq 0,$$

and

$$\int_{\mathbb{R}^d_+} \left(|u-k|_- \partial_t \varphi + \mathrm{sgn}_-(u-k)(f(u)-f(k))\,\nabla_{\mathbf{x}}\varphi\right) d\mathbf{x}dt \qquad (9)$$
$$+ \int_{\mathbb{R}^d} |u_0 - l|_- \varphi(0,\cdot) d\mathbf{x} + L \int_{\mathbb{R}^+ \times \partial\Omega} \varphi\, |u_B - k|_- d\gamma(\mathbf{x}) dt \geq 0,$$

where γ is the measure on $\partial\Omega$.

Initial-boundary value problem – heuristics

Assume that we are dealing with the flux depending on $\mathbf{x}$ i.e. $f = f(\mathbf{x}, \lambda)$. Denote by

$$S^- = \{\mathbf{x} \in \partial\Omega : \ \langle f'_\lambda(\mathbf{x}, \lambda), \vec{\nu} \rangle \leq 0 \ \text{ a.e. } \ \lambda \in I\},$$

where I contains all essential values of the functions u_B and u_0 (i.e. of appropriate entropy solution u), and $\vec{\nu}$ is the outer unit normal on $\partial\Omega$. The set S^- actually consists of all points such that all possible characteristics from that point enter into the (interior of the) set Ω. Therefore, for every $\mathbf{x} \in S^-$, the trace of the corresponding entropy solution is actually $u_B(\mathbf{x})$.
Similarly, for

$$S^+ = \{\mathbf{x} \in \partial\Omega : \ \langle f'_\lambda(\mathbf{x}, \lambda), \vec{\nu} \rangle \geq 0 \ \text{ a.e. } \ \lambda \in I\},$$

all possible characteristics issuing from $\mathbf{x} \in S^+$ leave the set Ω, and $u_B(\mathbf{x})$ does not influence on the weak entropy solution u to the initial boundary problem.

Initial-boundary value problem – Definition 2

We say that the function $u \in L^\infty(\mathbb{R}^+ \times \Omega; [a,b])$ is a weak entropy admissible solution to (1), (2), (3) if for every $k \in \mathbb{R}$ and every non-negative $\varphi \in C_c^1(\Omega \times \mathbb{R}^+)$ it holds

$$\int_{\Omega\times\mathbb{R}^+} \left(|u-k|_+ \partial_t\varphi + \operatorname{sgn}_+(u-k)(f(u)-f(k))\,\nabla_{\mathbf{x}}\varphi\right) d\mathbf{x}dt \tag{10}$$

$$-\int_a^b \int_{\substack{\mathbb{R}^+\times\partial\Omega \\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}} \varphi(t,\mathbf{x})\,\langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle\,\operatorname{sgn}_+(\lambda-k)\,\operatorname{sgn}_+(u_B(t,\mathbf{x})-\lambda)d\gamma(\mathbf{x})dtd\lambda$$

$$+\int_{\mathbb{R}^d} |u_0-k|_+\varphi(0,\cdot)d\mathbf{x} \geq 0,$$

and

$$\int_{\Omega\times\mathbb{R}^+} \left(|u-k|_- \partial_t\varphi + \operatorname{sgn}_-(u-k)(f(u)-f(k))\,\nabla_{\mathbf{x}}\varphi\right) d\mathbf{x}dt \tag{11}$$

$$-\int_a^b \int_{\substack{\mathbb{R}^+\times\partial\Omega \\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}} \varphi(t,\mathbf{x})\,\langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle\,\operatorname{sgn}_-(\lambda-k)\,\operatorname{sgn}_-(u_B(t,\mathbf{x})-\lambda)d\gamma(\mathbf{x})dtd\lambda$$

Transport-collapse operator

The idea of the transport collapse scheme for the initial value problem is to solve the kinetic equation when we omit its right-hand side:

$$\partial_t h + \mathrm{div}_{\mathbf{x},\lambda}[F(t,\mathbf{x},\lambda)h] = 0, \quad h|_{t=0} = \chi(\lambda, u_0(\mathbf{x})). \tag{12}$$

The solution of this equation is obtained via the method of characteristics and, since the equation is linear, it is given by

$$h(t,\mathbf{x},\lambda) = \chi(\lambda, u_0(\mathbf{x} - f'(\lambda)t)). \tag{13}$$

Convergence toward admissible solution

The transport collapse operator $T(t)$ is defined for every $u \in L^1(\mathbb{R}^d)$ by

$$T(t)u(\mathbf{x}) = \int \chi(\lambda, u(\mathbf{x} - f'(\lambda)t))d\lambda. \tag{14}$$

For any initial value $u_0 \in L^1(\mathbb{R}^d)$, the unique entropy solution to the Cauchy problem is given by

$$u(t, \mathbf{x}) = S(t)u_0(\mathbf{x}) = L^1 - \lim_{n \to \infty} \left(T(\frac{t}{n})^n \right)^n u_0(\mathbf{x}).$$

Boundary value problem

Assume that Ω is an open set such that for some $\sigma \in (0,1)$, no two outer normals from $\partial\Omega$ do not intersect in the set

$$\Omega_\sigma = \{\mathbf{x} \in \mathbb{R}^d : \quad dist(\mathbf{x}, \Omega) < \sigma\} \text{ i.e. in the set}$$
$$\Omega^\sigma = \Omega_\sigma \backslash \Omega$$

Denote by $\vec{\nu}(\mathbf{x})$, $\mathbf{x} \in \Omega_\sigma \setminus \Omega$ the unit outer normal on $\partial\Omega$ passing trough the point $\mathbf{x}$. We then extend the boundary data $u_B(t, \mathbf{x})$ for every fixed $t \geq 0$ along the normals $\vec{\nu}(\mathbf{x})$ in the set Ω_σ. More precisely, we set for $\mathbf{x} \in \Omega^\sigma = \Omega_\sigma \backslash \Omega$ (slightly abusing the notation)

$$u_B(t, \mathbf{x}) = u_B(t, \mathbf{x}_0), \quad \text{for } \mathbf{x}_0 \in \partial\Omega \text{ such that } \vec{\nu}(\mathbf{x}_0) = \vec{\nu}(\mathbf{x}). \qquad (15)$$

Finally, introduce the function

$$w_{u(t,\cdot)}(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \notin \Omega_\sigma \\ u(t,\mathbf{x}), & \mathbf{x} \in \Omega \\ u_B(t,\mathbf{x}), & \mathbf{x} \in \Omega_\sigma \setminus \Omega = \Omega^\sigma, \end{cases} \tag{16}$$

Remark that we can rewrite the function $w_{u(t,\cdot)}(\mathbf{x})$ in the form

$$w_{u(t,\cdot)}(\mathbf{x}) = u(t,\mathbf{x})\kappa_\Omega(\mathbf{x}) + u_B(t,\mathbf{x})\kappa_{\Omega^\sigma}(\mathbf{x}),$$

where κ_{Ω^σ} is the characteristic function of the set Ω^σ.

Fix $t > 0$ and $n \in \mathbf{N}$. We neglect the right-hand side in the kinetic equation and, on the first step, we take $\chi(\lambda, w_{u_0}(\mathbf{x}))$ as the initial data.

$$\partial_t h + f'(\lambda)\mathrm{div}_{\mathbf{x}} h = 0, \tag{17}$$

$$h|_{t=0} = \chi(\lambda, w_{u_0}(\mathbf{x})). \tag{18}$$

The solution to (17) is given by $h(t, \mathbf{x}, \lambda) = \chi(\lambda, \omega_{u_0}(\mathbf{x} - f'(\lambda)t))$ (see (13)). We construct the approximate solution u_n to (1), (2), (3) by the following procedure ($\mathbf{x} \in \Omega$):

- $$u_n(t', \mathbf{x}) = T(t'/n)(w_{u_0}(\mathbf{x})) := \int_0^b \chi(\lambda, \omega_{u_0}(\mathbf{x} - f'(\lambda)t'))d\lambda, \quad t' \in (0, t/n]. \tag{19}$$

- For $k = 1, \dots, n-1$, we take
$$u_n(kt/n + t', \mathbf{x}) = \int_0^b \chi(\lambda, \omega_{u_n(kt/n,\cdot)}(\mathbf{x} - f'(\lambda)t'))d\lambda, \quad t' \in (0, t/n]. \tag{20}$$

There exists a unique function u satisfying the initial boundary value problem in the sense of Definition 2.

The End

Thank you for listening.