Variant of optimality criteria method for multiple state optimal design problems

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WeConMApp

Optimal design problem

- Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded and $f \in \mathrm{H}^{-1}(\Omega)$. We consider stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega)\,. \end{cases}$$

- We assume that Ω is a mixture of two isotropic materials with conductivities $0 < \alpha < \beta$, i.e.

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}, \quad \text{where} \quad \chi \in \mathrm{L}^\infty(\Omega;\{0,1\})$$

and that the amount of the first material is given by $q_\alpha = \int_\Omega \chi\, d\mathbf{x}$. Then, the **multiple state optimal design problem** is

$$\begin{cases} J(\chi) = \displaystyle\int_\Omega \chi(\mathbf{x})g_\alpha(\mathbf{x},\mathsf{u}) + (1-\chi(\mathbf{x}))g_\beta(\mathbf{x},\mathsf{u})\, d\mathbf{x} \longrightarrow \min\,, \\ \chi \in \mathrm{L}^\infty(\Omega;\{0,1\})\,,\ \displaystyle\int_\Omega \chi\, d\mathbf{x} = q_\alpha\,, \end{cases} \tag{1}$$

where $\mathsf{u} = (u_1,\ldots,u_m)$ is the state function determined by

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1,\ldots,m$$

with $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$ and $f_i \in \mathrm{H}^{-1}(\Omega)$, while g_α, g_β are Caratheodory functions which satisfies growth condition

$$g_j(x,u) \le a|u|^s + b(x), \qquad j = \alpha,\beta,$$

for some $a > 0$, $b \in \mathrm{L}^1(\Omega)$ and $1 \le s < \frac{2d}{d-2}$, $d \ge 3$.

Relaxed problem

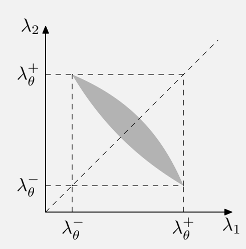
- Problem (1) does not have classical solution, therefore using relaxation by the homogenization method we get relaxed problem

$$\begin{cases} J(\theta,\mathbf{A}) = \displaystyle\int_\Omega (\theta(\mathbf{x})g_\alpha(\mathbf{x},\mathsf{u}) + (1-\theta(\mathbf{x}))g_\beta(\mathbf{x},\mathsf{u}))\, d\mathbf{x} \longrightarrow \min \\ (\theta,\mathbf{A}) \in \{(\theta,\mathbf{A}) \in \mathrm{L}^\infty(\Omega;[0,1]\times \mathrm{M}_d(\mathbf{R})) : \mathbf{A} \in \mathcal{K}(\theta)\ a.e.\}\,,\ \displaystyle\int_\Omega \theta\, d\mathbf{x} = q_\alpha\,, \\ \mathsf{u} = (u_1,u_2,\ldots,u_m),\ \text{where}\ u_i,\ i=1,\ldots,m\ \text{satisfies} \\ \qquad \begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega)\,. \end{cases} \end{cases} \tag{2}$$

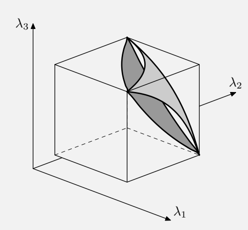
Set $\mathcal{K}(\theta)$ is given in terms of eigenvalues of matrix $\mathbf{A}$ (Murat & Tartar; Lurie & Cherkaev):

$$\begin{aligned} \lambda_\theta^- \le\ & \lambda_j \ \le \lambda_\theta^+ \quad j = 1,\ldots,d \\ \sum_{j=1}^d \frac{1}{\lambda_j - \alpha} &\le \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha} \\ \sum_{j=1}^d \frac{1}{\beta - \lambda_j} &\le \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+}\,, \\ \lambda_\theta^+ &= \theta\alpha + (1-\theta)\beta, \\ \frac{1}{\lambda_\theta^-} &= \frac{\theta}{\alpha} + \frac{1-\theta}{\beta} \end{aligned}$$

2D:



3D:



- Let us introduce adjoint states $p_1,\ldots,p_m$ as solutions of

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla p_i) = \theta\dfrac{\partial g_\alpha}{\partial u_i}(\cdot,\mathsf{u}) + (1-\theta)\dfrac{\partial g_\beta}{\partial u_i}(\cdot,\mathsf{u}) \\ p_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1,\ldots,m\,.$$

Result

Theorem 1. Let $(\theta^*,\mathbf{A}^*)$ be a local minimizer for relaxation problem (2) with corresponding states u_i^* and adjoint states p_i^*. We introduce symmetric matrix

$$\mathbf{N}^* = \mathrm{Sym}\ \sum_{i=1}^m \sigma_i^* \otimes \tau_i^*,$$

for $\sigma_i^* = \mathbf{A}^*\nabla u_i^*$, $\tau_i^* = \mathbf{A}^*\nabla p_i^*$ and function $g(\theta,\mathbf{N}) = \min_{\mathbf{A}\in\mathcal{K}(\theta)} (\mathbf{A}^{-1} : \mathbf{N})$. Then

$$(\mathbf{A}^*)^{-1}(\mathbf{x}) : \mathbf{N}^*(\mathbf{x}) = g(\theta^*(\mathbf{x}),\mathbf{N}^*(\mathbf{x})), \qquad \text{a.e. } x \in \Omega.$$

Moreover, if we define function

$$R^*(\mathbf{x}) := g_\alpha(\mathbf{x},\mathsf{u}^*(\mathbf{x})) - g_\beta(\mathbf{x},\mathsf{u}^*(\mathbf{x})) + l + \frac{\partial g}{\partial\theta}(\theta^*(\mathbf{x}),\mathbf{N}^*(\mathbf{x})),$$

the optimal θ^* satisfies (a.e. on Ω)

$$\begin{aligned} \theta^*(\mathbf{x}) = 0 \quad &\Longrightarrow \quad R^*(\mathbf{x}) > 0 \\ \theta^*(\mathbf{x}) = 1 \quad &\Longrightarrow \quad R^*(\mathbf{x}) < 0 \\ 0 \le \theta^*(\mathbf{x}) \le 1 \quad &\Longrightarrow \quad R^*(\mathbf{x}) = 0\,. \end{aligned}$$

- For two and three dimensional case we explicitly calculated partial derivative $\dfrac{\partial g}{\partial\theta}$, which enabled us update of design variables $(\theta^k,\mathbf{A}^k)$ in optimality criteria method.

Algorithm 1. Take some initial θ^0 and $\mathbf{A}^0$. For k from 0 to N:

1. Calculate u_i^k, $i = 1,\ldots,m$, the solution of

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}^k\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases}.$$

2. Calculate p_i^k, $i = 1,\ldots,m$, the solution of

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}^k\nabla p_i) = \theta^k\dfrac{\partial g_\alpha}{\partial u_i}(\cdot,\mathsf{u}^k) + (1-\theta^k)\dfrac{\partial g_\beta}{\partial u_i}(\cdot,\mathsf{u}^k) \\ p_i \in \mathrm{H}_0^1(\Omega),\ \mathsf{u}^k = (u_1^k,\ldots,u_m^k) \end{cases}$$

and define $\sigma_i^k := \mathbf{A}^k\nabla u_i^k$, $\tau_i^k := \mathbf{A}^k\nabla u_i^k$ and $\mathbf{N}^k := \mathrm{Sym}\sum_{i=1}^m(\sigma_i^k \otimes \tau_i^k)$.

3. For $\mathbf{x} \in \Omega$ let $\theta^{k+1}(\mathbf{x}) \in [0,1]$ be a zero of function

$$\theta \mapsto R^k(\theta,\mathbf{x}) := g_\alpha(\mathbf{x},\mathsf{u}^k(\mathbf{x})) - g_\beta(\mathbf{x},\mathsf{u}^k(\mathbf{x})) + l + \frac{\partial g}{\partial\theta}(\theta,\mathbf{N}^k(\mathbf{x}))\,, \tag{3}$$

and if a zero doesn't exist, take 0 (or 1) if the function is positive (or negative) on $[0,1]$.

4. Let $\mathbf{A}^{k+1}(\mathbf{x})$ be the minimizer in the definition of $g(\theta^{k+1}(\mathbf{x}),\mathbf{N}^k(\mathbf{x}))$.

Optimality criteria method

Example 1. Consider two-dimensional problem of weighted energy minimization

$$J(\theta,\mathbf{A}) = 2\int_\Omega f_1u_1\, d\mathbf{x} + \int_\Omega f_2u_2\, d\mathbf{x} \longrightarrow \min,$$

where $\Omega \subseteq \mathbf{R}^2$ is a ball $B(\mathbf{0},2)$, $\alpha = 1$, $\beta = 2$, while u_1 and u_2 are state functions for

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases}, \quad i = 1,2,$$

where we take $f_1 = \chi_{B(\mathbf{0},1)}$ and $f_2 \equiv 1$ for right-hand sides.

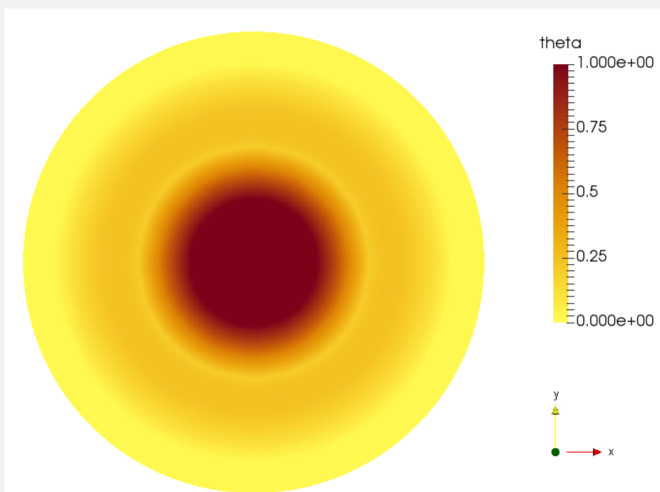


Figure 1: Optimal distribution of materials with volume fraction $\eta = 0.25$ of the first phase.

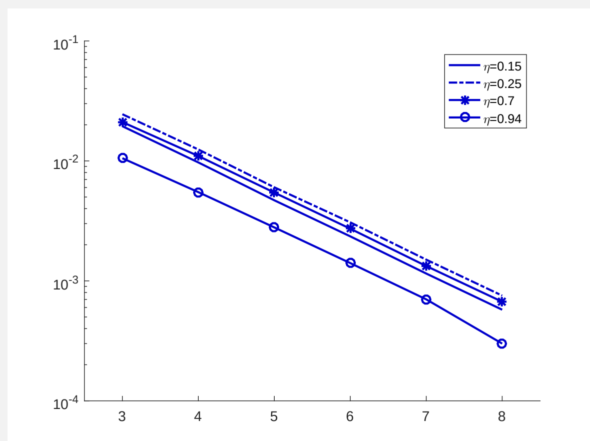


Figure 2: Dependence of L^1 error between numerical and exact solution with respect to mesh refinement for various choices of volume fractions η of the first phase.

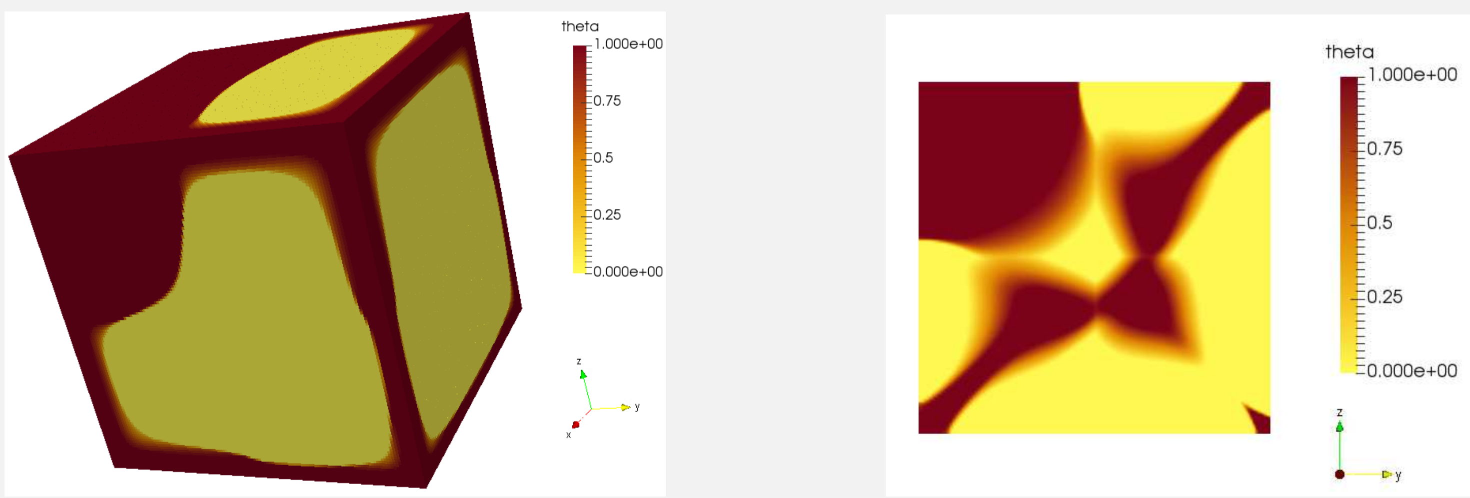
Example 2. Consider three-dimensional energy minimization problem

$$J(\theta,\mathbf{A}) = \int_\Omega (f_1u_1 + f_2u_2)\, d\mathbf{x} \longrightarrow \min,$$

on a cube $[-1,1]^3$, with $\alpha = 1$, $\beta = 2$ and two state equations

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases}, \quad i = 1,2.$$

We take function f_1 to be zero on the upper half ($z > 0$) and 10 on the lower half of the cube, and function f_2 to be zero on the left half ($y < 0$) and 10 on the right half of the cube.



(a) Outer look. (b) Intersection of the cube with $x = 0$ plane.

Figure 3: Optimal distribution of materials with volume fraction $\eta = 0.5$ of the first phase.