Second commutation lemma for fractional H-measures

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Joint work with Marko Erceg

WeConMApp

H-measures
- Work schedule
- Classical H-measures
- A variant of the first commutation lemma
- Extension of H-measures

Fractional H-measures
- Motivation
- Definition
- Second commutation lemma

Work schedule

- (i) we introduce the notion of *admissible* manifold and prove a variant of the first commutation lemma using it;
- (ii) we give an example of *admissible* manifold;
- (iii) we define H-measures on $\mathbf{R}^d \times P$ for any *admissible* manifold P;
- (iv) we discuss the possibility of using *non-admissible* manifolds and introduce so-called fractional H-measures;
- (v) we prove the second commutation lemma appropriate for usage with these measures, giving also one application.

H-measures were introduced independently by Luc Tartar and Patrick Gérard in the late 1980s and their existence is established by the following theorem.

Theorem 1. *If* (u_n) *is a sequence in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$ *such that* $\mathsf{u}_n \rightharpoonup 0$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and an* $r\times r$ *Hermitian complex matrix Radon measure* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d\times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1,\varphi_2\in \mathrm{C}_0(\mathbf{R}^d)$ *and* $\psi\in \mathrm{C}(\mathrm{S}^{d-1})$ *one has:*

$$\begin{aligned}\lim_{n'}\int_{\mathbf{R}^d}(\varphi_1\mathsf{u}_{n'})\otimes\mathcal{A}_\psi(\varphi_2\mathsf{u}_{n'})\,d\mathbf{x} &= \langle\boldsymbol{\mu},(\varphi_1\overline{\varphi_2})\boxtimes\overline{\psi}\rangle\\ &=\int_{\mathbf{R}^d\times\mathrm{S}^{d-1}}\varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})\psi(\boldsymbol{\xi})}\,d\boldsymbol{\mu}(\mathbf{x},\boldsymbol{\xi})\,,\end{aligned}$$

where $\mathcal{F}(\mathcal{A}_\psi v)(\boldsymbol{\xi})=\psi(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})\mathcal{F}v(\boldsymbol{\xi})$. ∎

Admissible manifolds

We need a metric d on $\mathbf{R}^d$ with the property:

$$(\forall\, R > 0)(\exists\, C_R > 0)(\forall\, \mathbf{x}, \mathbf{y} \in \mathbf{R}^d) \quad |\mathbf{x} - \mathbf{y}| \leqslant \mathbf{R} \quad \Longrightarrow \quad d(\mathbf{x}, \mathbf{y}) < C_R. \tag{1}$$

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A compact continuous manifold $P \subseteq \mathbf{R}^d$ is admissible if there exists

$$\{\varphi_{\boldsymbol{\nu}} : \mathbf{R}^+ \longrightarrow \mathbf{R}^d : \boldsymbol{\nu} \in P;\, \varphi_{\boldsymbol{\nu}}(1) = \boldsymbol{\nu}\}, \qquad (2)$$

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with properties:

(i) $(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d \backslash \{0\})(\exists!\, s \in \mathbf{R}^+)(\exists!\, \boldsymbol{\nu} \in P) \quad \boldsymbol{\xi} = \varphi_{\boldsymbol{\nu}}(s)$;

(ii) there exists a real nondecreasing function f, $\lim_{t\to\infty} f(t) = \infty$ such that

$$\begin{aligned}(\forall\, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in P)(\forall\, s_1, s_2 \in [1, \infty\rangle)\; & d(\varphi_{\boldsymbol{\nu}_1}(s_1), \varphi_{\boldsymbol{\nu}_2}(s_2)) \\ & \geqslant f(\min\{s_1, s_2\})|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2|,\end{aligned}$$

for some metric d with property (1);

(iii) $t_{\boldsymbol{\nu}}(s) = |\varphi_{\boldsymbol{\nu}}(s)|$ is strictly increasing and

$$(\forall\, s \in \mathbf{R}^+) \quad \sup_{\boldsymbol{\nu} \in P} t_{\boldsymbol{\nu}}(s) =: C_s < \infty\,.$$

A variant of the first commutation lemma

Lemma 1. *Let* P *be an admissible manifold. For* $b \in \mathrm{C}_0(\mathbf{R}^d)$, $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ *such that*

$$(\exists\, a_\infty \in \mathrm{C}(P)) \qquad \lim_{s\to\infty} a(\varphi_{\boldsymbol{\nu}}(s)) = a_\infty(\boldsymbol{\nu}), \text{ uniformly in } \boldsymbol{\nu} \in P,$$

and operators $\mathcal{A}$ *and* B *defined by*

$$\mathcal{F}(\mathcal{A}u) = a\mathcal{F}u, \quad Bu = bu,$$

a commutator $C := \mathcal{A}B - B\mathcal{A}$ *is compact on* $\mathrm{L}^2(\mathbf{R}^d)$. ∎

Example

In $[\mathrm{MI}]$ we used a manifold

$$P = \{\boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{\alpha_k} = 1\},$$

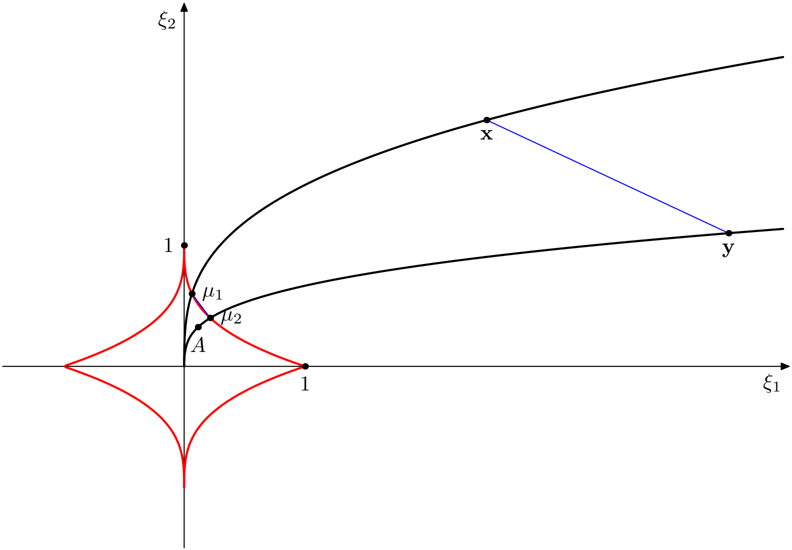
$\alpha_k \in \langle 0, 1]$, and a family of curves

$$\boldsymbol{\xi}(s) = \mathrm{diag}\,\{s^{\frac{1}{\alpha_1}}, \ldots, s^{\frac{1}{\alpha_d}}\}\boldsymbol{\mu}, \quad s > 0. \tag{3}$$

With the choice (3) we prove that P is admissible.

[MI] D. Mitrović, I. Ivec, *A generalization of H-measures and application on purely fractional scalar conservation laws*, *Comm. Pure Appl. Analysis* **10** (2011) (6) 1617–1627.

Picture



Admissible symbols

Let P be an admissible manifold.

$\tilde{\psi} \in \mathrm{C}(\mathbf{R}^d \backslash \{0\})$ is a P-admissible symbol if

$$\lim_{s \to \infty} \tilde{\psi}(\varphi_{\boldsymbol{\nu}}(s)) = \psi(\boldsymbol{\nu})$$

exists uniformly in $\boldsymbol{\nu} \in P$ and define a function $\psi \in \mathrm{C}(P)$.

Theorem 2. *Let P be an admissible manifold. If $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, then there exist a subsequence $(\mathsf{u}_{n'})$ and a Hermitian matrix Radon measure $\boldsymbol{\mu} = \{\mu^{ij}\}_{i,j=1,\dots,r}$ on $\mathbf{R}^d \times P$ so that for arbitrary $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$, a P-admissible symbol $\tilde\psi \in \mathrm{C}(\mathbf{R}^d \backslash \{0\})$, and $i, j = 1, \dots, r$ it holds:*

$$\lim_{n'\to\infty} \int_{\mathbf{R}^d} (\varphi_1 u^i_{n'})(\mathbf{x}) \overline{\mathcal{A}_{\tilde\psi}(\varphi_2 u^j_{n'})(\mathbf{x})}\, d\mathbf{x} = \langle \mu^{ij}, \varphi_1 \overline{\varphi_2 \psi} \rangle$$

$$= \int_{\mathbf{R}^d \times P} \varphi_1(\mathbf{x}) \overline{\varphi_2(\mathbf{x}) \psi(\boldsymbol{\xi})}\, d\mu^{ij}(\mathbf{x}, \boldsymbol{\xi}),$$

where $\psi \in \mathrm{C}(P)$ is given by the previous definition. ∎

Orthogonal manifold

The curves

$$\boldsymbol{\xi}(s) = \mathsf{diag}\,\{s^{\frac{1}{\alpha_1}}, \dots, s^{\frac{1}{\alpha_d}}\}\boldsymbol{\nu}, \quad s > 0$$

are also used in known variants of H-measures.

Classical H-measures: $\alpha_1 = \alpha_2 = \cdots = \alpha_d = 1$

Parabolic H-measures: $\alpha_1 = \frac{1}{2},\ \alpha_2 = \cdots = \alpha_d = 1$

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An ellipsoid

$$\frac{\xi_1^2}{\alpha_1} + \frac{\xi_2^2}{\alpha_2} + \cdots + \frac{\xi_d^2}{\alpha_d} = \frac{1}{\alpha_{\min}}$$

is orthogonal on the above curves.

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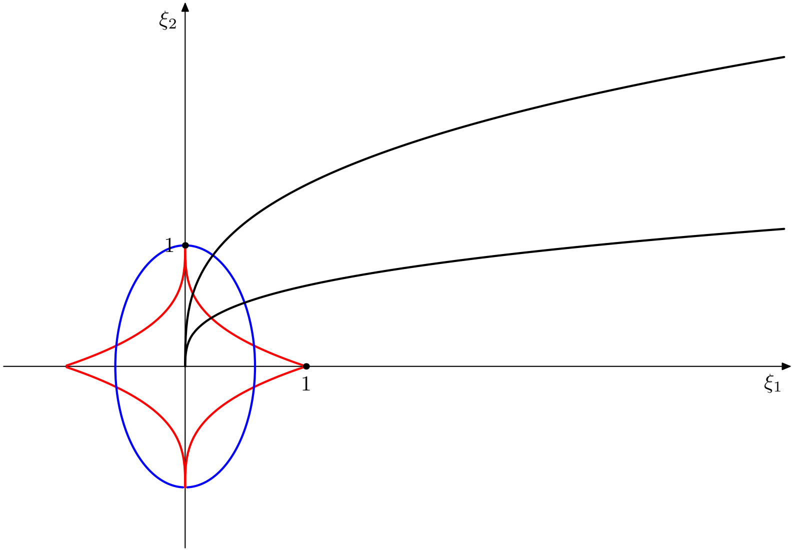

Good manifolds

If we define H-measures on $\mathbf{R}^d \times P$ for *admissible* manifold P, we can define them also on $\mathbf{R}^d \times Q$ (for a manifold Q not necessarily known to be *admissible*) if:

(iv) for each $\boldsymbol{\nu} \in P$ the curve $\varphi_{\boldsymbol{\nu}}$ intersects Q in a single point $\boldsymbol{\eta}$, and the function $\boldsymbol{\nu} \mapsto \boldsymbol{\eta}$ is continuous.

Manifolds P and Q are easily shown to be homeomorphic in that case.

Theorem 3. *Let Q be an ellipsoid*

$$\frac{\xi_1^2}{\alpha_1} + \frac{\xi_2^2}{\alpha_2} + \cdots + \frac{\xi_d^2}{\alpha_d} = \frac{1}{\alpha_{\min}},$$

and for each $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d) \in Q$ we define

$$\varphi_{\boldsymbol{\eta}}(s) = \mathsf{diag}\,\{s^{\frac{1}{\alpha_1}}, \ldots, s^{\frac{1}{\alpha_d}}\}\boldsymbol{\eta},$$

where $\alpha_k \in \langle 0, 1]$. Also, π_Q is a projection on Q along $\varphi_{\boldsymbol{\eta}}$.

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If $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)$, then there exist a subsequence $(\mathsf{u}_{n'})$ and a Hermitian matrix Radon measure $\boldsymbol{\mu} = \{\mu^{ij}\}_{i,j=1,\ldots,r}$ on $\mathbf{R}^d \times Q$ so that for $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$, $\psi \in \mathrm{C}(Q)$, and $i, j = 1, \ldots, r$:

$$\begin{aligned}\lim_{n'\to\infty} \int_{\mathbf{R}^d} (\varphi_1 u^i_{n'})(\mathbf{x}) \overline{\mathcal{A}_{\psi\circ\pi_Q}(\varphi_2 u^j_{n'})(\mathbf{x})}\, d\mathbf{x} &= \langle \mu^{ij}, \varphi_1 \overline{\varphi_2 \psi} \rangle \\ &= \int_{\mathbf{R}^d \times Q} \varphi_1(\mathbf{x}) \overline{\varphi_2(\mathbf{x})\psi(\boldsymbol{\xi})}\, d\mu^{ij}(\mathbf{x}, \boldsymbol{\xi}).\end{aligned}$$

∎

Properties of projections

The projection is given by the formula

$$\pi_Q(\boldsymbol{\xi}) = \left(\frac{\xi_1}{s(\boldsymbol{\xi})^{\frac{1}{\alpha_1}}}, \dots, \frac{\xi_d}{s(\boldsymbol{\xi})^{\frac{1}{\alpha_d}}} \right),$$

where $s(\boldsymbol{\xi})$ is the positive solution of the equation

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(a) $s\Big(\lambda^{\frac{1}{\alpha_1}}\xi_1, \dots, \lambda^{\frac{1}{\alpha_d}}\xi_d\Big) = \lambda s(\boldsymbol{\xi}), \quad \lambda \in \mathbf{R}^+$;

(b) $d_s(\boldsymbol{\xi}, \boldsymbol{\eta}) := s(\boldsymbol{\xi} - \boldsymbol{\eta})$ defines a metric on $\mathbf{R}^d$;

(c) $(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d \backslash \{0\}) \quad s(\boldsymbol{\xi}) = s(|\xi_1|, \dots, |\xi_d|)$;

(d) $|\eta_k| \geqslant |\xi_k|,\ k = 1, \dots, d \quad \Longrightarrow \quad s(\boldsymbol{\eta}) \geqslant s(\boldsymbol{\xi})$;

(e) $(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d \backslash \{0\}) \quad C_1 \sum_{k=1}^{d} |\xi_k|^{\alpha_k} \leqslant s(\boldsymbol{\xi}) \leqslant C_2 \sum_{k=1}^{d} |\xi_k|^{\alpha_k}$.

Anisotropic Tartar spaces

For $m \in \mathbf{N}$ and $\boldsymbol{\alpha} \in \langle 0, 1]^d$ we define

$$\mathrm{X}^{m\boldsymbol{\alpha}}(\mathbf{R}^d) := \{u \in \mathcal{S}' : k_{\boldsymbol{\alpha}}^m \hat{u} \in \mathrm{L}^1(\mathbf{R}^d)\},$$

where

$$k_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) := \Big(1 + \sum_{k=1}^{d} |\xi_k|^{\alpha_k}\Big).$$

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$\mathrm{X}^{m\boldsymbol{\alpha}}(\mathbf{R}^d)$ is a Banach space with the norm

$$\|u\|_{\mathrm{X}^{m\boldsymbol{\alpha}}} := \int_{\mathbf{R}^d} k_{\boldsymbol{\alpha}}^m |\hat{u}| \, d\boldsymbol{\xi}.$$

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Assumption: $\alpha_1, \alpha_2, \dots, \alpha_m < 1$, and $\alpha_{m+1} = \cdots = \alpha_d = 1$

Notation: $\mathbf{x} = (\bar{\mathbf{x}}, \mathbf{x}')$, $\bar{\mathbf{x}} = (x_1, \dots, x_m)$, $\mathbf{x}' = (x_{m+1}, \dots, x_d)$, $0 \leqslant m \leqslant d$

Second commutation lemma

Theorem 4. *Let P_ψ and M_ϕ be a Fourier and pointwise multiplier operators on $\mathrm{L}^2(\mathbf{R}^d)$ defined by $\mathcal{F}(P_\psi u) = \psi \mathcal{F} u, \quad M_\phi u = \phi u$, with associated symbols $\psi \in \mathrm{C}^1(P^d)$ and $\phi \in \mathrm{X}^{\boldsymbol{\alpha}}(\mathbf{R}^d)$ respectively. Then for a commutator $K := [P_\psi, M_\phi] = P_\psi M_\Phi - M_\phi P_\psi$ we have (up to a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$):*

$$\partial_j^{\alpha_j} K = P_{\frac{(2\pi i \xi_j)^{\alpha_j}}{2\pi i} \nabla^{\boldsymbol{\xi}'} \psi^P} M_{\nabla^{\mathbf{x}'} \phi}.$$

∎

An application

We study sequence of equations

$$iu_t^n + (a(x)u_{xx}^n)_{xx} = f^n,$$

where $a(x) = a(t,x) \in \mathrm{X}^{(\frac{1}{4},1)}(\mathbf{R}^2)$, $f \in \mathrm{L}^2(\mathbf{R}^2)$ and a is real.

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Using second commutation lemma and assumptions

$$f_n \rightharpoonup 0 \text{ in } \mathrm{L}^2, \quad u_{xx}^n \rightharpoonup 0 \text{ in } \mathrm{L}^2$$

we obtain

$$4\langle \mu, a\phi_x \boxtimes \psi\rangle - \langle \mu, a'\phi \boxtimes (\psi + \xi(\psi^P)_\xi)\rangle = 0,$$

where μ is a fractional ($\alpha_1 = \frac{1}{4}, \alpha_2 = 1$) H-measure associated with the sequence (u_{xx}^n).