The necessary and sufficient condition of optimality

Statement of the problem

Let \( \Omega \subset \mathbb{R}^3 \) be open and bounded set. It consists of two phases each with different isotropic conductivity: \( \alpha, \beta (0 < \alpha < \beta) \)
\( \rho_0 \) is the prescribed volume of the first phase \( \alpha > 0, \beta < \Omega \).
\( \chi \in L^\infty((0,1]) \) is a measurable characteristic function.

Conductivity can be expressed as

\[
A(\chi) = \chi \cdot 1 + (1 - \chi) \Omega,
\]

where

\[
\int_0^1 \rho_0 \chi(x) \, dx < \infty.
\]

State functions \( \alpha_i \in \mathcal{H}_0^1(0,1), \ i = 1, \ldots, m \) are solutions of the following boundary value problems:

\[
\begin{align*}
\frac{d}{dx} (\lambda \chi_\alpha \rho_\alpha) &= f_i \text{ in } (0,1), \\
\alpha_i &= 0 \text{ on } \Omega \setminus (0,1).
\end{align*}
\]

Energy functional:

\[
F(\chi) = \sum_{i=1}^m \frac{1}{n_i} \int_0^1 f_i(\chi) n_i(x) \, dx.
\]

Optimal problem design:

\[
\begin{align*}
F(\chi) &= \sum_{i=1}^m \frac{1}{n_i} \int_0^1 f_i(\chi) n_i(x) \, dx \rightarrow \max \\
\chi &\in L^\infty((0,1]), \ u \text{ solves (1) with } A = \chi \cdot 1 + (1 - \chi) \Omega.
\end{align*}
\]

If solution exists for (2) we call it classical solution

Important:

For general optimal design problems the classical solutions usually do not exist.

Analytical example on anisotropy for single state problem

For anisotropically symmetric problem such that:

\[
\Omega = \{ \mathbf{r} \in \mathbb{R}^3 | f(\mathbf{r}) \}
\]

\[f(r) \text{ are radial functions it can be proved that there exists radial solution } \chi_\alpha \text{ of (1).}
\]

In particular, it can be shown that

\[
\max_{\chi \in \mathcal{H}_0^1(0,1)} F(\chi) = \lambda_1^0(\Omega).
\]

Single state equation:

\[
\Omega = \{ \mathbf{r} \in \mathbb{R}^3 | f(\mathbf{r}) \}
\]

where \( \lambda_0^0(\Omega) = \frac{1}{\beta - \alpha} \). Optimisation problem:

\[
\begin{align*}
\text{For } \theta \in \mathbb{R} - \{ \theta | \theta \in (0,1], \ \lambda_0^0(\Omega) \} \frac{\partial}{\partial \theta} \chi \rightarrow \max \\
\text{subject to } \chi \in L^\infty((0,1]), \ f(\chi) \rightarrow \max.
\end{align*}
\]

Necessary and sufficient condition of optimality can also be expressed as non-linear system (minimun) \( \gamma, c, \tau, \rho, \rho_\pm \):

\[
\begin{align*}
S^\theta_{\tau, \gamma, \rho} = \frac{\partial}{\partial \rho_\pm} \rho_\pm \bigg| \frac{\partial}{\partial \rho_\pm} \rho_\pm \bigg| = 0, \\
\text{where } \rho_\pm = \rho(\tau, \gamma, \rho_\pm).
\end{align*}
\]

Consequently, necessary and sufficient condition of optimality for \( \theta^* \) states

\[
\begin{align*}
\gamma_{\tau, \rho} &> 0 \\
\gamma_{\tau, \rho} &< 0,
\end{align*}
\]

therefore, \( \gamma_{\tau, \rho} > 0 \) or \( \gamma_{\tau, \rho} < 0 \) by definition of \( \theta^* \) states.

Non-linear system (6) does not admit a solution (proved for \( d = 2, d = 3 \)).

For \( d = 3 \) case \( \beta - \alpha \)

Therefore, case 1) and 2) should be considered as only possible solutions. One can easily prove \( \rho_\pm = \rho(\tau, \gamma, \rho_\pm) \) is small, case \( \beta - \alpha \) is always solution (for arbitrary chosen parameters \( \alpha, \beta, r_1, r_2 \)).

Furthermore, one can numerically obtain critical value for which optimal design changes from case \( \beta - \alpha \) to \( \alpha - \beta \).

where

\[
\begin{align*}
\alpha &= \text{critical value for } \alpha - \beta, \\
\beta &= \text{critical value for } \beta - \alpha.
\end{align*}
\]

Remark:

- Problem can be easily generalized to multi-state problem for example \( m = 2 \):

\[
f(\chi) = \begin{cases} 
0, & \text{if } \chi \leq \alpha, \\
\frac{1}{\beta - \alpha}, & \text{if } \chi > \alpha.
\end{cases}
\]

- Existence of such solutions is important for any numerical method like shape derivative method.

Generalized (convex problem) B

Unfortunately, \( \bar{A} \) is not a convex set. To achieve convexity, an enlarged (artificial) set is introduced:

\[
\bar{A} = \{ (\theta, \chi) \in L^\infty((0,1], \chi \in \mathcal{H}_0^1(0,1)) | \chi(x) \leq \chi(0) \text{ for } x \in (0,1) \}
\]

and with it

\[
\text{max } \bar{F} = \text{max } \bar{F} \quad \text{subject to } \chi(x) = \chi(0) \text{ for } x \in (0,1)
\]

Using fuzzy one can rewrite problem (B) as min-max problem and prove:

Theorem

Optimization problem (B) is equivalent to following optimization problem:

\[
\begin{align*}
I(\theta, \chi) &= \sum_{i=1}^m \frac{1}{n_i} \int_0^1 f_i(\chi) n_i(x) \, dx \\
\text{subject to } \chi(x) &= \chi(0) \text{ for } x \in (0,1)
\end{align*}
\]

The necessary and sufficient condition of optimality

Define

\[
\psi = \sum_{i=1}^m \rho_i \sigma_i^m
\]

Lemma

The necessary and sufficient condition of optimality for solution \( \theta^* \) of optimal design problem (1) implies to the existence of a Lagrange multiplier \( \geq 0 \) such that:

\[
\begin{align*}
\psi &= \sum_{i=1}^m \rho_i \sigma_i^m \\
\psi &= \sum_{i=1}^m \rho_i \sigma_i^m \geq c \quad \Rightarrow \quad \theta^* = 0.
\end{align*}
\]

Gradient method using shape derivative

Perturbation of the set \( \Omega \) is given with:

\[
\Omega = \{ \mathbf{r} \in \mathbb{R}^3 | f(\mathbf{r}) \Omega \}
\]

where \( \psi = \Omega_{\theta}^\infty \) is a homogeneous manifold. This allows us to define shape derivative.

Definition (Shape derivative)

Let \( J(\theta) \) be a shape functional. \( J \) is said to be shape differentiable at \( \Omega \) in direction \( \psi \) if

\[
J' (\psi) = \lim_{\tau \to 0} \frac{J(\Omega + \tau \psi) - J(\Omega)}{\tau}
\]

exists and the mapping \( \psi \rightarrow J' (\psi) \) is linear and continuous. \( J' (\theta) \) is called the shape derivative.

For our optimal design problem - shape derivative is given with

\[
J' (\psi) = \int_{\Omega} \| \nabla \psi \|^2 + \| \nabla \psi \|^2 \nabla \cdot \nabla \psi
\]

where \( \psi \) is solution of IBVP (1) on domain \( \Omega \) with \( A \). Vector field \( \psi \in \mathcal{H}_0^1(\Omega) \) is constructed from:

\[
\int_{\Omega} \| \nabla \psi \|^2 + \int_{\Omega} \psi \cdot \theta (\psi, \chi) \psi
\]

The shape is evolved by gradually moving the boundary between phases.

Numerical results:

References: