Numerical approximation of classical optimal design on annuli

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The necessary and sufficient condition of optimality

Statement of the problem

Let $Ω ⊂ R^n$ be open and bounded set. It consists of two phases each with different isotropic conductivity: $α, β (0 < α < β)$. $g_0$ is the prescribed volume of the first phase $α (0 < g_0 < |Ω|)$. Conductivity can be expressed as

$$\sigma = \begin{cases} α, & \text{for } Γ \subset \Omega \\ β, & \text{for } Ω \setminus Γ \end{cases}$$

where $Γ = \{ x ∈ Ω | u(x) = 0 \}$ is the boundary of $Ω$. This set is a measurable characteristic function.

Relaxed design. Effective conductivity

For characteristic functions relaxation consists of:

$$\chi ∈ L^∞(Ω, (0, 1]) → \chi ∈ L^∞(Ω, [0, 1])$$

with $f_α(0, 1) = f_β(0, 1)$. Set of effective conductivities $K(θ)$:

$$K(θ) = \{ A ∈ K(θ) | A^α = αχ + (1 − α)\mathbb{I} \}$$

with

$$A^α = \begin{cases} 0, & \chi = 1 \\ 1, & \chi = 0 \end{cases}$$

and $H$ is Hilbert space.

Optimal solution problem:

$$\max_{θ ∈ [0, 1]} J(θ, A) = \max_{θ ∈ [0, 1]} \left\{ \int_Ω f_1(u_1) \text{d}x + \int_Ω f_2(u_2) \text{d}x \right\}$$

where $u_1 = u_1(θ, A)$ and $u_2 = u_2(θ, A)$.

Direct calculations.

Necessary and sufficient condition of optimality can also be expressed as a nonlinear system (minimizing $c, ε, r, χ_ε, r$):

$$\begin{cases} S_ε f_1(u_1)(θ) \text{d}x = \theta \epsilon_1(θ) \text{d}x \\ S_ε f_2(u_2)(θ) \text{d}x = \theta \epsilon_2(θ) \text{d}x \\ r(θ, χ_ε, r) = 0 \\ σ(ε, r, r, χ) = 0 \\ \sigma(ε, r, r, χ) = 0 \end{cases}$$

where $S_ε f_1(u_1)(θ) \text{d}x = \int_Ω f_1(u_1) \text{d}x - \int_Ω f_1_ε(u_1) \text{d}x$ and $S_ε f_2(u_2)(θ) \text{d}x = \int_Ω f_2(u_2) \text{d}x - \int_Ω f_2_ε(u_2) \text{d}x$.

Therefore, cases 1) and 2) should be considered as only possible solutions. One can easily prove if $g_0$ is very small, case alpah-beta is always solution (for arbitrary chosen parameters $α, β, m, r$).

Furthermore, one can numerically obtain critical value for which optimal design changes from case alpha-beta to alpha-beta-alpha.

**Remark:**

- Problem can be easily generalized to multi-state problem for example $m = 2$.
- $f_0(x) = 1, f_1(x) = \frac{1}{x}$
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Visual representation of a set $K(θ)$

$$K(θ) = \{ A ∈ K(θ) | A^α = αχ + (1 − α)\mathbb{I} \}$$

with

$$A^α = \begin{cases} 0, & \chi = 1 \\ 1, & \chi = 0 \end{cases}$$

and $H$ is Hilbert space.

Graduated (convex) problem B

Unfortunately, $A$ is not a convex set. To achieve convexity, an enlarged (artificial) set is introduced:

$$B = \{ (θ, A) ∈ L^∞(Ω, [0, 1]) × Sym^d | \lambda^α ≤ A(x) ≤ \lambda^β, A = A(x) \}$$

with it

$$\max_{θ ∈ [0, 1]} J(θ, A) = \max_{θ ∈ [0, 1]} \int_Ω f_1(u_1) \text{d}x + \int_Ω f_2(u_2) \text{d}x$$

Using figures one can rewrite problem (B) as max-min problem and prove:

**Theorem**

Optimization problem (B) is equivalent to following optimization problem:

$$\max_{θ ∈ [0, 1]} \int_Ω f_1(u_1) \text{d}x + \int_Ω f_2(u_2) \text{d}x$$

**Lemma**

The necessary and sufficient condition of optimality for solution $θ^*$ of optimal design problem (I) simplifies to the existence of a Lagrange multiplier $c > 0$ such that

$$\psi = \sum_{i=1}^m μ_i(σ_i)^2 = c = c^* > 0$$

Numerical results:

- Perturbation of the set $Ω$ is given with $Ω_r = (I + r\cdot\epsilon_0)Ω$ where $ψ ∈ H^∞(Ω_r)$.
- It is small (i.e. $\Psi[\psi]_H = |Ω_r|)$ mapping $I + r\epsilon_0$ is homeomorphism. This allows us to define shape derivative.

**Definition (Shape derivative)**

Let $J = J(θ)$ be a shape functional. $J$ is said to be shape differentiable at $θ$ in direction $ψ$ if

$$J(θ + rψ) = J(θ) + rJ(θ, ψ) + o(r)$$

exists and the mapping $ψ → J(θ, ψ)$ is linear and continuous.

$J′(θ)$, $ψ$ is called the shape derivative.

For our optimal design problem shape derivative is given with

$$J′(θ, ψ) = \int_Ω (− div(ψ) + ψ \cdot \nabla v_1)^\top \nabla v_0 \text{d}x + 2\int_Ω v_2(ψ) \text{d}x$$

where $w_1$ is solution of BVP (1) on domain $Ω$ with $A$. Vector field $ψ \in H^1(Ω)$ is constructed from:

$$\int_Ω v_0 \nabla [u^1] + \int_Ω [v_0] \nabla u^1$$

The shape is evolved by gradually moving the boundary between phases.

**References**


