Parameter dependent systems of ODE’s

The system

A finite dimensional linear control system
\[ \begin{align*}
  x'(t) &= A(t)x(t) + B(t)u(t), \quad 0 < t < T, \\
  x(0) &= x_0, \\
\end{align*} \]

is large.

A general greedy algorithm

Choose \( x_1, x_2, \ldots, x_n \) such that \( \|x_i\|_X = \max_{x \in K} \|x\|_X \).

Having found \( x_1, x_2, \ldots, x_n \), denote
\( V_n = \text{span}\{x_1, \ldots, x_n\} \).

Choose the next element
\( x_{n+1} = \text{argmax}_{x \in K} \text{dist}(x, V_n) \).

The algorithm stops when \( \sigma_n(K) = \max_{x \in K} \text{dist}(x, V_n) \) becomes less than the given tolerance \( \varepsilon \).

The problem

Each control can be uniquely determined by the relation
\[ u_\nu = B^* e^{(T-t)A^*} x_\nu, \]
where \( x_\nu \in \mathbb{R}^d \) is the unique minimiser of a quadratic functional associated to the adjoint problem.

Method

Based on greedy algorithms and reduced bases methods for parameter dependent PDEs [1, 2].

The greedy approach

The (unknown) quantity \( \text{dist} \) is approximated on the manifold \( \varphi_0(\mathcal{N}) \): \( \nu \in \mathcal{N} \rightarrow \varphi_0^* \in \mathbb{R}^d \).

Perform a greedy algorithm to the manifold \( \varphi_0(\mathcal{N}) \): \( \nu \in \mathcal{N} \rightarrow \varphi_0^* \in \mathbb{R}^d \).

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The greedy control algorithm results in an optimal decay of the approximation rates.

Numerical examples

We consider the system (1) with
\[ A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
\[ \Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

The system corresponds to the discretisation of the wave equation problem with the control on the right boundary:
\[ \begin{align*}
  &\frac{\partial v}{\partial t} - \Delta v = 0, \quad (t, x) \in (0, T) \times (0, 1), \\
  &v(t, 0) = 0, \quad v(t, 1) = u(t) \\
  &v(0, x) = v_0, \quad \frac{\partial v}{\partial t}(x, 0) = v_1. \\
\end{align*} \]

We take the following values:
\[ T = 3, \quad N = 20, \quad v_0 = \sin(\pi x), \quad v_1 = 0, \quad x_1 = 0, \quad \nu \in [1, 10] = \mathcal{N} \]

References


