Controllability of PDEs and Applications, Marseille 2015.

Greedy control

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Parameter dependent systems of ODE's

The topic

– control of a parameter dependent system in a robust manner.

The system

A finite dimensional linear control system

$$\begin{cases} x'(t) = \mathbf{A}(\nu)x(t) + \mathbf{B}u(t), \; 0 < t < T, \\ x(0) = x^0. \end{cases} \quad (1)$$

- $A(\nu)$ is a $N \times N-$matrix,
- B is a $N \times M$ control operator, $M \leq N$,
- ν is a parameter living in a compact set $\mathcal{N}$ of $\mathbb{R}^d$.

Assumptions:

– the system is (uniform) controllable for all $\nu \in \mathcal{N}$,
– system dimension N is large.

The problem

Fix a control time $T > 0$, an arbitrary initial data x^0, and a final target $x^1 \in \mathbf{R}^N$.

Given $\varepsilon > 0$ we aim at determining a family of parameters $\nu_1, \dots, \nu_n$ in $\mathcal{N}$ so that the corresponding controls $u_1, \dots, u_n$ are such that for every $\nu \in \mathcal{N}$ there exists $u^\star_\nu \in \operatorname{span}\{u_1, \dots, u_n\}$ steering the system (1) to the state $x^\star_\nu(T)$ within the ε distance from the target x^1.

Method

– based on greedy algorithms and reduced bases methods for parameter dependent PDEs [1, 2].

The greedy approach

X – a Banach space
$K \subset X$ – a compact subset.

The method approximates K by a a series of finite dimensional linear spaces V_n (a linear method).

A general greedy algorithm

The first step

Choose $x_1 \in K$ such that

$$\|x_1\|_X = \max_{x \in K} \|x\|_X.$$

The general step

Having found $x_1..x_n$, denote

$$V_n = \operatorname{span}\{x_1, \dots, x_n\}.$$

Choose the next element

$$x_{n+1} := \operatorname*{argmax}_{x \in K} \operatorname{dist}(x, V_n) .$$

The algorithm stops

when $\sigma_n(K) := \max_{x \in K} \operatorname{dist}(x, V_n)$ becomes less than the given tolerance ε.

The Kolmogorov n width, $d_n(K)$ – measures optimal approximation of K by a n-dimensional subspace.

$$d_n(K) := \inf_{\dim Y = n} \sup_{x \in K} \inf_{y \in Y} \|x - y\|_X .$$

The greedy approximation rates have same decay as the Kolmogorov widths.

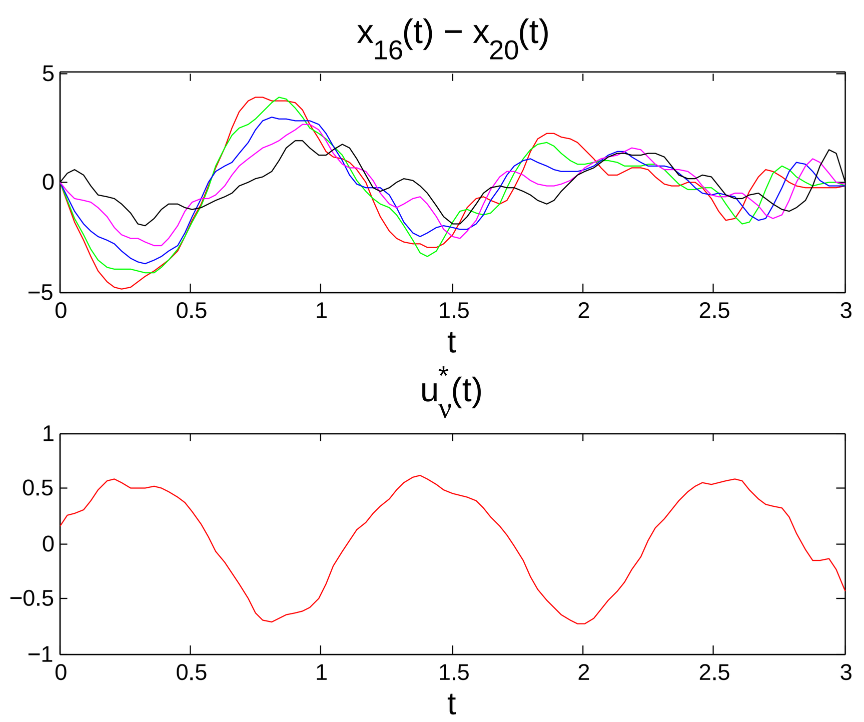

Figure 1: Evolution of a) last 5 system components and b) the approximate control for $\nu = \pi$.

Greedy control

Each control can be uniquely determined by the relation

$$\mathrm{u}_\nu = \mathbf{B}^* e^{(T-t)\mathbf{A}^*_\nu} \varphi^0_\nu,$$

where $\varphi^0_\nu \in \mathbf{R}^N$ is the unique minimiser of a quadratic functional associated to the adjoint problem.

This minimiser can be expressed as the solution of the linear system

$$\Lambda_\nu \varphi^0_\nu = \mathrm{x}^1 - e^{T\mathbf{A}_\nu}\mathbf{x}^0,$$

where Λ_ν is the controllability Gramian

$$\Lambda_\nu = \int_0^T e^{(T-t)\mathbf{A}_\nu} \mathbf{B}_\nu \mathbf{B}^*_\nu e^{(T-t)\mathbf{A}^*_\nu} dt .$$

Perform a greedy algorithm to the manifold $\varphi^0(\mathcal{N})$:

$$\nu \in \mathcal{N} \to \varphi^0_\nu \in \mathbf{R^N} .$$

The (unknown) quantity $\operatorname{dist}(\varphi^0_\nu, \varphi^0_i)$ to be maximised by the greedy algorithm is replaced by a surrogate (Fig. 2):

$$\begin{aligned} \operatorname{dist}(\varphi^0_\nu, \varphi^0_i) &\sim \operatorname{dist}(\Lambda_\nu \varphi^0_\nu, \Lambda_\nu \varphi^0_i) \\ &= \operatorname{dist}(\mathrm{x}^1 - e^{T\mathbf{A}_\nu}\mathbf{x}^0, \Lambda_\nu \varphi^0_i) . \end{aligned}$$

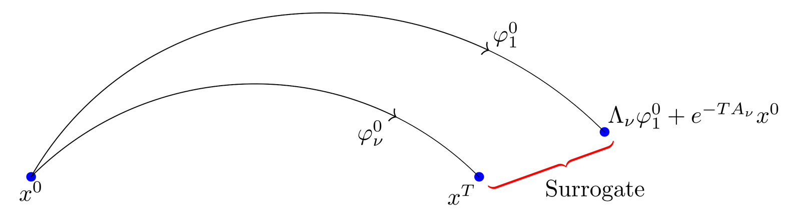

Figure 2: The surrogate of $\operatorname{dist}(\varphi^0_\nu, \varphi^0_i)$

The greedy control algorithm results in an optimal decay of the approximation rates.

Numerical examples

We consider the system (1) with

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & -I \\ \nu(N/2+1)^2 \tilde{\mathbf{A}} & \mathbf{0} \end{pmatrix},$$

$$\tilde{\mathbf{A}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The system corresponds to the discretisation of the wave equation problem with the control on the right boundary:

$$\begin{cases} \partial_{tt} v - \nu \partial_{xx} v = 0, & (t, x) \in \langle 0, T \rangle \times \langle 0, 1 \rangle \\ v(t, 0) = 0, & v(t, 1) = u(t) \\ v(0, x) = v_0, & \partial_t v(x, 0) = v_1 . \end{cases} \quad (2)$$

We take the following values:

$$T = 3, \; N = 20, \; v_0 = \sin(\pi x), \; v_1 = 0, \; x^1 = 0$$
$$\nu \in [1, 10] = \mathcal{N}$$

The greedy control has been applied with $\varepsilon = 0.5$ and the uniform discretisation of $\mathcal{N}$ in $k = 100$ values.

The offline algorithm stopped after 10 iterations.

The 20-D controls manifold is well approximated by a 10-D subspace (Fig. 1, 3).

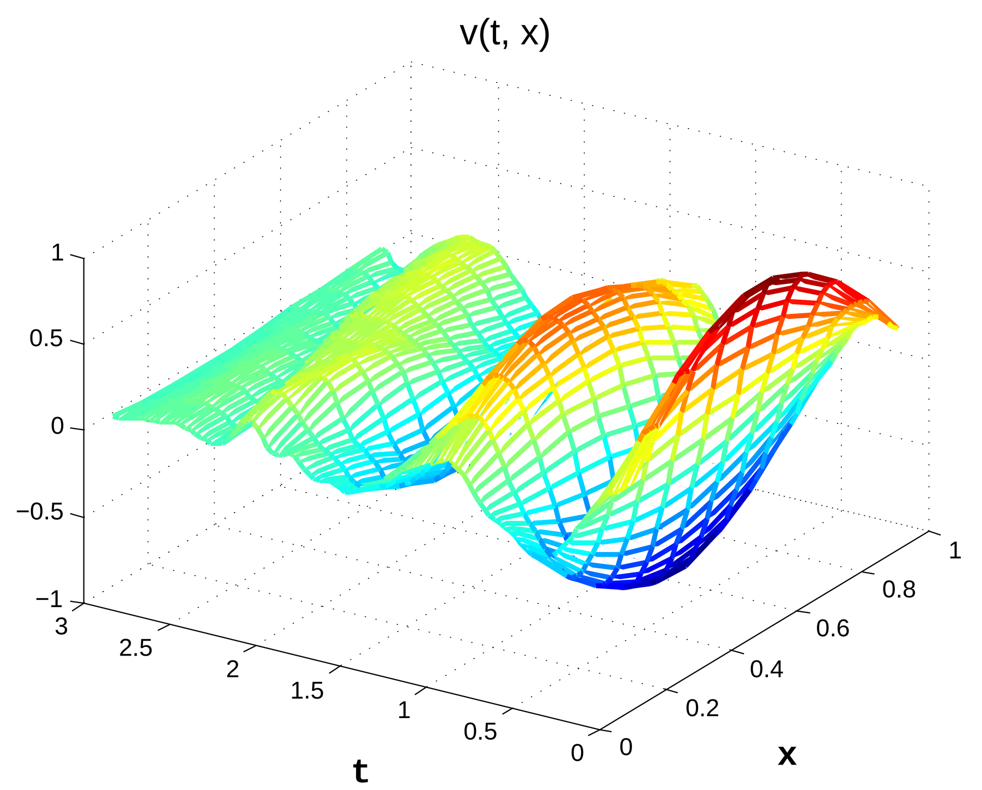

Figure 3: Evolution of the solution to the semi-discretised problem (2) governed by the approximate control $u^\star_\nu$ for $\nu = \pi$.

References

[1] A. Cohen, R. DeVore: Kolmogorov widths under holomorphic mappings, submitted

[2] A. Cohen, R. DeVore: Approximation of high-dimensional parametric PDEs, Acta Numerica, 24 (2015), 1–159.

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