Compensated compactness in the L^p–L^q setting

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February 11, 2016

Div-rot lemma in L^2

Theorem 1. *Assume that Ω is open and bounded subset of $\mathbf{R}^3$, and that it holds:*

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } \mathrm{L}^2(\Omega; \mathbf{R}^3),$$

$$\mathbf{v}_n \rightharpoonup \mathbf{v} \text{ in } \mathrm{L}^2(\Omega; \mathbf{R}^3),$$

$$\operatorname{rot} \mathbf{u}_n \text{ bounded in } \mathrm{L}^2(\Omega; \mathbf{R}^3), \ \operatorname{div} \mathbf{v}_n \text{ bounded in } \mathrm{L}^2(\Omega).$$

Then

$$\mathbf{u}_n \cdot \mathbf{v}_n \rightharpoonup \mathbf{u} \cdot \mathbf{v}$$

in the sense of distributions. ∎

Quadratic theorem

Theorem 2. *(**Quadratic theorem**) Assume that $\Omega \subseteq \mathbf{R}^d$ is open and that $\Lambda \subseteq \mathbf{R}^r$ is defined by*

$$\Lambda := \left\{ \boldsymbol{\lambda} \in \mathbf{R}^r \,:\, (\exists\, \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}) \quad \sum_{k=1}^{d} \xi_k \mathbf{A}^k \boldsymbol{\lambda} = 0 \right\},$$

where Q is a real quadratic form on $\mathbf{R}^r$, which is nonnegative on Λ, i.e.

$$(\forall\, \boldsymbol{\lambda} \in \Lambda) \quad Q(\boldsymbol{\lambda}) \geqslant 0\,.$$

Furthermore, assume that the sequence of functions $(\mathbf{u}_n)$ satisfies

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \textit{weakly in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{R}^r)\,,$$

$$\left(\sum_k \mathbf{A}^k \partial_k \mathbf{u}_n \right) \quad \textit{relatively compact in} \quad \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{R}^q)\,.$$

Then every subsequence of $(Q \circ \mathbf{u}_n)$ which converges in distributions to it's limit L, satisfies

$$L \geqslant Q \circ \mathbf{u}$$

in the sense of distributions. ∎

The most general version of the classical L^2 results has recently been proved by E. Yu. Panov (2011):

Assume that the sequence $(\mathbf{u}_n)$ is bounded in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)$, $2 \le p < \infty$, and converges weakly in $\mathcal{D}'(\mathbf{R}^d)$ to a vector function $\mathbf{u}$.
Let $q = p'$ if $p < \infty$, and $q > 1$ if $p = \infty$. Assume that the sequence

$$\sum_{k=1}^{\nu} \partial_k(\mathbf{A}^k \mathbf{u}_n) + \sum_{k,l=\nu+1}^{d} \partial_{kl}(\mathbf{B}^{kl} \mathbf{u}_n)$$

is precompact in the anisotropic Sobolev space $\mathrm{W}^{-1,-2;q}_{loc}(\mathbf{R}^d;\mathbf{R}^m)$, where $m \times r$ matrices $\mathbf{A}^k$ and $\mathbf{B}^{kl}$ have variable coefficients belonging to $\mathrm{L}^{2\bar{q}}(\mathbf{R}^d)$, $\bar{q} = \frac{p}{p-2}$ if $p > 2$, and to the space $\mathrm{C}(\mathbf{R}^d)$ if $p = 2$.

We introduce the set $\Lambda(\mathbf{x})$

$$\Lambda(\mathbf{x}) = \Bigg\{ \boldsymbol{\lambda} \in \mathbf{C}^r \Big| \, (\exists \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}) : \qquad (1)$$

$$\Bigg(i \sum_{k=1}^{\nu} \xi_k \mathbf{A}^k(\mathbf{x}) - 2\pi \sum_{k,l=\nu+1}^{d} \xi_k \xi_l \mathbf{B}^{kl}(\mathbf{x}) \Bigg) \boldsymbol{\lambda} = \mathbf{0}_m \Bigg\},$$

and consider the bilinear form on $\mathbf{C}^r$

$$q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\eta}) = \mathbf{Q}(\mathbf{x}) \boldsymbol{\lambda} \cdot \boldsymbol{\eta}, \qquad (2)$$

where $\mathbf{Q} \in \mathrm{L}^{\bar{q}}_{loc}(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p > 2$ and $\mathbf{Q} \in \mathrm{C}(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p = 2$.
Finally, let $q(\mathbf{x}, \mathbf{u}_n, \mathbf{u}_n) \rightharpoonup \omega$ weakly in the space of distributions.

Result by Panov

The following theorem holds

Theorem 3. *[P, 2011] Assume that* $(\forall \boldsymbol{\lambda} \in \Lambda(\mathbf{x}))\ q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\lambda}) \geq 0$ *(a.e.* $\mathbf{x} \in \mathbf{R}^d$*) and* $\mathbf{u}_n \rightharpoonup \mathbf{u}$*, then* $q(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})) \leq \omega$*.* ∎

The connection between q and Λ given in the previous theorem, we shall call *the consistency condition*.

H-distributions

H-distributions were introduced by N. Antonić and D. Mitrović (2011) as an extension of H-measures to the $\mathrm{L}^p - \mathrm{L}^q$ context.

M. Lazar and D. Mitrović (2012) extended and applied them on a velocity averaging problem.

We need Fourier multiplier operators with symbols defined on a manifold P determined by d-tuple $\boldsymbol{\alpha} \in (\mathbf{R}^+)^d$:

$$\mathrm{P} = \Big\{ \boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{2\alpha_k} = 1 \Big\},$$

where $\alpha_k \in \mathbf{N}$ or $\alpha_k \geq d$. In order to associate an L^p Fourier multiplier to a function defined on P, we extend it to $\mathbf{R}^d \backslash \{0\}$ by means of the projection

$$(\pi_{\mathrm{P}}(\boldsymbol{\xi}))_j = \xi_j \left(|\xi_1|^{2\alpha_1} + \cdots + |\xi_d|^{2\alpha_d} \right)^{-1/2\alpha_j}, \quad j = 1, \ldots, d.$$

We need the following extension of the results given above.

Theorem 4. *Let* (u_n) *be a bounded sequence in* $\mathrm{L}^p(\mathbf{R}^d)$, $p > 1$, *and let* (v_n) *be a bounded sequence of uniformly compactly supported functions in* $\mathrm{L}^q(\mathbf{R}^d)$, $1/q + 1/p < 1$. *Then, after passing to a subsequence (not relabelled), for any* $\bar{s} \in (1, \frac{pq}{p+q})$ *there exists a continuous bilinear functional* B *on* $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d) \otimes \mathrm{C}^d(\mathrm{P})$ *such that for every* $\varphi \in \mathrm{L}^{\bar{s}'}(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^d(\mathrm{P})$, *it holds*

$$B(\varphi, \psi) = \lim_{n \to \infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_n(\mathbf{x}) \big(\mathcal{A}_{\psi_\mathrm{P}} v_n\big)(\mathbf{x}) d\mathbf{x} \,,$$

where $\mathcal{A}_{\psi_\mathrm{P}}$ *is the Fourier multiplier operator on* $\mathbf{R}^d$ *associated to* $\psi \circ \pi_\mathrm{P}$. *The bilinear functional* B *can be continuously extended as a linear functional on* $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d; \mathrm{C}^d(\mathrm{P}))$. ∎

Localisation principle

Lemma
Assume that sequences $(\mathbf{u}_n)$ and $(\mathbf{v}_n)$ are bounded in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)$ and $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^r)$, respectively, and converge toward $\mathbf{0}$ and $\mathbf{v}$ in the sense of distributions.
Furthermore, assume that sequence $(\mathbf{u}_n)$ satisfies:

$$\mathbf{G}_n := \sum_{k=1}^{d} \partial_k^{\alpha_k}(\mathbf{A}^k \mathbf{u}_n) \to \mathbf{0} \text{ in } \mathrm{W}^{-\alpha_1,\ldots,-\alpha_d;p}(\Omega;\mathbf{R}^m), \tag{3}$$

where either $\alpha_k \in \mathbf{N}$, $k = 1,\ldots,d$ or $\alpha_k > d$, $k = 1,\ldots,d$, and elements of matrices $\mathbf{A}^k$ belong to $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d)$, $\bar{s} \in (1, \frac{pq}{p+q})$.
Finally, by $\boldsymbol{\mu}$ denote a matrix H-distribution corresponding to subsequences of $(\mathbf{u}_n)$ and $(\mathbf{v}_n - \mathbf{v})$. Then the following relation holds

$$\Big(\sum_{k=1}^{d} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k\Big)\boldsymbol{\mu} = \mathbf{0}.$$

Strong consistency condition

Introduce the set

$$\Lambda_{\mathcal{D}} = \Big\{ \boldsymbol{\mu} \in \ \mathrm{L}^{\bar{s}}(\mathbf{R}^d; (\mathrm{C}^d(\mathrm{P}))')^r : \Big(\sum_{k=1}^{n} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k \Big) \boldsymbol{\mu} = \mathbf{0}_m \Big\},$$

where the given equality is understood in the sense of $\mathrm{L}^{\bar{s}}(\mathbf{R}^d; (\mathrm{C}^d(\mathrm{P}))')^m$.

Let us assume that coefficients of the bilinear form q from (2) belong to space $\mathrm{L}^t_{loc}(\mathbf{R}^d)$, where $1/t + 1/p + 1/q < 1$.

Definition

We say that set $\Lambda_{\mathcal{D}}$, bilinear form q from (2) and matrix $\boldsymbol{\mu} = [\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r], \boldsymbol{\mu}_j \in \mathrm{L}^{\bar{s}}(\mathbf{R}^d; (\mathrm{C}^d(\mathrm{P}))')^r$ satisfy the strong consistency condition if $(\forall j \in \{1, \dots, r\})\ \boldsymbol{\mu}_j \in \Lambda_{\mathcal{D}}$, and it holds

$$\langle \phi \mathbf{Q} \otimes 1, \boldsymbol{\mu} \rangle \geq \mathbf{0}, \quad \phi \in \mathrm{L}^{\bar{s}}(\mathbf{R}^d; \mathbf{R}^+_0).$$

Theorem 5. *Assume that sequences* $(\mathbf{u}_n)$ *and* $(\mathbf{v}_n)$ *are bounded in* $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)$ *and* $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^r)$*, respectively, and converge toward* $\mathbf{u}$ *and* $\mathbf{v}$ *in the sense of distributions.*
Assume that (3) *holds and that*

$$q(\mathbf{x};\mathbf{u}_n,\mathbf{v}_n) \rightharpoonup \omega \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

If the set $\Lambda_{\mathcal{D}}$*, the bilinear form* (2)*, and matrix* H*-distribution* $\boldsymbol{\mu}$*, corresponding to subsequences of* $(\mathbf{u}_n - \mathbf{u})$ *and* $(\mathbf{v}_n - \mathbf{v})$*, satisfy the strong consistency condition, then*

$$q(\mathbf{x};\mathbf{u},\mathbf{v}) \leq \omega \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

∎

Application to the parabolic type equation

Now, let us consider the non-linear parabolic type equation

$$L(u) = \partial_t u - \mathsf{div}\,\mathsf{div}\,(g(t,\mathbf{x},u)\mathbf{A}(t,\mathbf{x})),$$

on $(0,\infty)\times\Omega$, where Ω is an open subset of $\mathbf{R}^d$. We assume that

$$u \in \mathrm{L}^p((0,\infty)\times\Omega), \quad g(t,\mathbf{x},u(t,\mathbf{x})) \in \mathrm{L}^q((0,\infty)\times\Omega), \quad 1 < p, q,$$
$$\mathbf{A} \in \mathrm{L}^s_{loc}((0,\infty)\times\Omega)^{d\times d}, \quad \text{where} \quad 1/p + 1/q + 1/s < 1,$$

and that the matrix $\mathbf{A}$ is strictly positive definite, i.e.

$$\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi} > 0, \quad \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{\mathbf{0}\}, \quad (a.e.(t,\mathbf{x}) \in (0,\infty)\times\Omega).$$

Furthermore, assume that g is a Carathèodory function and non-decreasing with respect to the third variable.

Then we have the following theorem.

Theorem 6. *Assume that sequences* (u_r) *and* $g(\cdot, u_r)$ *are such that* $u_r, g(u_r) \in \mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d)$ *for every* $r \in \mathbf{N}$*; assume that they are bounded in* $\mathrm{L}^p(\mathbf{R}^+ \times \mathbf{R}^d)$, $p \in (1, 2]$, *and* $\mathrm{L}^q(\mathbf{R}^+ \times \mathbf{R}^d)$, $q > 2$, *respectively, where* $1/p + 1/q < 1$*; furthermore, assume* $u_r \rightharpoonup u$ *and, for some,* $f \in \mathrm{W}^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d)$, *the sequence*

$$L(u_r) = f_r \to f \quad \textit{strongly in } \mathrm{W}^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d).$$

Under the assumptions given above, it holds

$$L(u) = f \ \text{ in } \ \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d).$$

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