Classical Optimal Design in Two-phase Conductivity Problems

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Compliance maximization

State equation ($\Omega \subseteq \mathbf{R}^d$ open and bounded)

$$\begin{cases} -\operatorname{div}(\mathbf{A}\nabla u) = 1 = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

Two phases: $0 < \alpha < \beta$
$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$, $\int_\Omega \chi \, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$
Cost functional:

$$J(\chi) = \int_\Omega u(\mathbf{x}) d\mathbf{x} \longrightarrow \max$$

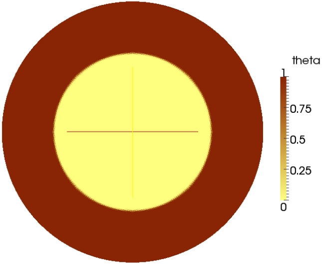
Interpretations:

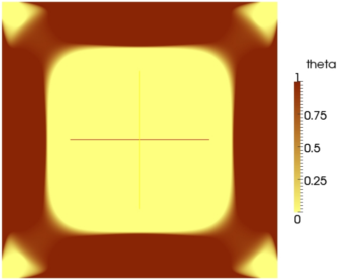
- Maximize the amount of heat kept inside body
- Maximize the torsional rigidity of a rod made of two materials
- Maximize the flow rate of two viscous immiscible fluids through pipe

In general, compliance functional

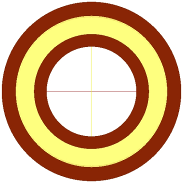
$$J(\chi) = \int_\Omega f(\mathbf{x}) u(\mathbf{x}) \, d\mathbf{x} \longrightarrow \max$$

Classical vs. relaxed optimal design





Intuition for annulus?



In general, there might exist no classical optimal design. The relaxation is needed, introducing composite materials

$\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$	$\cdots$	$\theta \in \mathrm{L}^\infty(\Omega; [0,1])$
		$\mathbf{A} \in \mathcal{K}(\theta)$ a.e. on Ω
classical design		relaxed design

Effective conductivities – set $\mathcal{K}(\theta)$

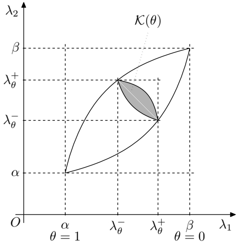
$\mathcal{K}(\theta)$ is given in terms of eigenvalues:

$$
\begin{aligned}
\lambda_\theta^- \leq \quad \lambda_j \quad &\leq \lambda_\theta^+ \quad j = 1, \ldots, d \\
\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \quad &\leq \quad \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha} \\
\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \quad &\leq \quad \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+} \,,
\end{aligned}
$$

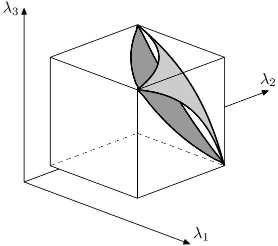
where

$$
\begin{aligned}
\lambda_\theta^+ \quad &= \quad \theta\alpha + (1-\theta)\beta \\
\frac{1}{\lambda_\theta^-} \quad &= \quad \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}
\end{aligned}
$$

2D:



3D:



Multiple state optimal design problem

State equations

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad i = 1, \ldots, m$$

State function $\mathrm{u} = (u_1, \ldots, u_m)$

$$\begin{cases} I(\chi) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \max \\ \qquad \mathrm{u} = (u_1, \ldots, u_m) \text{ state function for } \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \\ \qquad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \, , \ \displaystyle\int_\Omega \chi \, d\mathbf{x} = q_\alpha \, , \end{cases}$$

for some given weights $\mu_i > 0$. Relaxed designs:

$$\mathcal{A} := \left\{ (\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \, , \ \mathbf{A}(\mathbf{x}) \in \mathcal{K}(\theta(\mathbf{x})) \text{ a.e. on } \Omega \right\}$$

Relaxation ...

$$\begin{cases} J(\theta, \mathbf{A}) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \max \\ \qquad (\theta, \mathbf{A}) \in \mathcal{A} \end{cases}$$

Single vs. multiple state problems

A. Single state equation

[Murat & Tartar, 1985] There exists relaxed solution $(\theta^*, \mathbf{A}^*)$ among simple laminates ... conductivity λ_θ^- in one direction (∇u), and λ_θ^+ in orthogonal directions. As a consequence, θ^* is also a solution of

$$I(\theta) = \int_\Omega fu\, d\mathbf{x} \to \max$$

$$\theta \in \mathrm{L}^\infty(\Omega; [0,1])\,, \ \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,,$$

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^- \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

can be rewritten as a convex minimization problem

B. Multiple state equations

It is not enough to use only simple laminates, but composite materials that correspond to a non-affine boundary of $\mathcal{K}(\theta)$ ... higher order sequential laminates. The above simpler relaxation fails.

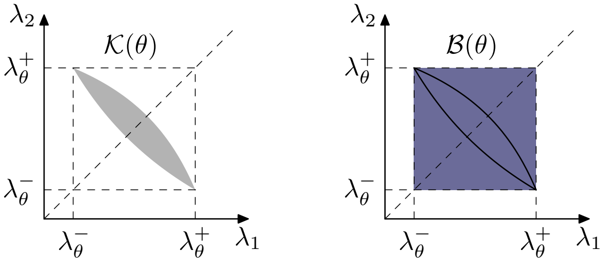
The aim of this talk

- in spherically symmetric case, simpler relaxation is correct
- present some problems with classical optimal design

Extended set of admissible designs

We shall enlarge the set $\mathcal{A}$ of admissible designs

$$\mathcal{A} = \left\{ (\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \ \mathbf{A} \in \mathcal{K}(\theta) \text{ (a.e. on } \Omega) \right\}$$



$$\mathcal{B} = \left\{ (\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \ \mathbf{A} \in \mathcal{B}(\theta) \text{ (a.e. on } \Omega) \right\}$$

$$\mathcal{C} := \left\{ (\theta, \mathbf{B}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}) : (\theta, \mathbf{B}^{-1}) \in \mathcal{B} \right\}.$$

Extended set of admissible designs

$\mathcal{B}$ and $\mathcal{C}$ are convex sets: e.g. $\mathcal{B}$ can be rewritten as

$$\lambda_{\min}(\mathbf{A}) \geq \lambda_\theta^- \,,\; \lambda_{\max}(\mathbf{A}) \leq \lambda_\theta^+ \,, \quad \text{a.e. on } \Omega \,,$$

where $\lambda_{\min}$ and $\lambda_\cdot^+$ are concave, and $\lambda_{\max}$ and $\lambda_\cdot^-$ are convex functions.

$$\begin{aligned}
-J(\theta, \mathbf{A}) &= -\sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \\
&= -\sum_{i=1}^m \mu_i \int_\Omega \mathbf{A}\nabla u_i \cdot \nabla u_i - 2 f_i u_i \, d\mathbf{x} \\
&= -\min_{v \in \mathrm{H}_0^1(\Omega;\mathbf{R}^m)} \sum_{i=1}^m \mu_i \int_\Omega \mathbf{A}\nabla v_i \cdot \nabla v_i - 2 f_i v_i \, d\mathbf{x} \\
&= -\max_{\boldsymbol{\sigma} \in \mathcal{S}} \left(-\sum_{i=1}^m \mu_i \int_\Omega \mathbf{A}^{-1} \sigma_i \cdot \sigma_i \, d\mathbf{x} \right) \,,
\end{aligned}$$

where $\mathcal{S} = \{\sigma \in \mathrm{L}^2(\Omega; \mathbf{R}^d)^m : -\operatorname{div} \sigma_i = f_i, i = 1, \ldots, m\}$.

Representation by a convex optimization problem

Lemma

There exists a unique $\boldsymbol{\sigma}^* \in \mathcal{S} = \{\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega; \mathbf{R}^d)^m : -\mathrm{div}\, \boldsymbol{\sigma}_i = f_i, i = 1..m\}$ *such that*

$$\max_{(\theta,\mathbf{A})\in\mathcal{B}} J(\theta, \mathbf{A}) = \max_{(\theta,\mathbf{A})\in\mathcal{B}} \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{A}^{-1}\boldsymbol{\sigma}_i^* \cdot \boldsymbol{\sigma}_i^* \, d\mathbf{x} = \max_{(\theta,\mathbf{B})\in\mathcal{C}} \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{B}\boldsymbol{\sigma}_i^* \cdot \boldsymbol{\sigma}_i^* \, d\mathbf{x} \,. \tag{1}$$

Moreover, if $(\theta^*, \mathbf{A}^*)$ *is an optimal design for problem* $\max_\mathcal{B} J$ *and* u^* *the corresponding state function, then* $\mathbf{A}^*\nabla u_i^* = \boldsymbol{\sigma}_i^*\,,\ i = 1, \ldots, m\,.$

Above maximization problems are easily solved:
Design $(\theta^*, \mathbf{A}^*)$ is optimal if and only if (almost everywhere in Ω)

$$(\mathbf{A}^*)^{-1}\boldsymbol{\sigma}_i^* = \frac{1}{\lambda^-_{\theta^*}}\, \boldsymbol{\sigma}_i^* \;\; i = 1..m\,.$$

If u^* is the corresponding state function, we have

$$\boldsymbol{\sigma}_i^* = \lambda^-_{\theta^*}\nabla u_i^* \text{ or equivalently } \mathbf{A}^*\nabla u_i^* = \lambda^-_{\theta^*}\nabla u_i^*\,,\ i = 1..m\,.$$

Simpler relaxation problem

... in terms of only local fraction θ belonging to the set

$$\mathcal{T} := \left\{ \theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \right\}$$

Theorem

Let $(\theta^*, \mathbf{A}^*)$ be an optimal design for the problem $\max_\mathcal{B} J$. Then θ^* solves

$$\begin{aligned} & I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \max \\ & \theta \in \mathcal{T} \ \ \textit{and } \mathrm{u} \textit{ determined uniquely by} \\ & \begin{cases} -\mathrm{div}\,(\lambda_\theta^- \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \ldots, m\,, \end{aligned} \tag{2}$$

Conversely, if $\widetilde{\theta}$ is a solution of optimal design problem (2), and $\widetilde{\mathrm{u}}$ is the corresponding state function, then for any measurable $\widetilde{\mathbf{A}} \in \mathcal{B}(\widetilde{\theta})$ such that $\widetilde{\mathbf{A}} \nabla \widetilde{u}_i = \lambda_-(\widetilde{\theta}) \nabla \widetilde{u}_i$ almost everywhere on Ω, e.g. for $\widetilde{\mathbf{A}} = \lambda_-(\widetilde{\theta})\mathbf{I}$, $(\widetilde{\theta}, \widetilde{\mathbf{A}})$ is an optimal design for the problem $\max_\mathcal{B} J$.

Necessary and sufficient optimality conditions

Similar to Lemma above, one can rephrase the simpler relaxation problem (2): there exists a unique $\boldsymbol{\sigma}^* \in \mathcal{S} = \{\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega; \mathbf{R}^d)^m : -\mathrm{div}\, \boldsymbol{\sigma}_i = f_i, i = 1..m\}$ such that

$$\max_{\mathcal{T}} I = \max_{\theta \in \mathcal{T}} \sum_{i=1}^{m} \mu_i \int_\Omega \frac{\beta - \alpha}{\alpha\beta}\, \theta |\boldsymbol{\sigma}_i^*|^2 \, d\mathbf{x} \,.$$

Moreover, $\boldsymbol{\sigma}^*$ is the same as for the problem $\max_{\mathcal{B}} J$.

Lemma

The necessary and sufficient condition of optimality for solution $\theta^* \in \mathcal{T}$ of optimal design problem (2) simplifies to the existence of a Lagrange multiplier $c \geq 0$ such that

$$\sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 > c \quad \Rightarrow \quad \theta^* = 1 \,,$$
$$\sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 < c \quad \Rightarrow \quad \theta^* = 0 \,.$$

Spherically symmetric case

Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric: in spherical coordinates given by $r \in \omega$ (an interval), and the right-hand side $\mathrm{f} = \mathrm{f}(r)$, $r \in \omega$ be a radial function.
Since $\boldsymbol{\sigma}^*$ is unique, it must be radial: $\boldsymbol{\sigma}_i^* = \sigma_i^*(r)\mathbf{e}_r$.

Lemma

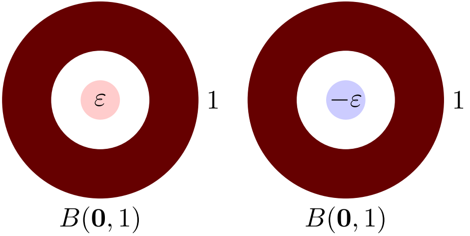
For any maximizer $(\theta^*, \mathbf{A}^*)$ *for the problem* $\max_{\mathcal{B}} J$*, there exist a radial maximizer* $(\widetilde{\theta}, \widetilde{\mathbf{A}}) \in \mathcal{B}$ *where*

$$\widetilde{\theta}(r) = \fint_{\partial B(\mathbf{0},r)} \theta^* \, dS \,.$$

Theorem

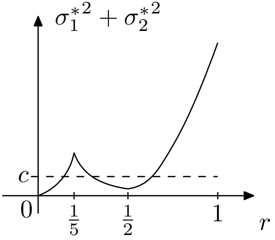
a. *If* $\widetilde{\theta}$ *is a maximizer of* I *over* $\mathcal{T}$*, then for a simple laminate* $\widetilde{\mathbf{A}} \in \mathcal{K}(\widetilde{\theta})$ *with layers orthogonal to* $\mathbf{e}_r$*,* $(\widetilde{\theta}, \widetilde{\mathbf{A}})$ *is a maximizer of* J *over* $\mathcal{A}$*.*
b. *For any maximizer* $(\theta^*, \mathbf{A}^*)$ *of* J *over* $\mathcal{A}$*,* θ^* *is a maximizer of* I *over* $\mathcal{T}$*.*

Back to the example $\varepsilon > 0$



$$f_{1,2}(r) = \begin{cases} \pm\varepsilon, & 0 \le r \le \frac{1}{5} \\ 0, & \frac{1}{5} < r \le \frac{1}{2} \\ 1, & \frac{1}{2} < r \le 1\,. \end{cases}$$

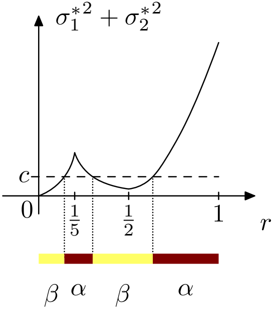
$-\operatorname{div} \boldsymbol{\sigma}_i = f_i, i = 1, 2$ in polar coordinates: $-\frac{1}{r}\,(r\sigma_i)' = f_i$. Due to regularity at r=0, we can calculate unique solutions σ_1^* and σ_2^*:



$$\begin{aligned} {\sigma_1^*}^2 + {\sigma_2^*}^2 > c \quad &\Rightarrow \quad \theta^* = 1\,, \\ {\sigma_1^*}^2 + {\sigma_2^*}^2 < c \quad &\Rightarrow \quad \theta^* = 0\,. \end{aligned}$$

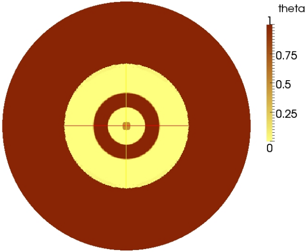
For any c, the solution θ^* is unique and classical (more precisely, the uniqueness of solution for $\max_{\mathcal{B}} J$ follows).

How to determine Lagrange multiplier c?



$$\sigma_1^{*2} + \sigma_2^{*2} > c \quad \Rightarrow \quad \theta^* = 1\,,$$
$$\sigma_1^{*2} + \sigma_2^{*2} < c \quad \Rightarrow \quad \theta^* = 0\,.$$

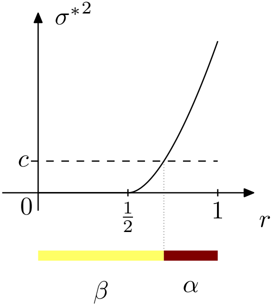
Amount q_α of the first phase uniquely determines c (as usual).



The case $\varepsilon = 0$

$$f_{1,2}(r) := f(r) = \begin{cases} 0, & 0 < r \leq \frac{1}{2} \\ 1, & \frac{1}{2} < r \leq 1\,. \end{cases}$$

Small q_α: unique classical solution



$$\sigma^{*2} > c \quad \Rightarrow \quad \theta^* = 1\,,$$
$$\sigma^{*2} < c \quad \Rightarrow \quad \theta^* = 0\,.$$

If $q_\alpha > \frac{3}{4}\pi$ then c have to be zero. Now, solution is not unique – it is only important to put α in annulus $B\left(\mathbf{0}, \frac{1}{2}\right)^c$.

Uniqueness

Conditions of optimality:

$$\sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 > c \quad \Rightarrow \quad \theta^* = 1\,,$$
$$\sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 < c \quad \Rightarrow \quad \theta^* = 0\,.$$

In case of spherical symmetry $\boldsymbol{\sigma}_i^* = \sigma_i^*(r)\mathbf{e}_r$, we denote

$$\psi(r) := \sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 = \sum_{i=1}^{m} \mu_i (\sigma_i^*)^2\,.$$

Corollary

For spherically symmetric case, if ψ is piecewise strictly monotone on ω then the problem $\max_{\mathcal{T}} I$ has a unique solution θ^*, which is a characteristic function. Consequently, the solutions of the problems $\max_{\mathcal{B}} J$ and $\max_{\mathcal{A}} J$ are unique and classical.

Example

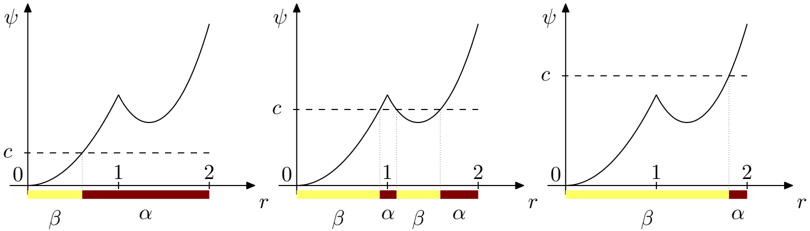
Two state equations on a ball $\Omega = B(\mathbf{0}, 2R) \subseteq \mathbf{R}^d$, $d = 2$ or 3.

- $f_1 = \chi_{B(\mathbf{0},R)}$, $f_2 = \chi_{B(\mathbf{0},R)^c}$,
- $\begin{cases} -\mathrm{div}\,(\lambda_\theta^- \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, 2$
- $\mu_1 \int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \to \max$

For studying conditions of optimality, we introduce

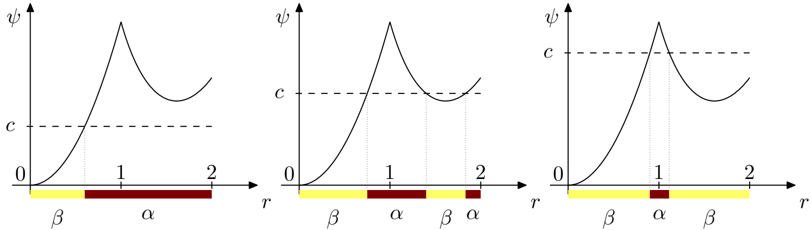
$$\psi(r) = \mu_1 \left(\sigma_1^*(r)\right)^2 + \left(\sigma_2^*(r)\right)^2 \ .$$

The case $0 < \mu_1 < 3$ for $d = 2$, or $0 < \mu_1 < \frac{49}{15}$ for $d = 3$:

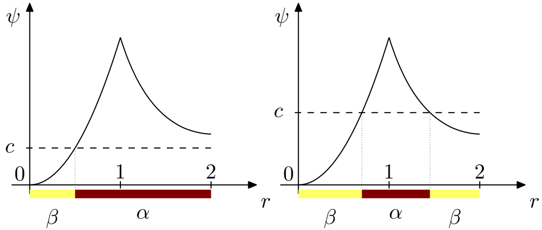


Example

The case $3 \leq \mu_1 < 15$ for $d = 2$, or $\frac{49}{15} \leq \mu_1 < 35$:

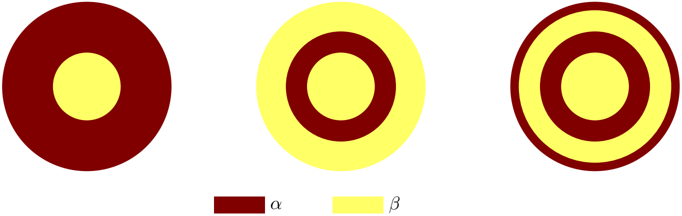


The case $\mu_1 \geq 15$ for $d = 2$, or $\mu_1 \geq 35$ for $d = 3$:



Multiple states

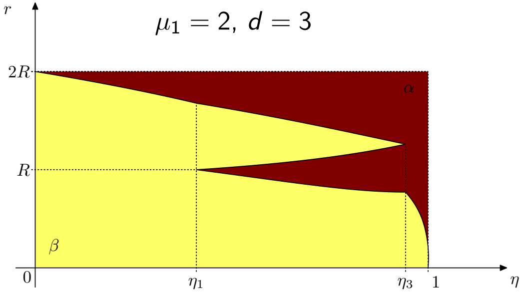
Three optimal configurations, depending on μ_1 and q_α:



Overall percentage of the first material: $\eta = \frac{q_\alpha}{|\Omega|}$.

Radii are obtained by solving algebraic equations in terms of μ_1 and η:

- $d = 2$ – explicitly
- $d = 3$ – numerically.



Conclusion

General strategy for solving $\max_{\mathcal{A}} J$ in spherically symmetric case:

1. Solve $-\operatorname{div} \boldsymbol{\sigma}_i = f_i, i = 1..m$ – candidates for $\boldsymbol{\sigma}^*$ (in case of ball there is only one candidate).
2. Study conditions of optimality (they usually give unique solution θ^* – radial, but also classical).
3. Construct solution to $\max_{\mathcal{A}} J$ (commonly, it would be classical solution; for minimization problem the situation is quite different).

Thank you for your attention!