Multiple state optimal design problems with explicit solution

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88th GAMM Annual Meeting, Weimar, March **2017**

Multiple state problem optimizing energy

Fill $\Omega \subseteq \mathbf{R}^d$ with two isotropic materials with conductivity $0 < \alpha < \beta$, quantity q_α of the first material is given:

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}, \quad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$$

$$\int_\Omega \chi \, d\mathbf{x} = q_\alpha$$

State equations

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad i = 1, \ldots, m\,,$$

Goal functional is a conic sum of energies ($\mu_i > 0$)

$$I(\chi) = \sum_{i=1}^m \mu_i \int_\Omega f_i(\mathbf{x}) u_i(\mathbf{x}) d\mathbf{x} \longrightarrow \min / \max$$

Relaxation via homogenization theory:

classical design		relaxed design
$\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$	$\cdots$	$\theta \in \mathrm{L}^\infty(\Omega; [0,1])$
		$\mathbf{A} \in \mathcal{K}(\theta)$ a.e. on Ω
$I(\chi)$		$J(\theta, \mathbf{A})$ – given by the same formula

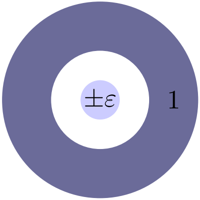
Example – maximization

$\Omega := B(\mathbf{0}, 1) \subseteq \mathbf{R}^2$, $q_\alpha := 0.8|\Omega|$

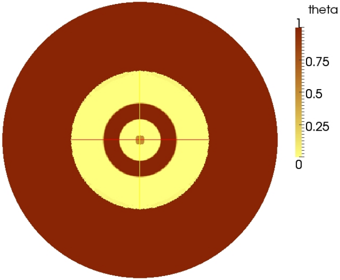
$I(\chi) = \sum_{i=1}^{2} \int_\Omega f_i(\mathbf{x}) u_i(\mathbf{x}) d\mathbf{x} \longrightarrow \max$

$$\begin{matrix} f_1 = \chi_A + \varepsilon \chi_B \\ f_2 = \chi_A - \varepsilon \chi_B \end{matrix}, \text{ where}$$

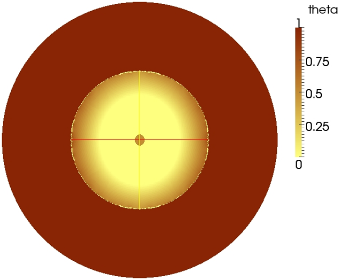
$A := B\left(\mathbf{0}, \frac{1}{2}\right)^c$, $B := B\left(\mathbf{0}, \frac{1}{5}\right)$



$B(\mathbf{0}, 1)$



Numerical solution, $\varepsilon = 0.01$



Numerical solution, $\varepsilon = 0$

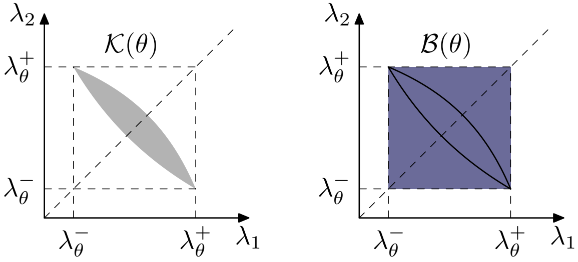
Representation by a concave maximization problem

The space of admissible local fractions

$$\mathcal{T} := \left\{ \theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \right\}$$

Admissible (relaxed) designs

$$\mathcal{A} = \{(\theta, \mathbf{A}) \in \mathcal{T} \times \mathrm{L}^\infty(\Omega; \mathrm{Sym}) : \mathbf{A} \in \mathcal{K}(\theta) \text{ (a.e. on } \Omega)\}$$



$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta$$
$$\frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}$$

$$\mathcal{B} = \{(\theta, \mathbf{A}) \in \mathcal{T} \times \mathrm{L}^\infty(\Omega; \mathrm{Sym}) : \mathbf{A} \in \mathcal{B}(\theta) \text{ (a.e. on } \Omega)\}$$

Maximization over $\mathcal{B}$

Murat and Tartar (1985), Casado-Díaz (2015) – one state equation.

Lemma

There exists a unique $\boldsymbol{\sigma}^* \in \mathcal{S} = \{\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega; \mathbf{R}^d)^m : -\mathrm{div}\, \boldsymbol{\sigma}_i = f_i, i = 1..m\}$ *such that*

$$\max_{(\theta,\mathbf{A})\in\mathcal{B}} J(\theta, \mathbf{A}) = \max_{(\theta,\mathbf{A})\in\mathcal{B}} \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{A}^{-1}\boldsymbol{\sigma}_i^* \cdot \boldsymbol{\sigma}_i^* \, d\mathbf{x} \,. \tag{1}$$

Moreover, if $(\theta^*, \mathbf{A}^*)$ *is an optimal design for problem* $\max_{\mathcal{B}} J$ *and* u^* *the corresponding state function, then* $\mathbf{A}^*\nabla u_i^* = \boldsymbol{\sigma}_i^*\,,\ i = 1, \dots, m\,.$

Above maximization problems is easily solved:
Design $(\theta^*, \mathbf{A}^*)$ is optimal if and only if (almost everywhere in Ω)

$$\mathbf{A}^*\boldsymbol{\sigma}_i^* = \lambda_{\theta^*}^-\boldsymbol{\sigma}_i^*\,,\ i = 1..m\,.$$

If u^* is the corresponding state function, we have

$$\boldsymbol{\sigma}_i^* = \lambda_{\theta^*}^-\nabla u_i^* \text{ or equivalently } \mathbf{A}^*\nabla u_i^* = \lambda_{\theta^*}^-\nabla u_i^*\,,\ i = 1..m\,.$$

Simpler relaxation problem

Theorem

Let $(\theta^*, \mathbf{A}^*)$ be an optimal design for the problem $\max_{\mathcal{B}} J$. Then θ^* solves

$$
\begin{aligned}
& I(\theta) = \sum_{i=1}^{m} \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \max \\
& \theta \in \mathcal{T} \quad \textit{and } \mathrm{u} \textit{ determined uniquely by} \\
& \begin{cases} -\mathrm{div}\,(\lambda_\theta^- \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \ldots, m \, ,
\end{aligned}
\tag{2}
$$

Conversely, if $\widetilde{\theta}$ is a solution of optimal design problem (2), and $\widetilde{\mathrm{u}}$ is the corresponding state function, then for any measurable $\widetilde{\mathbf{A}} \in \mathcal{B}(\widetilde{\theta})$ such that $\widetilde{\mathbf{A}} \nabla \widetilde{u}_i = \lambda_-(\widetilde{\theta}) \nabla \widetilde{u}_i$, e.g. for $\widetilde{\mathbf{A}} = \lambda_-(\widetilde{\theta})\mathbf{I}$, $(\widetilde{\theta}, \widetilde{\mathbf{A}})$ is an optimal design for the problem $\max_{\mathcal{B}} J$.

Necessary and sufficient optimality conditions

Similar to Lemma above, one can rephrase the simpler relaxation problem (2): there exists a unique $\boldsymbol{\sigma}^* \in \mathcal{S} = \{\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega; \mathbf{R}^d)^m : -\operatorname{div} \boldsymbol{\sigma}_i = f_i, i = 1..m\}$ such that

$$\max_{\mathcal{T}} I = \max_{\theta \in \mathcal{T}} \sum_{i=1}^m \mu_i \int_\Omega \frac{\beta - \alpha}{\alpha\beta} \theta |\boldsymbol{\sigma}_i^*|^2 \, d\mathbf{x} \, .$$

Moreover, $\boldsymbol{\sigma}^*$ is the same as for $\max_{\mathcal{B}} J$.

Lemma

The necessary and sufficient condition of optimality for solution $\theta^* \in \mathcal{T}$ of optimal design problem (2) simplifies to the existence of a Lagrange multiplier $c \geq 0$ such that

$$\sum_{i=1}^m \mu_i |\boldsymbol{\sigma}_i^*|^2 > c \quad \Rightarrow \quad \theta^* = 1 \, ,$$

$$\sum_{i=1}^m \mu_i |\boldsymbol{\sigma}_i^*|^2 < c \quad \Rightarrow \quad \theta^* = 0 \, .$$

Spherically symmetric case

Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric (ball or annulus), and let the right-hand sides be radial functions: $f_i = f_i(r)$.
Since $\boldsymbol{\sigma}^*$ is unique, it must be radial: $\boldsymbol{\sigma}_i^* = \sigma_i^*(r)\mathbf{e}_r$.

Theorem

For any maximizer θ^* for $\max_{\mathcal{T}} I$, the radial function

$$\widetilde{\theta}(r) = \fint_{\partial B(\mathbf{0},r)} \theta^* \, dS$$

is also a maximizer.

- *If $\widetilde{\theta}$ is a maximizer of I over $\mathcal{T}$, then for a simple laminate $\widetilde{\mathbf{A}} \in \mathcal{K}\big(\widetilde{\theta}\big)$ with layers orthogonal to $\mathbf{e}_r$, $(\widetilde{\theta}, \widetilde{\mathbf{A}})$ is a maximizer of J over $\mathcal{A}$.*
- *For any maximizer $(\theta^*, \mathbf{A}^*)$ of J over $\mathcal{A}$, θ^* is a maximizer of I over $\mathcal{T}$.*

For problems on a ball, $\boldsymbol{\sigma}^*$ is a unique (radial) solution od $-\mathrm{div}\, \boldsymbol{\sigma}_i = f_i, i = 1..m$, and so conditions of optimality easily determine optimal θ^*.

Energy minimization

A. Single state equation:
[Murat & Tartar, 1985]

$$I(\theta) = \int_\Omega fu\,d\mathbf{x} \longrightarrow \min$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

B. Multiple state equations:

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i\,d\mathbf{x} \longrightarrow \min$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \quad i = 1, \ldots, m$$

$$\min_{\mathcal{A}} J \quad \Longleftrightarrow \quad \min_{\mathcal{T}} I$$

A: Holds always!

B: Holds in spherically symmetric case or when $m < d$.

$m < d$

Theorem

If $m < d$ *then* $\min_{\mathcal{A}} J = \min_{\mathcal{T}} I$ *and:*

- *There is unique* $\mathrm{u}^* \in \mathrm{H}^1_0(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{A}} J$ *and* $\min_{\mathcal{T}} I$.
- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{A}} J$*, then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$.
- *Conversely, if* θ^* *is a solution of optimal design problem* $\min_{\mathcal{T}} I$*, then any* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$ *satisfying* $\mathbf{A}^* \nabla u_i^* = \lambda^+_{\theta^*} \nabla u_i^*$, $i = 1, \dots, m$ *(e.g. simple laminates) is an optimal design for the problem* $\min_{\mathcal{A}} J$.

Spherical symmetry

$\Omega \subseteq \mathbf{R}^d$ is spherically symmetric and right-hand sides $f_i = f_i(r)$, $i = 1, \ldots, m$ are radial functions.

Theorem

There is a unique radial u^* *which is the state for any solution of* $\min_{\mathcal{A}} J$ *and* $\min_{\mathcal{T}} I$. *Moreover,*

- *If* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$ *is a solution of the relaxed problem* $\min_{\mathcal{A}} J$ *then* θ^* *is optimal for* $\min_{\mathcal{T}} I$, *and* $\mathbf{A}^* \nabla u_i^* = \lambda_{\theta^*}^+ \nabla u_i^*$, $i = 1, \ldots, m$.
- *There exists a radial minimizer* θ^* *of* I *over* $\mathcal{T}$ *and for any radial minimizer* θ^* *of* I *over* $\mathcal{T}$ *there exists a simple laminate* $\mathbf{A}^* \in \mathcal{K}(\theta^*)$ *such that* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for* $\min_{\mathcal{A}} J$.

Optimality conditions for $\min_{\mathcal{T}} I$

$$\min_{\theta\in\mathcal{T}} I(\theta) = -\max_{\theta\in\mathcal{T}} \min_{\mathrm{v}\in \mathrm{H}^1_0(\Omega;\mathbf{R}^m)} \sum_{i=1}^m \mu_i \int_\Omega \lambda_\theta^+ |\nabla v_i|^2 - 2f_i v_i \, d\mathbf{x}$$

Saddle points exist ... share the same v (aka u*).

$$\min_{\theta\in\mathcal{T}} I(\theta) = -\max_{\theta\in\mathcal{T}} \sum_{i=1}^m \mu_i \int_\Omega \lambda_\theta^+ |\nabla u_i^*|^2 - 2f_i u_i^* \, d\mathbf{x}$$

Lemma

$\theta^* \in \mathcal{T}$ is a solution $\min_{\mathcal{T}} I$ if and only if there exists a Lagrange multiplier $c \geq 0$ such that

$$\sum_{i=1}^m \mu_i |\nabla u_i^*|^2 > c \quad \Rightarrow \quad \theta^* = 0\,,$$
$$\sum_{i=1}^m \mu_i |\nabla u_i^*|^2 < c \quad \Rightarrow \quad \theta^* = 1\,.$$

Example – energy minimization

- $\Omega = B(\mathbf{0}, 2)\,,\ f_1 = \chi_{B(\mathbf{0},1)}\,,\ f_2 \equiv 1\,,$
- $\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, 2$
- $\mu \int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \to \min$

Solving state equation in polar coordinates

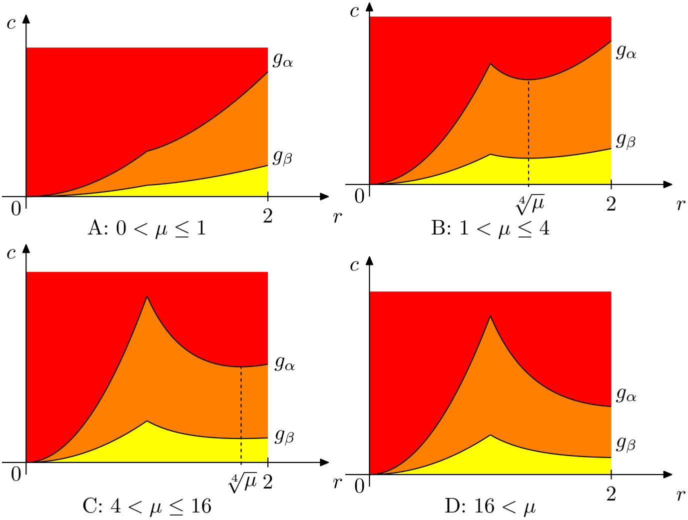
$$u_i'(r) = \frac{\sigma_i(r)}{\theta(r)\alpha + (1 - \theta(r))\beta}\,,\ i = 1, 2\,,$$

with

$$\sigma_1(r) = \begin{cases} -\dfrac{r}{2}\,, & 0 \le r < 1\,, \\ -\dfrac{1}{2r}\,, & 1 \le r \le 2\,, \end{cases} \quad \text{and} \quad \sigma_2(r) = -\frac{r}{2}\,.$$

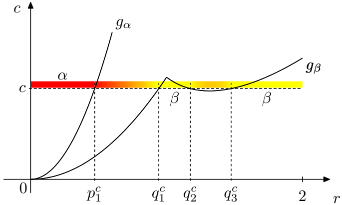
Define $\psi := \mu\sigma_1^2 + \sigma_2^2$, $g_\alpha := \frac{\psi}{\alpha^2}$, $g_\beta := \frac{\psi}{\beta^2}$.

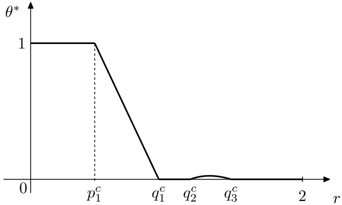
Geometric interpretation of optimality conditions

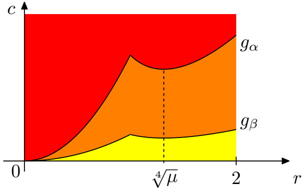


Optimal θ^* for case B

Optimal state u^* is unknown but $\sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 = \mu |u_1^*|^2 + |u_2^*|^2 \in [g_\beta, g_\alpha]$.
By necessary conditions of optimality, on a set where $c > g_\alpha$ we have $\theta^* = 1$, on a set where $c < g_\beta$ we have $\theta^* = 0$, and if $g_\beta < c < g_\alpha$ we have $\theta^* \in \langle 0, 1 \rangle$, and θ^* is uniquely determined from $\frac{\psi}{\lambda_+(\theta^*)^2} = c$.







B: $1 < \mu \leq 4$

All possible optimal configurations (for various q_α):

- α
- $\alpha - mix$
- $\alpha - mix - \alpha - mix$
- $\alpha - mix - \beta$
- $\alpha - mix - \beta - mix - \beta$