Multiple state optimal design problems with explicit solution

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Multiple state problem optimizing energy

Fill $\Omega \subseteq \mathbb{R}^d$ with two isotropic materials with conductivity $0 < \alpha < \beta$, quantity $q_\alpha$ of the first material is given:

$$A = \chi \alpha I + (1 - \chi) \beta I, \quad \chi \in L^\infty(\Omega; \{0, 1\})$$

$$\int_{\Omega} \chi \, dx = q_\alpha$$

State equations

$$\begin{align*}
-\text{div} (A \nabla u_i) &= f_i \\
u_i &\in H^1_0(\Omega)
\end{align*} \quad i = 1, \ldots, m, $$

Goal functional is a conic sum of energies ($\mu_i > 0$)

$$I(\chi) = \sum_{i=1}^{m} \mu_i \int_{\Omega} f_i(x) u_i(x) \, dx \longrightarrow \min / \max$$

Relaxation via homogenization theory:

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<th>classical design</th>
<th>relaxed design</th>
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<tr>
<td>$\chi \in L^\infty(\Omega; {0, 1})$</td>
<td>$\theta \in L^\infty(\Omega; [0, 1])$</td>
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<tr>
<td>$A \in \mathcal{K}(\theta)$ a.e. on $\Omega$</td>
<td>$J(\theta, A)$ – given by the same formula</td>
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Example – maximization

\[ \Omega := B(0, 1) \subseteq \mathbb{R}^2, \quad q_\alpha := 0.8|\Omega| \]

\[ I(\chi) = \sum_{i=1}^{2} \int_{\Omega} f_i(x) u_i(x) dx \longrightarrow \max \]

\[ f_1 = \chi_A + \varepsilon \chi_B \]

\[ f_2 = \chi_A - \varepsilon \chi_B \]

where

\[ A := B(0, \frac{1}{2})^c, \quad B := B(0, \frac{1}{5}) \]

Numerical solution, \( \varepsilon = 0.01 \)

Numerical solution, \( \varepsilon = 0 \)
The space of admissible local fractions

\[ \mathcal{T} := \left\{ \theta \in L^\infty(\Omega; [0, 1]) : \int_\Omega \theta \, dx = q_\alpha \right\} \]

Admissible (relaxed) designs

\[ \mathcal{A} = \{ (\theta, A) \in \mathcal{T} \times L^\infty(\Omega; \text{Sym}) : A \in \mathcal{K}(\theta) \text{ (a.e. on } \Omega) \} \]

\[ \mathcal{B} = \{ (\theta, A) \in \mathcal{T} \times L^\infty(\Omega; \text{Sym}) : A \in \mathcal{B}(\theta) \text{ (a.e. on } \Omega) \} \]

\[ \lambda_1^+ = \theta \alpha + (1 - \theta) \beta \]

\[ \lambda_1^- = \frac{\theta}{\alpha} + \frac{1 - \theta}{\beta} \]
Lemma

There exists a unique $\sigma^* \in S = \{ \sigma \in L^2(\Omega; \mathbb{R}^d)^m : -\text{div } \sigma_i = f_i, i = 1..m \}$ such that

$$\max_{(\theta,A) \in B} J(\theta, A) = \max_{(\theta,A) \in B} \sum_{i=1}^{m} \mu_i \int_{\Omega} A^{-1} \sigma^*_i \cdot \sigma^*_i \, dx. \quad (1)$$

Moreover, if $(\theta^*, A^*)$ is an optimal design for problem $\max_{B} J$ and $u^*$ the corresponding state function, then $A^* \nabla u^*_i = \sigma_i^*$, $i = 1, \ldots, m$.

Above maximization problems is easily solved:

Design $(\theta^*, A^*)$ is optimal if and only if (almost everywhere in $\Omega$)

$$A^* \sigma_i^* = \lambda_{\theta^*} \sigma_i^*, \quad i = 1..m.$$

If $u^*$ is the corresponding state function, we have

$$\sigma_i^* = \lambda_{\theta^*} \nabla u_i^* \text{ or equivalently } A^* \nabla u_i^* = \lambda_{\theta^*} \nabla u_i^*, \quad i = 1..m.$$
Simpler relaxation problem

**Theorem**

Let \((\theta^*, A^*)\) be an optimal design for the problem \(\max_B J\). Then \(\theta^*\) solves

\[
I(\theta) = \sum_{i=1}^{m} \mu_i \int_{\Omega} f_i u_i \, dx \rightarrow \max
\]

\(\theta \in T\) and \(u\) determined uniquely by

\[
\begin{align*}
-\text{div} \left( \lambda^- \nabla u_i \right) &= f_i \\
u_i &\in H^1_0(\Omega) \quad i = 1, \ldots, m
\end{align*}
\]

(2)

Conversely, if \(\tilde{\theta}\) is a solution of optimal design problem (2), and \(\tilde{u}\) is the corresponding state function, then for any measurable \(\tilde{A} \in B(\tilde{\theta})\) such that \(\tilde{A} \nabla \tilde{u}_i = \lambda_-(\tilde{\theta}) \nabla \tilde{u}_i\), e.g. for \(\tilde{A} = \lambda_-(\tilde{\theta}) I\), \((\tilde{\theta}, \tilde{A})\) is an optimal design for the problem \(\max_B J\).
Similar to Lemma above, one can rephrase the simpler relaxation problem (2): there exists a unique $\sigma^* \in S = \{\sigma \in L^2(\Omega; \mathbb{R}^d)^m : -\text{div} \sigma_i = f_i, i = 1..m\}$ such that

$$\max_{T} I = \max_{\theta \in T} \sum_{i=1}^{m} \mu_i \int_{\Omega} \frac{\beta - \alpha}{\alpha \beta} \theta |\sigma_i^*|^2 \, dx.$$ 

Moreover, $\sigma^*$ is the same as for $\max_B J$.

**Lemma**

The necessary and sufficient condition of optimality for solution $\theta^* \in T$ of optimal design problem (2) simplifies to the existence of a Lagrange multiplier $c \geq 0$ such that

$$\sum_{i=1}^{m} \mu_i |\sigma_i^*|^2 > c \Rightarrow \theta^* = 1,$$

$$\sum_{i=1}^{m} \mu_i |\sigma_i^*|^2 < c \Rightarrow \theta^* = 0.$$
Spherically symmetric case

Let $\Omega \subseteq \mathbb{R}^d$ be spherically symmetric (ball or annulus), and let the right-hand sides be radial functions: $f_i = f_i(r)$. Since $\sigma^*$ is unique, it must be radial: $\sigma_i^* = \sigma_i^*(r)e_r$.

**Theorem**

For any maximizer $\theta^*$ for $\max_T I$, the radial function

$$\tilde{\theta}(r) = \int_{\partial B(0,r)} \theta^* \, dS$$

is also a maximizer.

- If $\tilde{\theta}$ is a maximizer of $I$ over $T$, then for a simple laminate $\tilde{A} \in \mathcal{K}(\tilde{\theta})$ with layers orthogonal to $e_r$, $(\tilde{\theta}, \tilde{A})$ is a maximizer of $J$ over $A$.
- For any maximizer $(\theta^*, A^*)$ of $J$ over $A$, $\theta^*$ is a maximizer of $I$ over $T$.

For problems on a ball, $\sigma^*$ is a unique (radial) solution of $-\text{div} \sigma_i = f_i$, $i = 1..m$, and so conditions of optimality easily determine optimal $\theta^*$. 
Energy minimization

A. Single state equation:  
[Murat & Tartar, 1985]

\[ I(\theta) = \int_{\Omega} f u \, dx \rightarrow \min \]

\( \theta \in \mathcal{T} \), and \( u \) determined uniquely by

\[
\begin{aligned}
- \text{div} (\lambda_\theta^{+} \nabla u) &= f \\
 u &\in H^1_0(\Omega)
\end{aligned}
\]

B. Multiple state equations:

\[ I(\theta) = \sum_{i=1}^{m} \mu_i \int_{\Omega} f_i u_i \, dx \rightarrow \min \]

\( \theta \in \mathcal{T} \), and \( u_i \) determined uniquely by

\[
\begin{aligned}
- \text{div} (\lambda_\theta^{+} \nabla u_i) &= f_i \\
u_i &\in H^1_0(\Omega) \\
i = 1, \ldots, m
\end{aligned}
\]

\[
\begin{aligned}
\min_{\mathcal{A}} J &\quad \iff \quad \min_{\mathcal{T}} I \\
A: \text{Holds always!} &\quad B: \text{Holds in spherically symmetric case or when } m < d.
\end{aligned}
\]
Theorem

If \( m < d \) then \( \min_{\mathcal{A}} J = \min_{\mathcal{T}} I \) and:

- There is unique \( u^* \in H^1_0(\Omega; \mathbb{R}^m) \) which is the state for every solution of \( \min_{\mathcal{A}} J \) and \( \min_{\mathcal{T}} I \).

- If \((\theta^*, A^*)\) is an optimal design for the problem \( \min_{\mathcal{A}} J \), then \( \theta^* \) is optimal design for \( \min_{\mathcal{T}} I \).

- Conversely, if \( \theta^* \) is a solution of optimal design problem \( \min_{\mathcal{T}} I \), then any \((\theta^*, A^*) \in \mathcal{A}\) satisfying \( A^* \nabla u^*_i = \lambda^+_\theta^* \nabla u^*_i, \ i = 1, \ldots, m \) (e.g. simple laminates) is an optimal design for the problem \( \min_{\mathcal{A}} J \).
\( \Omega \subseteq \mathbb{R}^d \) is spherically symmetric and right-hand sides \( f_i = f_i(r), \ i = 1, \ldots, m \) are radial functions.

**Theorem**

There is a unique radial \( u^* \) which is the state for any solution of \( \min_{\mathcal{A}} J \) and \( \min_{\mathcal{T}} I \). Moreover,

- If \((\theta^*, \mathbf{A}^*) \in \mathcal{A} \) is a solution of the relaxed problem \( \min_{\mathcal{A}} J \) then \( \theta^* \) is optimal for \( \min_{\mathcal{T}} I \), and \( \mathbf{A}^* \nabla u_i^* = \lambda_{\theta^*}^+ \nabla u_i^* \), \( i = 1, \ldots, m \).

- There exists a radial minimizer \( \theta^* \) of \( I \) over \( \mathcal{T} \) and for any radial minimizer \( \theta^* \) of \( I \) over \( \mathcal{T} \) there exists a simple laminate \( \mathbf{A}^* \in \mathcal{K}(\theta^*) \) such that \((\theta^*, \mathbf{A}^*) \) is an optimal design for \( \min_{\mathcal{A}} J \).
Optimality conditions for $\min_I l$

$$\min_{\theta \in T} l(\theta) = -\max_{\theta \in T} \min_{v \in H^1_0(\Omega; \mathbb{R}^m)} \sum_{i=1}^m \mu_i \int_{\Omega} \lambda^+_\theta |\nabla v_i|^2 - 2f_i v_i \, dx$$

Saddle points exist . . . share the same $v$ (aka $u^*$).

$$\min_{\theta \in T} l(\theta) = -\max_{\theta \in T} \sum_{i=1}^m \mu_i \int_{\Omega} \lambda^+_\theta |\nabla u_i^*|^2 - 2f_i u_i^* \, dx$$

**Lemma**

$\theta^* \in T$ is a solution $\min_T l$ if and only if there exists a Lagrange multiplier $c \geq 0$ such that

$$\sum_{i=1}^m \mu_i |\nabla u_i^*|^2 > c \quad \Rightarrow \quad \theta^* = 0,$$

$$\sum_{i=1}^m \mu_i |\nabla u_i^*|^2 < c \quad \Rightarrow \quad \theta^* = 1.$$
Example – energy minimization

\[ \Omega = B(0, 2), \ f_1 = \chi_{B(0,1)}, \ f_2 \equiv 1, \]

\[
\begin{cases}
-\text{div} \ (\lambda_\theta^+ \nabla u_i) = f_i \\
\quad u_i \in H^1_0(\Omega)
\end{cases} \quad i = 1, 2
\]

\[
\mu \int_{\Omega} f_1 u_1 \, dx + \int_{\Omega} f_2 u_2 \, dx \to \text{min}
\]

Solving state equation in polar coordinates

\[
u_i'(r) = \frac{\sigma_i(r)}{\theta(r)\alpha + (1 - \theta(r))\beta}, \quad i = 1, 2,
\]

with

\[
\sigma_1(r) = \begin{cases}
-\frac{r}{2}, & 0 \leq r < 1, \\
-\frac{1}{2r}, & 1 \leq r \leq 2,
\end{cases}
\]

and \( \sigma_2(r) = -\frac{r}{2} \).

Define \( \psi := \mu \sigma_1^2 + \sigma_2^2, \ g_\alpha := \frac{\psi}{\alpha^2}, \ g_\beta := \frac{\psi}{\beta^2}. \)
Geometric interpretation of optimality conditions

A: $0 < \mu \leq 1$

B: $1 < \mu \leq 4$

C: $4 < \mu \leq 16$

D: $16 < \mu$

$A: 0 < \mu \leq 1$

$B: 1 < \mu \leq 4$

$C: 4 < \mu \leq 16$

$D: 16 < \mu$
Optimal $\theta^*$ for case B

Optimal state $u^*$ is unknown but $\sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 = \mu |u_1^*|^2 + |u_2^*|^2 \in [g_\beta, g_\alpha]$. By necessary conditions of optimality, on a set where $c > g_\alpha$ we have $\theta^* = 1$, on a set where $c < g_\beta$ we have $\theta^* = 0$, and if $g_\beta < c < g_\alpha$ we have $\theta^* \in \langle 0, 1 \rangle$, and $\theta^*$ is uniquely determined from $\psi = \lambda_+ (\theta^*)^2 = c$.

All possible optimal configurations (for various $q_\alpha$):

- $\alpha$
- $\alpha - \text{mix}$
- $\alpha - \text{mix} - \alpha - \text{mix}$
- $\alpha - \text{mix} - \beta$
- $\alpha - \text{mix} - \beta - \text{mix} - \beta$