Classical solutions in optimal design problems

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Compliance maximization

State equation ($\Omega \subseteq \mathbf{R}^d$ open and bounded)

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = 1 = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

Two phases: $0 < \alpha < \beta$
$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$, $\int_\Omega \chi\, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$
Cost functional:

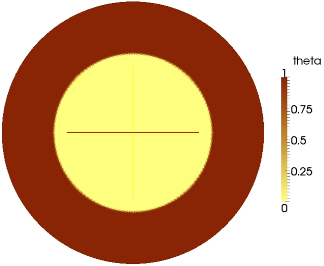
$$J(\chi) = \int_\Omega u(\mathbf{x})d\mathbf{x} \longrightarrow \max$$

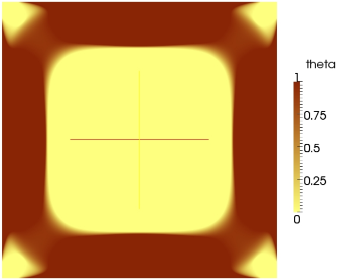
Interpretations:

- Maximize the amount of heat kept inside body
- Maximize the torsional rigidity of a rod made of two materials
- Maximize the flow rate of two viscous immiscible fluids through pipe

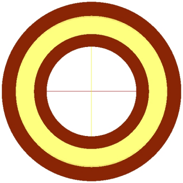
In general, compliance functional

$$J(\chi) = \int_\Omega f(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} \longrightarrow \max$$





Intuition for annulus?



In general, there might exist no classical optimal design. The relaxation is needed, introducing composite materials

$$\chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \quad \cdots \quad \theta \in \mathrm{L}^\infty(\Omega; [0,1])$$

$$\mathbf{A} \in \mathcal{K}(\theta) \quad \text{a.e. on } \Omega$$

classical design — relaxed design

Effective conductivities – set $\mathcal{K}(\theta)$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

$$\lambda_\theta^- \le \quad \lambda_j \quad \le \lambda_\theta^+ \quad j = 1, \dots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \quad \le \quad \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

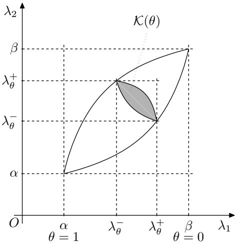
$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \quad \le \quad \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+} \,,$$

where

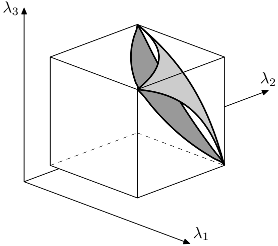
$$\lambda_\theta^+ \quad = \quad \theta\alpha + (1-\theta)\beta$$

$$\frac{1}{\lambda_\theta^-} \quad = \quad \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}$$

2D:

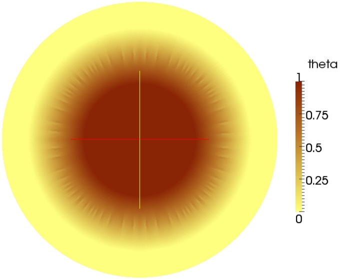


3D:

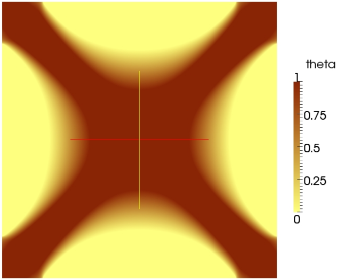


Compliance minimization

Murat and Tartar, 1985



Lurie and Cherkaev, 1986



Multiple state optimal design problem

State equations

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad i = 1, \ldots, m$$

State function $\mathrm{u} = (u_1, \ldots, u_m)$

$$\begin{cases} I(\chi) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \max \\ \qquad \mathrm{u} = (u_1, \ldots, u_m) \text{ state function for } \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \\ \qquad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \,, \; \displaystyle\int_\Omega \chi \, d\mathbf{x} = q_\alpha \,, \end{cases}$$

for some given weights $\mu_i > 0$. Relaxed designs:

$$\mathcal{A} := \left\{ (\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \; \mathbf{A}(\mathbf{x}) \in \mathcal{K}(\theta(\mathbf{x})) \text{ a.e. on } \Omega \right\}$$

Relaxation [Allaire, 2002] $\quad \ldots \quad \begin{cases} J(\theta, \mathbf{A}) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \max \\ \qquad (\theta, \mathbf{A}) \in \mathcal{A} \end{cases}$

Single vs. multiple state problems

A. Single state equation

[Murat & Tartar, 1985] There exists relaxed solution $(\theta^*, \mathbf{A}^*)$ among simple laminates ...conductivity λ_θ^- in one direction (∇u), and λ_θ^+ in orthogonal directions. As a consequence, θ^* is also a solution of

$$I(\theta) = \int_\Omega fu\, d\mathbf{x} \to \max$$

$$\theta \in \mathrm{L}^\infty(\Omega; [0,1])\,, \quad \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,,$$

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^- \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

can be rewritten as a convex minimization problem

B. Multiple state equations

It is not enough to use only simple laminates, but composite materials that correspond to a non-affine boundary of $\mathcal{K}(\theta)$ ...higher order sequential laminates.
The above simpler relaxation fails.
The aim of this talk

- in spherically symmetric case, simpler relaxation is correct
- present some problems with classical optimal design

Motivation: random right-hand side

Let $(S, \mathcal{M}, \mu)$ be a probability space.
Suppose that $f \in \mathrm{L}^1(S; \mathrm{H}^{-1}(\Omega))$, and denote $\overline{f} := \int_S f \, d\mu$.
In other words we consider $s \in S$ to be a parameter in boundary value problem

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f(s, \cdot) \\ u \in \mathrm{H}^1_0(\Omega) \end{cases} \tag{1}$$

A priori estimate for the solution implies that solution u belongs to $\mathrm{L}^1(S; \mathrm{H}^1_0(\Omega))$.
We consider the following optimal design problem [Buttazzo, Maestre 2011]:
Given $f \in \mathrm{L}^1(S; \mathrm{H}^{-1}(\Omega))$, one seeks for a characteristic function χ on Ω that optimizes

$$J(\chi) = \int_S \int_\Omega f(s, \mathbf{x}) u(s, \mathbf{x}) \, d\mathbf{x} \, d\mu \to \min / \max \,,$$

where $u \in \mathrm{L}^1(S; \mathrm{H}^1_0(\Omega))$ is determined by (1) with $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$.
Moreover, we assume that quantity of the first material is given: $\displaystyle\int_\Omega \theta \, d\mathbf{x} = q_\alpha$.

Discrete probability space

$S := \{s_1, s_2, \ldots, s_m\}$, $\mu(s_i) = \mu_i \geq 0$, $\sum_i \mu_i = 1$.
Then $f \in \mathrm{L}^1(S; \mathrm{H}^{-1}(\Omega))$ is characterized by functionals $f_i := f(s_i, \cdot) \in \mathrm{H}^{-1}(\Omega)$, which uniquely determine state functions u_i:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad i = 1, \ldots, m\,.$$

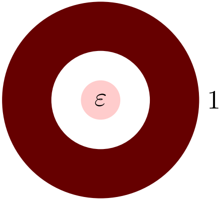
Finally, the goal functional becomes

$$J(\theta, \mathbf{A}) = \sum_i \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \max\,,$$

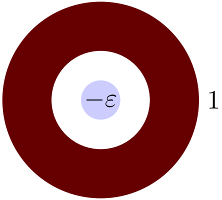
Therefore, we can use any method for numerical solution of this new multiple state optimal design problem.

Example

We consider $\Omega := B(\mathbf{0},1) \subseteq \mathbf{R}^2$, $q_\alpha := 0.8|\Omega|$, $S = \{1,2\}$, $\mu_1 = \mu_2 = \frac{1}{2}$.



$B(\mathbf{0},1)$

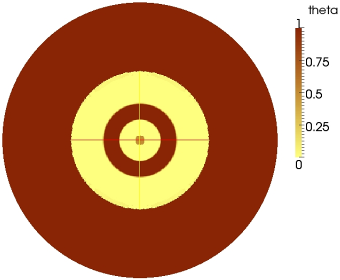


$B(\mathbf{0},1)$

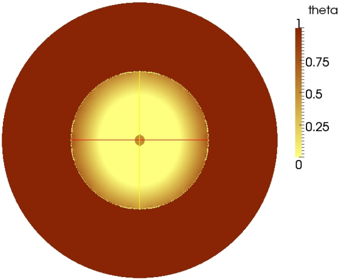
$$f_1 = \chi_A + \varepsilon\chi_B$$

$$f_2 = \chi_A - \varepsilon\chi_B, \text{ where}$$

$$A := B\left(\mathbf{0}, \tfrac{1}{2}\right)^c, \ B := B\left(\mathbf{0}, \tfrac{1}{5}\right)$$



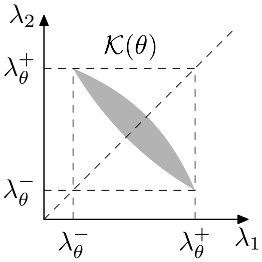
Numerical solution, $\varepsilon = 0.01$

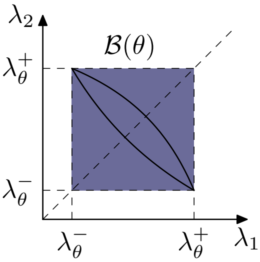


Numerical solution, $\varepsilon = 0$

We shall enlarge the set $\mathcal{A}$ of admissible designs

$$\mathcal{A} = \left\{ (\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,,\; \mathbf{A} \in \mathcal{K}(\theta) \text{ (a.e. on } \Omega) \right\}$$

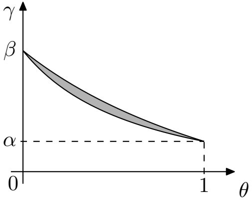




$$\mathcal{B} = \left\{ (\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,,\; \mathbf{A} \in \mathcal{B}(\theta) \text{ (a.e. on } \Omega) \right\}$$

$$\mathcal{C} := \left\{ (\theta, \mathbf{B}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}) : (\theta, \mathbf{B}^{-1}) \in \mathcal{B} \right\}.$$

$\mathcal{A}$ is not convex: e.g. isotropic materials $(\theta, \gamma \mathbf{I}) \in \mathcal{A}$ can be easily characterised:



Extended set of admissible designs

$\mathcal{B}$ and $\mathcal{C}$ are convex sets: e.g. $\mathcal{B}$ can be rewritten as

$$\lambda_{\min}(\mathbf{A}) \geq \lambda^-_\theta\,,\ \lambda_{\max}(\mathbf{A}) \leq \lambda^+_\theta\,,\quad \text{a.e. on } \Omega\,,$$

where $\lambda_{\min}$ and $\lambda^+_\cdot$ are concave, and $\lambda_{\max}$ and $\lambda^-_\cdot$ are convex functions.

$$\begin{aligned}
-J(\theta,\mathbf{A}) &= -\sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \\
&= -\sum_{i=1}^m \mu_i \int_\Omega \mathbf{A}\nabla u_i \cdot \nabla u_i - 2 f_i u_i \, d\mathbf{x} \\
&= -\min_{v \in \mathrm{H}^1_0(\Omega;\mathbf{R}^m)} \sum_{i=1}^m \mu_i \int_\Omega \mathbf{A}\nabla v_i \cdot \nabla v_i - 2 f_i v_i \, d\mathbf{x} \\
&= -\max_{\boldsymbol{\sigma}\in\mathcal{S}} \left(-\sum_{i=1}^m \mu_i \int_\Omega \mathbf{A}^{-1}\sigma_i \cdot \sigma_i \, d\mathbf{x} \right) ,
\end{aligned}$$

where $\mathcal{S} = \{\sigma \in \mathrm{L}^2(\Omega;\mathbf{R}^d)^m : -\operatorname{div}\sigma_i = f_i, i = 1,\dots,m\}$.

Representation by a convex optimization problem

Lemma

There exists a unique $\boldsymbol{\sigma}^* \in \mathcal{S} = \{\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega; \mathbf{R}^d)^m : -\mathrm{div}\, \boldsymbol{\sigma}_i = f_i, i = 1..m\}$ *such that*

$$\max_{(\theta,\mathbf{A})\in\mathcal{B}} J(\theta, \mathbf{A}) = \max_{(\theta,\mathbf{A})\in\mathcal{B}} \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{A}^{-1}\boldsymbol{\sigma}_i^* \cdot \boldsymbol{\sigma}_i^* \, d\mathbf{x} = \max_{(\theta,\mathbf{B})\in\mathcal{C}} \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{B}\boldsymbol{\sigma}_i^* \cdot \boldsymbol{\sigma}_i^* \, d\mathbf{x} \,. \tag{2}$$

Moreover, if $(\theta^*, \mathbf{A}^*)$ *is an optimal design for problem* $\max_{\mathcal{B}} J$ *and* u^* *the corresponding state function, then* $\mathbf{A}^*\nabla u_i^* = \boldsymbol{\sigma}_i^*\,,\ i = 1, \dots, m\,.$

Above maximization problems are easily solved:
Design $(\theta^*, \mathbf{A}^*)$ is optimal if and only if (almost everywhere in Ω)

$$\mathbf{A}^*\boldsymbol{\sigma}_i^* = \lambda_{\theta^*}^- \boldsymbol{\sigma}_i^* \quad i = 1..m\,.$$

If u^* is the corresponding state function, we have

$$\boldsymbol{\sigma}_i^* = \lambda_{\theta^*}^- \nabla u_i^* \text{ or equivalently } \mathbf{A}^*\nabla u_i^* = \lambda_{\theta^*}^- \nabla u_i^*\,,\ i = 1..m\,.$$

Simpler relaxation problem

... in terms of only local fraction θ belonging to the set

$$\mathcal{T} := \left\{ \theta \in \mathrm{L}^\infty(\Omega;[0,1]) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \right\}$$

Theorem

Let $(\theta^*, \mathbf{A}^*)$ be an optimal design for the problem $\max_{\mathcal{B}} J$. Then θ^* solves

$$\begin{aligned}
& I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \max \\
& \theta \in \mathcal{T} \quad \textit{and } \mathrm{u} \textit{ determined uniquely by} \\
& \begin{cases} -\mathrm{div}\,(\lambda_\theta^- \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad\qquad i = 1, \ldots, m\,,
\end{aligned} \tag{3}$$

Conversely, if $\widetilde{\theta}$ is a solution of optimal design problem (3), and $\widetilde{\mathrm{u}}$ is the corresponding state function, then for any measurable $\widetilde{\mathbf{A}} \in \mathcal{B}(\widetilde{\theta})$ such that $\widetilde{\mathbf{A}}\nabla\widetilde{u}_i = \lambda_-(\widetilde{\theta})\nabla\widetilde{u}_i$ almost everywhere on Ω, e.g. for $\widetilde{\mathbf{A}} = \lambda_-(\widetilde{\theta})\mathbf{I}$, $(\widetilde{\theta}, \widetilde{\mathbf{A}})$ is an optimal design for the problem $\max_{\mathcal{B}} J$.

Necessary and sufficient optimality conditions

Similar to Lemma above, one can rephrase the simpler relaxation problem (3): there exists a unique $\boldsymbol{\sigma}^* \in \mathcal{S} = \{\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega; \mathbf{R}^d)^m : -\mathrm{div}\, \boldsymbol{\sigma}_i = f_i, i = 1..m\}$ such that

$$\max_{\mathcal{T}} I = \max_{\theta \in \mathcal{T}} \sum_{i=1}^m \mu_i \int_\Omega \frac{\beta - \alpha}{\alpha\beta}\, \theta |\boldsymbol{\sigma}_i^*|^2 \, d\mathbf{x} \,.$$

Moreover, $\boldsymbol{\sigma}^*$ is the same as for the problem $\max_{\mathcal{B}} J$.

Lemma

The necessary and sufficient condition of optimality for solution $\theta^* \in \mathcal{T}$ of optimal design problem (3) simplifies to the existence of a Lagrange multiplier $c \geq 0$ such that

$$\sum_{i=1}^m \mu_i |\boldsymbol{\sigma}_i^*|^2 > c \quad \Rightarrow \quad \theta^* = 1 \,,$$

$$\sum_{i=1}^m \mu_i |\boldsymbol{\sigma}_i^*|^2 < c \quad \Rightarrow \quad \theta^* = 0 \,.$$

Spherically symmetric case

Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric: in spherical coordinates given by $r \in \omega$ (an interval), and the right-hand side $\mathrm{f} = \mathrm{f}(r)$, $r \in \omega$ be a radial function.
Since $\boldsymbol{\sigma}^*$ is unique, it must be radial: $\boldsymbol{\sigma}_i^* = \sigma_i^*(r)\mathbf{e}_r$.

Theorem

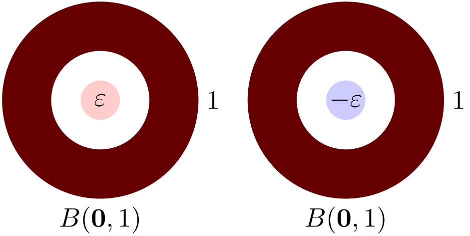
For any maximizer $(\theta^*, \mathbf{A}^*)$ for the problem $\max_{\mathcal{B}} J$, there exist a radial maximizer $(\widetilde{\theta}, \widetilde{\mathbf{A}}) \in \mathcal{B}$ where

$$\widetilde{\theta}(r) = \fint_{\partial B(\mathbf{0},r)} \theta^* \, dS \,.$$

Corollary

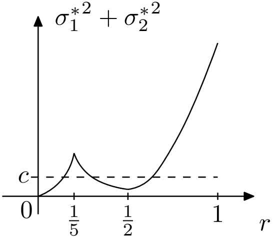
For any radial solution θ^* for $\max_{\mathcal{T}} I$, there exist a radial conductivity $\mathbf{A}^* \in \mathcal{K}(\theta^*)$ such that $(\theta^*, \mathbf{A}^*)$ is maximizer for $\max_{\mathcal{A}} J$.
Conversely, if $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$ is a radial maximizer for $\max_{\mathcal{A}} J$ then θ^* is a maximizer for problem $\max_{\mathcal{T}} I$.

Back to the example $\varepsilon > 0$



$$f_{1,2}(r) = \begin{cases} \pm\varepsilon, & 0 \le r \le \frac{1}{5} \\ 0, & \frac{1}{5} < r \le \frac{1}{2} \\ 1, & \frac{1}{2} < r \le 1\,. \end{cases}$$

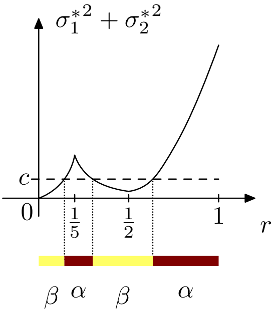
$-\mathrm{div}\,\boldsymbol{\sigma}_i = f_i, i = 1,2$ in polar coordinates: $-\dfrac{1}{r}\,(r\sigma_i)' = f_i$. Due to regularity at r=0, we can calculate unique solutions σ_1^* and σ_2^*:

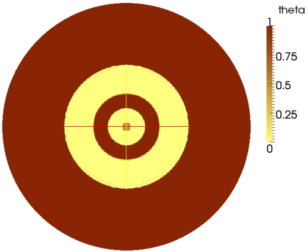


$$\begin{aligned} {\sigma_1^*}^2 + {\sigma_2^*}^2 > c \quad &\Rightarrow \quad \theta^* = 1\,, \\ {\sigma_1^*}^2 + {\sigma_2^*}^2 < c \quad &\Rightarrow \quad \theta^* = 0\,. \end{aligned}$$

For any c, the solution θ^* is unique and classical (more precisely, the uniqueness of solution for $\max_{\mathcal{B}} J$ follows).

How to determine Lagrange multiplier c?



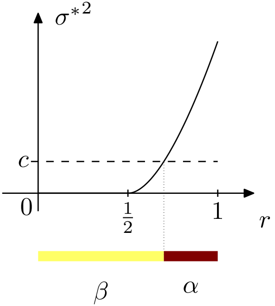


$$\begin{aligned} \sigma_1^{*2} + \sigma_2^{*2} > c \quad &\Rightarrow \quad \theta^* = 1\,, \\ \sigma_1^{*2} + \sigma_2^{*2} < c \quad &\Rightarrow \quad \theta^* = 0\,. \end{aligned}$$

Quantity of given materials uniquely determines c (as usual).

The case $\varepsilon = 0$

Small q_α: unique classical solution



$$\begin{aligned} \sigma^{*2} > c \quad &\Rightarrow \quad \theta^* = 1\,, \\ \sigma^{*2} < c \quad &\Rightarrow \quad \theta^* = 0\,. \end{aligned}$$

If $q_\alpha > \frac{3}{4}\pi$ then c have to be zero. Now, solution is not unique – it is only important to put α in annulus $B\left(\mathbf{0}, \frac{1}{2}\right)^c$.

Example 2

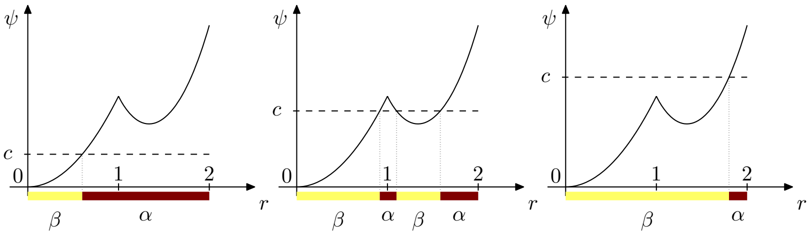
Two state equations on a ball $\Omega = B(\mathbf{0}, 2)$

- $f_1 = \chi_{B(\mathbf{0},1)}$, $f_2 = \chi_{B(\mathbf{0},1)^c}$,
- $\begin{cases} -\mathrm{div}\,(\lambda_\theta^- \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, 2$
- $\mu_1 \int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \to \max$

For studying conditions of optimality, we introduce

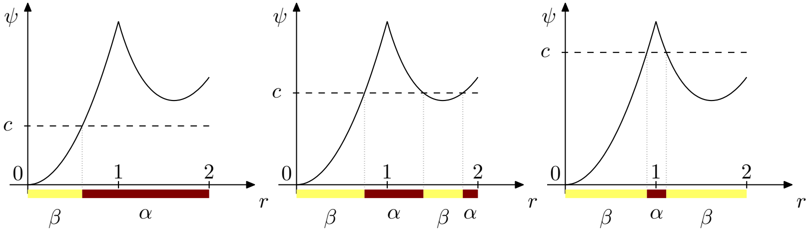
$$\psi(r) = \mu_1 \left(\sigma_1^*(r)\right)^2 + \left(\sigma_2^*(r)\right)^2 .$$

The case $0 < \mu_1 < \frac{49}{3}$:

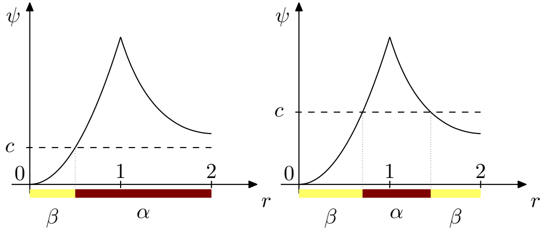


Example 2

The case $\frac{49}{3} \leq \mu_1 < 119$:

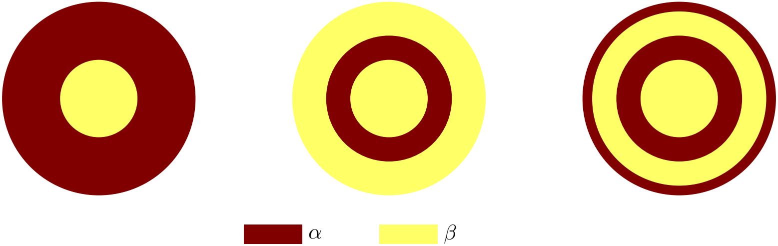


The case $\mu_1 \geq 119$:



Multiple states

Three optimal configurations, depending on μ_1 and q_α:



Radii are solutions of some algebraic equations (solved numerically).

Conclusion

General strategy for solving $\max_{\mathcal{A}} J$ in spherically symmetric case:

1. Solve $-\operatorname{div} \boldsymbol{\sigma}_i = f_i, i = 1..m$ – candidates for $\boldsymbol{\sigma}^*$ (in case of ball there is only one candidate).
2. Study conditions of optimality (they usually give unique solution θ^* – radial, but also classical).
3. Construct solution to $\max_{\mathcal{A}} J$ (commonly, it would be classical solution; for minimization problem the situation is quite different).
4. It is also possible to comment the possible non-uniqueness of relaxation problem.

Thank you for your attention!