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Mechanics through Mathematical Modelling

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Compliance maximization

State equation ($\Omega \subseteq \mathbb{R}^d$ open and bounded)

\[
\begin{cases}
-\text{div}(A \nabla u) = 1 = f \\
u \in H^1_0(\Omega)
\end{cases}
\]

Two phases: $0 < \alpha < \beta$

$A = \chi \alpha I + (1 - \chi) \beta I$, $\chi \in L^\infty(\Omega; \{0, 1\})$, $\int_\Omega \chi \, dx = q_\alpha$, for given $0 < q_\alpha < |\Omega|

Cost functional:

$$J(\chi) = \int_\Omega u(x) \, dx \longrightarrow \text{max}$$

Interpretations:

- Maximize the amount of heat kept inside body
- Maximize the torsional rigidity of a rod made of two materials
- Maximize the flow rate of two viscous immiscible fluids through pipe

In general, compliance functional

$$J(\chi) = \int_\Omega f(x)u(x) \, dx \longrightarrow \text{max}$$
In general, there might exist no classical optimal design. The relaxation is needed, introducing composite materials

\[ \chi \in L^\infty(\Omega; \{0, 1\}) \quad \cdots \quad \theta \in L^\infty(\Omega; [0, 1]) \]

\[ \mathbf{A} \in \mathcal{K}(\theta) \quad \text{a.e. on } \Omega \]

**classical design**

**relaxed design**
Effective conductivities – set $\mathcal{K}(\theta)$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

$$
\lambda^-_\theta \leq \lambda_j \leq \lambda^+_\theta \quad j = 1, \ldots, d
$$

$$
\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda^-_\theta - \alpha} + \frac{d - 1}{\lambda^+_\theta - \alpha}
$$

$$
\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda^-_\theta} + \frac{d - 1}{\beta - \lambda^+_\theta},
$$

where

$$
\lambda^+_\theta = \theta \alpha + (1 - \theta) \beta
$$

$$
\frac{1}{\lambda^-_\theta} = \frac{\theta}{\alpha} + \frac{1 - \theta}{\beta}
$$
Compliance minimization

Murat and Tartar, 1985

Lurie and Cherkaev, 1986
Multiple state optimal design problem

State equations

\[
\begin{aligned}
-\text{div}(A \nabla u_i) &= f_i \\
&\quad u_i \in H^1_0(\Omega) \\
&\quad i = 1, \ldots, m
\end{aligned}
\]

State function \( u = (u_1, \ldots, u_m) \)

\[
\begin{aligned}
I(\chi) &= \sum_{i=1}^{m} \mu_i \int_{\Omega} f_i u_i \, dx \\ 
&\quad u = (u_1, \ldots, u_m) \text{ state function for } A = \chi \alpha l + (1 - \chi) \beta l \\
&\quad \chi \in L^\infty(\Omega; \{0, 1\}), \quad \int_{\Omega} \chi \, dx = q_\alpha ,
\end{aligned}
\]

for some given weights \( \mu_i > 0 \). Relaxed designs:

\[
\mathcal{A} := \left\{ (\theta, A) \in L^\infty(\Omega; [0, 1] \times M_d(\mathbb{R})) : \int_{\Omega} \theta \, dx = q_\alpha , \ A(x) \in \mathcal{K}(\theta(x)) \text{ a.e. on } \Omega \right\}
\]

Relaxation [Allaire, 2002] …

\[
\begin{aligned}
J(\theta, A) &= \sum_{i=1}^{m} \mu_i \int_{\Omega} f_i u_i \, dx \\ 
&\quad (\theta, A) \in \mathcal{A}
\end{aligned}
\]
A. Single state equation

[Murat & Tartar, 1985] There exists relaxed solution \((\theta^*, A^*)\) among simple laminates . . . conductivity \(\lambda^-\) in one direction \((\nabla u)\), and \(\lambda^+\) in orthogonal directions. As a consequence, \(\theta^*\) is also a solution of

\[
I(\theta) = \int_\Omega fu \, d\mathbf{x} \to \max
\]

\[
\theta \in L^\infty(\Omega; [0, 1]), \quad \int_\Omega \theta \, d\mathbf{x} = q_\alpha,
\]

\[
\begin{cases}
-\text{div} (\lambda^- \nabla u) = f \\
u \in H^1_0(\Omega)
\end{cases}
\]

can be rewritten as a convex minimization problem

B. Multiple state equations

It is not enough to use only simple laminates, but composite materials that correspond to a non-affine boundary of \(K(\theta)\) . . . higher order sequential laminates. The above simpler relaxation fails.

The aim of this talk

- in spherically symmetric case, simpler relaxation is correct
- present some problems with classical optimal design
Motivation: random right-hand side

Let \((S, \mathcal{M}, \mu)\) be a probability space.
Suppose that \(f \in L^1(S; H^{-1}(\Omega))\), and denote \(\bar{f} := \int_S f \, d\mu\).
In other words we consider \(s \in S\) to be a parameter in boundary value problem

\[
\begin{aligned}
-\text{div}(A \nabla u) &= f(s, \cdot) \\
 u &\in H^1_0(\Omega)
\end{aligned}
\] (1)

A priori estimate for the solution implies that solution \(u\) belongs to \(L^1(S; H^1_0(\Omega))\).
We consider the following optimal design problem [Buttazzo, Maestre 2011]:
Given \(f \in L^1(S; H^{-1}(\Omega))\), one seeks for a characteristic function \(\chi\) on \(\Omega\) that optimizes

\[
J(\chi) = \int_S \int_\Omega f(s, x) u(s, x) \, dx \, d\mu \to \min / \max,
\]

where \(u \in L^1(S; H^1_0(\Omega))\) is determined by (1) with \(A = \chi \alpha I + (1 - \chi) \beta I\).
Moreover, we assume that quantity of the first material is given: \(\int_\Omega \theta \, dx = q_\alpha\).
Discrete probability space

\[ S := \{s_1, s_2, \ldots, s_m\}, \mu(s_i) = \mu_i \geq 0, \sum_i \mu_i = 1. \]

Then \( f \in L^1(S; H^{-1}(\Omega)) \) is characterized by functionals \( f_i := f(s_i, \cdot) \in H^{-1}(\Omega) \), which uniquely determine state functions \( u_i \):

\[
\begin{cases}
-\text{div} (A \nabla u_i) = f_i \\
u_i \in H^1_0(\Omega)
\end{cases}
\]

\( i = 1, \ldots, m \).

Finally, the goal functional becomes

\[
J(\theta, A) = \sum_i \mu_i \int_{\Omega} f_i u_i \, d\mathbf{x} \to \max,
\]

Therefore, we can use any method for numerical solution of this new multiple state optimal design problem.
We consider $\Omega := \mathbb{B}(0,1) \subseteq \mathbb{R}^2$, $q_\alpha := 0.8|\Omega|$, $S = \{1,2\}$, $\mu_1 = \mu_2 = \frac{1}{2}$.

$f_1 = \chi_A + \epsilon \chi_B$

$f_2 = \chi_A - \epsilon \chi_B$, where

$A := \mathbb{B}(0,\frac{1}{2})^c$, $B := \mathbb{B}(0,\frac{1}{5})$

Numerical solution, $\epsilon = 0.01$

Numerical solution, $\epsilon = 0$
We shall enlarge the set $\mathcal{A}$ of admissible designs

$$\mathcal{A} = \left\{ (\theta, A) \in L^\infty(\Omega; [0, 1] \times \text{Sym}) : \int_{\Omega} \theta \, dx = q_\alpha, \ A \in \mathcal{K}(\theta) \ (\text{a.e. on } \Omega) \right\}$$

$$\mathcal{B} = \left\{ (\theta, A) \in L^\infty(\Omega; [0, 1] \times \text{Sym}) : \int_{\Omega} \theta \, dx = q_\alpha, \ A \in \mathcal{B}(\theta) \ (\text{a.e. on } \Omega) \right\}$$

$$\mathcal{C} := \left\{ (\theta, B) \in L^\infty(\Omega; [0, 1] \times \text{Sym}) : (\theta, B^{-1}) \in \mathcal{B} \right\}.$$

$\mathcal{A}$ is not convex: e.g. isotropic materials $(\theta, \gamma I) \in \mathcal{A}$ can be easily characterised:
\( B \) and \( C \) are convex sets: e.g. \( B \) can be rewritten as

\[
\lambda_{\min}(A) \geq \lambda^- \quad \text{and} \quad \lambda_{\max}(A) \leq \lambda^+ \quad \text{a.e. on } \Omega,
\]

where \( \lambda_{\min} \) and \( \lambda^+ \) are concave, and \( \lambda_{\max} \) and \( \lambda^- \) are convex functions.

\[
-J(\theta, A) = -\sum_{i=1}^{m} \mu_i \int_{\Omega} f_i u_i \, dx
\]

\[
= -\sum_{i=1}^{m} \mu_i \int_{\Omega} A \nabla u_i \cdot \nabla u_i - 2f_i u_i \, dx
\]

\[
= -\min_{v \in H_0^1(\Omega;\mathbb{R}^m)} \sum_{i=1}^{m} \mu_i \int_{\Omega} A \nabla v_i \cdot \nabla v_i - 2f_i v_i \, dx
\]

\[
= -\max_{\sigma \in S} \left( -\sum_{i=1}^{m} \mu_i \int_{\Omega} A^{-1} \sigma_i \cdot \sigma_i \, dx \right),
\]

where \( S = \{ \sigma \in L^2(\Omega;\mathbb{R}^d)^m : -\text{div} \sigma_i = f_i, i = 1, \ldots, m \} \).
Lemma

There exists a unique $\sigma^* \in S = \{\sigma \in L^2(\Omega; \mathbb{R}^d)^m : -\text{div} \, \sigma_i = f_i, i = 1..m\}$ such that

$$\max_{(\theta, A) \in B} J(\theta, A) = \max_{(\theta, A) \in B} \sum_{i=1}^{m} \mu_i \int_{\Omega} A^{-1} \sigma^*_i \cdot \sigma^*_i \, dx = \max_{(\theta, B) \in C} \sum_{i=1}^{m} \mu_i \int_{\Omega} B \sigma^*_i \cdot \sigma^*_i \, dx. \quad (2)$$

Moreover, if $(\theta^*, A^*)$ is an optimal design for problem $\max_B J$ and $u^*$ the corresponding state function, then $A^* \nabla u^*_i = \sigma^*_i, i = 1, \ldots, m$.

Above maximization problems are easily solved:

Design $(\theta^*, A^*)$ is optimal if and only if (almost everywhere in $\Omega$)

$$A^* \sigma^*_i = \lambda_{\theta^*}^* \sigma^*_i \quad i = 1..m.$$ 

If $u^*$ is the corresponding state function, we have

$$\sigma^*_i = \lambda_{\theta^*}^* \nabla u^*_i \quad \text{or equivalently} \quad A^* \nabla u^*_i = \lambda_{\theta^*}^* \nabla u^*_i, \quad i = 1..m.$$
... in terms of only local fraction $\theta$ belonging to the set

$$\mathcal{T} := \left\{ \theta \in L^\infty(\Omega; [0, 1]) : \int_\Omega \theta \, dx = q_\alpha \right\}$$

**Theorem**

Let $(\theta^*, A^*)$ be an optimal design for the problem $\max_B J$. Then $\theta^*$ solves

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, dx \longrightarrow \max$$

$\theta \in \mathcal{T}$ and $u$ determined uniquely by

$$\begin{cases}
-\text{div} \left( \lambda_\theta \nabla u_i \right) = f_i \\
\lambda_\theta \nabla u_i \in \mathcal{H}^1(\Omega) \quad i = 1, \ldots, m,
\end{cases}$$

Conversely, if $\tilde{\theta}$ is a solution of optimal design problem (3), and $\tilde{u}$ is the corresponding state function, then for any measurable $\tilde{A} \in \mathcal{B}(\tilde{\theta})$ such that $\tilde{A} \nabla \tilde{u}_i = \lambda_\theta (\tilde{\theta}) \nabla \tilde{u}_i$ almost everywhere on $\Omega$, e.g. for $\tilde{A} = \lambda_\theta (\tilde{\theta}) I$, $(\tilde{\theta}, \tilde{A})$ is an optimal design for the problem $\max_B J$. 

Marko Vrdoljak

Classical solutions in optimal design problems
Similar to Lemma above, one can rephrase the simpler relaxation problem (3): there exists a unique \( \sigma^* \in S = \{ \sigma \in L^2(\Omega; \mathbb{R}^d)^m : -\text{div} \sigma_i = f_i, i = 1..m \} \) such that

\[
\max_{\mathcal{T}} l = \max_{\theta \in \mathcal{T}} \sum_{i=1}^{m} \mu_i \int_{\Omega} \frac{\beta - \alpha}{\alpha \beta} \theta |\sigma^*_i|^2 \, dx.
\]

Moreover, \( \sigma^* \) is the same as for the problem \( \max_{\mathcal{B}} J \).

**Lemma**

The necessary and sufficient condition of optimality for solution \( \theta^* \in \mathcal{T} \) of optimal design problem (3) simplifies to the existence of a Lagrange multiplier \( c \geq 0 \) such that

\[
\sum_{i=1}^{m} \mu_i |\sigma^*_i|^2 > c \quad \Rightarrow \quad \theta^* = 1,
\]

\[
\sum_{i=1}^{m} \mu_i |\sigma^*_i|^2 < c \quad \Rightarrow \quad \theta^* = 0.
\]
Spherically symmetric case

Let \( \Omega \subseteq \mathbb{R}^d \) be spherically symmetric: in spherical coordinates given by \( r \in \omega \) (an interval), and the right-hand side \( f = f(r) \), \( r \in \omega \) be a radial function. Since \( \sigma^* \) is unique, it must be radial: \( \sigma^*_i = \sigma^*_i(r) e_r \).

Theorem

For any maximizer \((\theta^*, A^*)\) for the problem \( \max_B J \), there exist a radial maximizer \((\tilde{\theta}, \tilde{A}) \in B \) where

\[
\tilde{\theta}(r) = \int_{\partial B(0,r)} \theta^* \ dS.
\]

Corollary

For any radial solution \( \theta^* \) for \( \max_T I \), there exist a radial conductivity \( A^* \in K(\theta^*) \) such that \((\theta^*, A^*)\) is maximizer for \( \max_A J \). Conversely, if \((\theta^*, A^*) \in A \) is a radial maximizer for \( \max_A J \) then \( \theta^* \) is a maximizer for problem \( \max_T I \).
Back to the example $\varepsilon > 0$

$$f_{1,2}(r) = \begin{cases} 
\pm \varepsilon, & 0 \leq r \leq \frac{1}{5} \\
0, & \frac{1}{5} < r \leq \frac{1}{2} \\
1, & \frac{1}{2} < r \leq 1. 
\end{cases}$$

$-\text{div} \sigma_i = f_i$, $i = 1, 2$ in polar coordinates: $-\frac{1}{r} (r \sigma_i)' = f_i$. Due to regularity at $r=0$, we can calculate unique solutions $\sigma_1^*$ and $\sigma_2^*$:

$$\sigma_1^* + \sigma_2^* > c \Rightarrow \theta^* = 1,$$

$$\sigma_1^* + \sigma_2^* < c \Rightarrow \theta^* = 0.$$  

For any $c$, the solution $\theta^*$ is unique and classical (more precisely, the uniqueness of solution for $\max_B J$ follows).
How to determine Lagrange multiplier $c$?

Quantity of given materials uniquely determines $c$ (as usual).

$$\sigma_1^* + \sigma_2^* > c \implies \theta^* = 1,$$

$$\sigma_1^* + \sigma_2^* < c \implies \theta^* = 0.$$
Small $q_\alpha$: unique classical solution

\[
\sigma^* > c \quad \Rightarrow \quad \theta^* = 1, \\
\sigma^* < c \quad \Rightarrow \quad \theta^* = 0.
\]

If $q_\alpha > \frac{3}{4} \pi$ then $c$ have to be zero. Now, solution is not unique – it is only important to put $\alpha$ in annulus $B \left( 0, \frac{1}{2} \right)^c$. 
Example 2

Two state equations on a ball $\Omega = B(0, 2)$

- $f_1 = \chi_{B(0,1)}$, $f_2 = \chi_{B(0,1)^c}$,
- \[
\begin{cases}
-\text{div} \ (\lambda_\theta \nabla u_i) = f_i \\
u_i \in H^1_0(\Omega)
\end{cases}
\quad i = 1, 2
\]
- $\mu_1 \int_{\Omega} f_1 u_1 \, dx + \int_{\Omega} f_2 u_2 \, dx \to \text{max}$

For studying conditions of optimality, we introduce

$$
\psi(r) = \mu_1 \left( \sigma_1^*(r) \right)^2 + \left( \sigma_2^*(r) \right)^2.
$$

The case $0 < \mu_1 < \frac{49}{3}$:
Example 2

The case \( \frac{49}{3} \leq \mu_1 < 119:\)

![Graph 1](image1)

The case \( \mu_1 \geq 119:\)

![Graph 2](image2)
Multiple states

Three optimal configurations, depending on $\mu_1$ and $q_\alpha$:

Radii are solutions of some algebraic equations (solved numerically).
Conclusion

General strategy for solving $\max_A J$ in spherically symmetric case:

1. Solve $-\text{div} \sigma_i = f_i, i = 1..m$ – candidates for $\sigma^*$ (in case of ball there is only one candidate).

2. Study conditions of optimality (they usually give unique solution $\theta^*$ – radial, but also classical).

3. Construct solution to $\max_A J$ (commonly, it would be classical solution; for minimization problem the situation is quite different).

4. It is also possible to comment the possible non-uniqueness of relaxation problem.

Thank you for your attention!