Classical Optimal Design on Annulus and Numerical Solution by Shape Derivative Method

Marko Vrdoljak

Joint work with Petar Kunštek

Department of Mathematics
Faculty of Science
University of Zagreb, Croatia

89th GAMM Annual Meeting,

München, March **2018**

Energy maximization problem

State equation ($\Omega \subseteq \mathbf{R}^d$ open and bounded)

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = 1 \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

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$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$, $\int_\Omega \chi\, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$

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- Maximize the amount of heat kept inside body
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In general, energy functional

$$J(\chi) = \int_\Omega f(\mathbf{x}) u(\mathbf{x})\, d\mathbf{x} \longrightarrow \max$$

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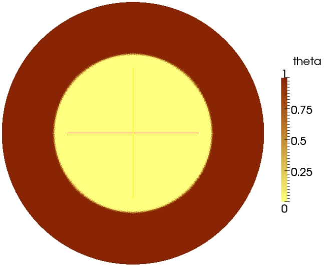
Classical vs. relaxed optimal design



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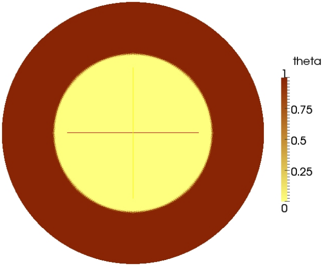
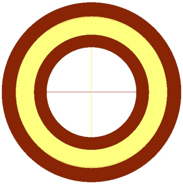


Intuition for annulus?



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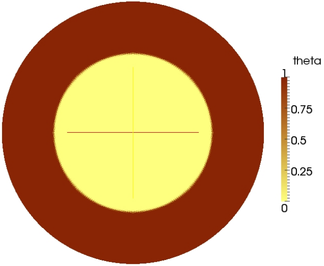

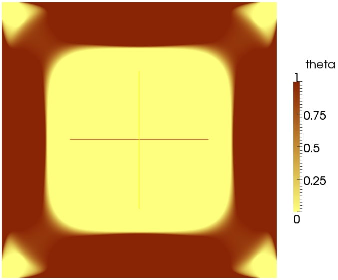

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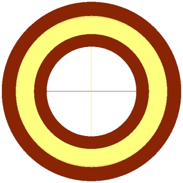

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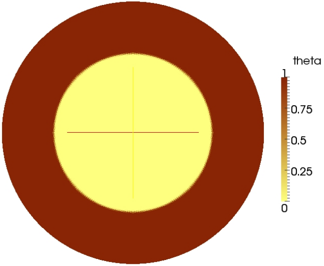

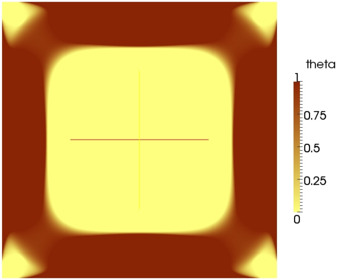

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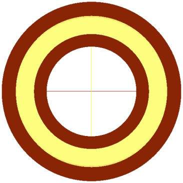

In general, there might exist no classical optimal design. The relaxation is needed, introducing composite materials.

classical design		relaxed design
$\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$	$\cdots$	$\theta \in \mathrm{L}^\infty(\Omega; [0,1])$
		$\mathbf{A} \in \mathcal{K}(\theta)$ a.e. on Ω

Multiple state optimal design problem

State equations

$$\begin{cases} -\operatorname{div}\left(\mathbf{A}\nabla u_i\right) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad\qquad i = 1, \ldots, m$$

State function $\mathsf{u} = (u_1, \ldots, u_m)$

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$$\begin{cases} I(\chi) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \max \\ \qquad \mathsf{u} = (u_1, \ldots, u_m) \text{ state function for } \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \\ \qquad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}), \ \displaystyle\int_\Omega \chi \, d\mathbf{x} = q_\alpha , \end{cases}$$

for some given weights $\mu_i > 0$.

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for some given weights $\mu_i > 0$. Relaxed designs:

$$\mathcal{A} := \left\{ (\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \, , \; \mathbf{A}(\mathbf{x}) \in \mathcal{K}(\theta(\mathbf{x})) \text{ a.e. on } \Omega \right\}$$

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However, if Ω is a ball and f_i are radial functions, solution is usually classical. Minimization of the same functional - classical optimal designs are rare exceptions (**Juan Casado-Díaz**); for multiple state problems – joint works with **Krešimir Burazin** and **Ivana Crnjac**.

Single vs. multiple state problems

A. Single state equation

[Murat & Tartar, 1985]

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$$I(\theta) = \int_\Omega fu\, d\mathbf{x} \to \max$$
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$$\frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}$$
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In spherically symmetric case (f_i are radial functions), simpler relaxation problem is equivalent to the true relaxation problem (simple laminates are enough).

Simpler relaxation problem

... in terms of only local fraction θ belonging to the set

$$\mathcal{T} := \left\{ \theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \right\}$$

$$I(\theta) = \sum_{i=1}^{m} \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \max$$

$\theta \in \mathcal{T}$ and u determined uniquely by (1)

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^- \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \dots, m\,,$$

Minimax formulation

$$
\begin{aligned}
I(\theta) &= \sum_{i=1}^{m} \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \\
&= -\sum_{i=1}^{m} \mu_i \int_\Omega \lambda_\theta^- |\nabla u_i|^2 - 2 f_i u_i \, d\mathbf{x} \\
&= -\min_{v \in \mathrm{H}_0^1(\Omega;\mathbf{R}^m)} \sum_{i=1}^{m} \mu_i \int_\Omega \lambda_\theta^- |\nabla v_i|^2 - 2 f_i v_i \, d\mathbf{x} \\
&= -\max_{\boldsymbol{\sigma} \in \mathcal{S}} \left(-\sum_{i=1}^{m} \mu_i \int_\Omega \frac{|\sigma_i|^2}{\lambda_\theta^-} \, d\mathbf{x} \right) \\
&= \min_{\boldsymbol{\sigma} \in \mathcal{S}} \left(\sum_{i=1}^{m} \mu_i \int_\Omega \frac{|\sigma_i|^2}{\lambda_\theta^-} \, d\mathbf{x} \right) ,
\end{aligned}
$$

where $\mathcal{S} = \{\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega; \mathbf{R}^d)^m : -\mathrm{div}\, \sigma_i = f_i, i = 1, \ldots, m\}$.

Necessary and sufficient optimality conditions

By minimax theorem there exists a unique $\boldsymbol{\sigma}^* \in \mathcal{S} = \{\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega; \mathbf{R}^d)^m : -\operatorname{div} \boldsymbol{\sigma}_i = f_i, i = 1..m\}$ such that

$$\max_{\mathcal{T}} I = \max_{\theta \in \mathcal{T}} \sum_{i=1}^{m} \mu_i \int_\Omega \frac{\beta - \alpha}{\alpha\beta} \theta |\boldsymbol{\sigma}_i^*|^2 \, d\mathbf{x} \,.$$

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Lemma

The necessary and sufficient condition of optimality for solution $\theta^* \in \mathcal{T}$ of optimal design problem (1) simplifies to the existence of a Lagrange multiplier $c \geq 0$ such that

$$\sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 > c \quad \Rightarrow \quad \theta^* = 1 \, ,$$
$$\sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 < c \quad \Rightarrow \quad \theta^* = 0 \, .$$

Spherically symmetric case – uniqueness

Conditions of optimality:

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In case of spherical symmetry $\boldsymbol{\sigma}_i^* = \sigma_i^*(r)\mathbf{e}_r$, where σ_i^* solves $-\frac{1}{r}(r\sigma_i)' = f_i\,.$

Let us denote

$$\psi(r) := \sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 = \sum_{i=1}^{m} \mu_i (\sigma_i^*)^2\,.$$

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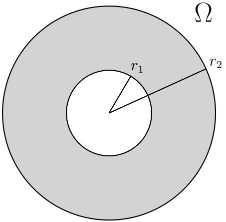
Corollary

For spherically symmetric case, if ψ is piecewise strictly monotone on ω then the problem $\max_{\mathcal{T}} I$ has a unique solution θ^*, which is a radial characteristic function. Consequently, the solution of the true relaxation problem is unique, classical and radial.



Single state equation:

$$\begin{cases} -\mathrm{div}\left(\lambda_\theta^- \nabla u\right) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where $\lambda_{\theta(\mathbf{x})}^- = \left(\frac{\theta(\mathbf{x})}{\alpha} + \frac{1-\theta(\mathbf{x})}{\beta}\right)^{-1}$.

Optimization problem:

$$\begin{cases} I(\theta) = \int_\Omega u \,\mathrm{d}\mathbf{x} \to \max \\ s.t. \quad \theta \in \mathrm{L}^\infty(\Omega, [0,1]), \ \int_\Omega \theta = q_\alpha, \ \text{where } u \text{ satisfies (2)} \end{cases} \tag{3}$$

Single state optimal design problem

One can rewrite (2) in polar coordinates :

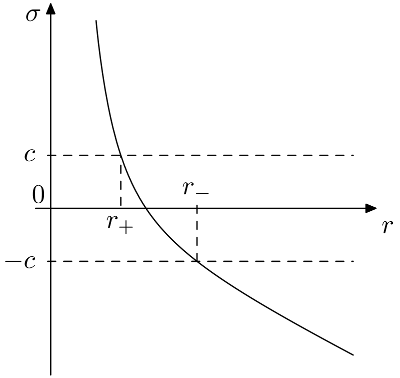
$$-\frac{1}{r^{d-1}}(r^{d-1}\underbrace{\lambda_\theta^- u'(r)}_{\sigma})' = 1 \text{ in } \langle r_1, r_2\rangle\,, \quad u(r_1) = u(r_2) = 0.$$

Observe that σ satisfies

$$\sigma = -\frac{r}{d} + \frac{\gamma}{r^{d-1}}, \quad \gamma > 0$$

$\sigma(r) : \langle 0, \infty\rangle \to \mathbb{R}$ is a strictly decreasing function, for any γ.

$\Longrightarrow$ Optimal design is unique, classical and radial.



The necessary and sufficient condition of optimality for θ^* states

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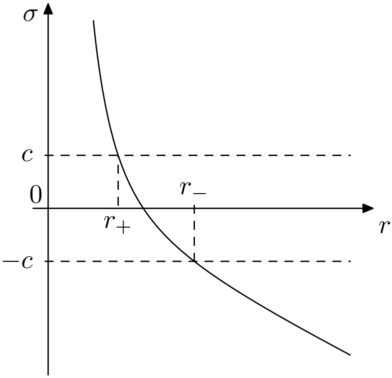

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There are only three possible candidates for optimal design:

1) $\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_-\rangle \\ 1, & r \in [r_-, r_2] \end{cases}$

2) $\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_2\rangle \end{cases}$

3) $\theta^*(r) = \begin{cases} 0, & r \in [r_1, r_-\rangle \\ 1, & r \in [r_-, r_2\rangle \end{cases}$

Simplification to a non-linear system

From condition of optimality a non-linear system (with unknowns γ, c, r_+, r_-) is created:

$$\begin{cases} S_d \int\limits_{r_1}^{r_2} \theta(\rho)\rho^{d-1}\,\mathrm{d}\rho = q_\alpha \\ u(r_2) = 0 \iff \gamma \int\limits_{r_1}^{r_2} \left(\frac{1}{a(\rho)\rho^{d-1}}\right)\mathrm{d}\rho = \int\limits_{r_1}^{r_2} \frac{\rho}{a(\rho)}\,\mathrm{d}\rho \\ \sigma(r_+) = c, \quad \sigma(r_-) = -c, \qquad \text{where } c > 0 \end{cases} \tag{NS}$$

where

$$\sigma(r) = \frac{\gamma}{r^{d-1}} - \frac{r}{d}, \quad \& \quad a(r) = \left(\frac{\theta(r)}{\alpha} + \frac{1-\theta(r)}{\beta}\right)^{-1}.$$

Theorem (Optimal design for annulus $d = 2, 3,\ f = 1$)

With previous assumptions the problem admits classical solution with two possible designs:

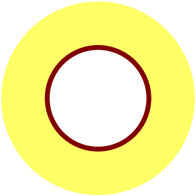
1) $$\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_-\rangle \\ 1, & r \in [r_-, r_2] \end{cases}$$ **alpha-beta-alpha**

2) $$\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_2\rangle \end{cases}$$ **alpha-beta**

More precisely, if q_α is small enough, design 2) is optimal.

alpha-beta

($q_\alpha <$ critical value)



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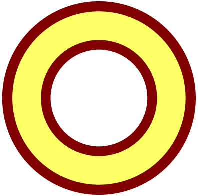
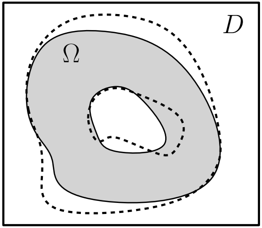


Shape derivative

Perturbation of the set Ω is given with

$$\Omega_t = (\mathrm{Id} + t\psi)\Omega$$

where $\psi \in W^{k,\infty}(\mathbf{R}^d, \mathbf{R}^d)$.



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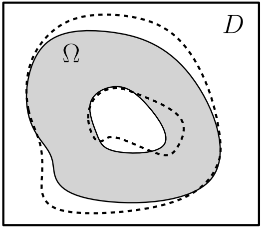
$$\Omega_t = (\mathrm{Id} + t\psi)\Omega$$

where $\psi \in W^{k,\infty}(\mathbf{R}^d, \mathbf{R}^d)$.



Definition (Shape derivative)

Let $J = J(\Omega)$ be a shape functional. J is said to be shape differentiable at Ω in direction ψ if

$$J'(\Omega, \psi) := \lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}$$

exists and the mapping $\psi \mapsto J'(\Omega, \psi)$ is linear and continuous.
$J'(\Omega, \psi)$ is called the **shape derivative**.

Single state problem (general f)

In case of our single state optimal design problem:

$$\begin{cases} J(\Omega_\alpha) = \displaystyle\int_\Omega f u \, \mathrm{d}\mathbf{x} \to \max \\ u \text{ is determined by } \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \\ \chi \in L^\infty(\Omega, \{0,1\}) \text{ is a characteristic function of } \Omega_\alpha, \quad |\Omega_\alpha| = q_\alpha \end{cases} \tag{4}$$

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The shape derivative is given by:

$$\begin{aligned} J'(\Omega_\alpha, \psi) = &\int_\Omega \mathbf{A}(-\operatorname{div}(\psi) + \nabla\psi + \nabla\psi^\tau)\nabla u_0 \cdot \nabla u_0 \,\mathrm{d}\mathbf{x} \\ &+ \int_\Omega 2(\operatorname{div}(\psi) f + \nabla f \cdot \psi) u_0 \,\mathrm{d}\mathbf{x} \end{aligned}$$

where u_0 is the corresponding state function.

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where u_0 is the corresponding state function. The construction of ψ:

$$\int_\Omega \nabla\psi : \nabla\varphi + \int_\Omega \psi \cdot \varphi = \mathcal{L}'(\Omega_\alpha, \varphi), \quad \forall \varphi \in H_0^1(\Omega).$$

Gradient method, Lagrange approach

Algorithm 1: k-th step of gradient method

1 Input : Γ_k - interface is discretized in points (it is used to create new mesh $\mathcal{T}_k$)

2 Construct vector spaces `Vh` on mesh $\mathcal{T}_k$ (`Vh`=P1, P2 . . .)

3 Determine vector field $\psi \in$ `Vh`

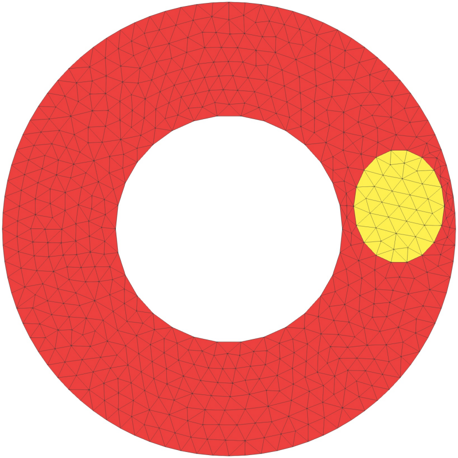
4 Determine $t_0 > 0$ (if too small, increase of J is insignificant; upper bound is dictated by mesh $\mathcal{T}_k$)

5 Move mesh: $\mathcal{T}_{k+1} = (\mathrm{Id} + t_0\psi)\mathcal{T}_k$

6 Output: $\Gamma_{k+1} = (\mathrm{Id} + t_0\psi)\Gamma_k$

Upper bound in part 4 is calculated by `checkmovemesh`. This ensures that moving of a mesh doesn't create wrong ordering of elements (volume of triangle should not be negative). Part 5 is implemented using `movemesh`. At the end of the step it is recommended to use `adaptmesh`.

Numerical results



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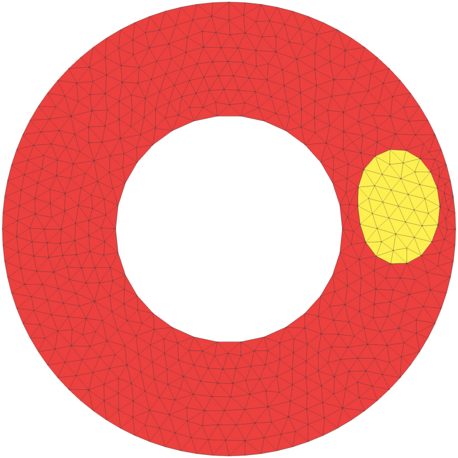

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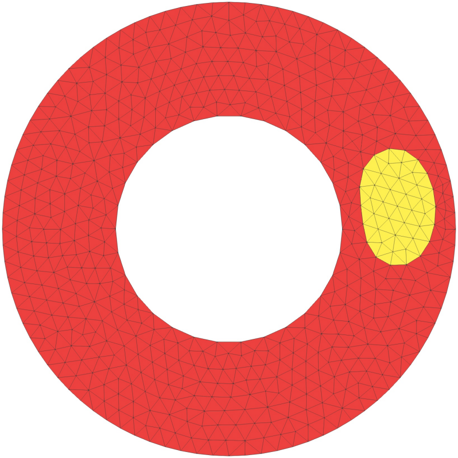

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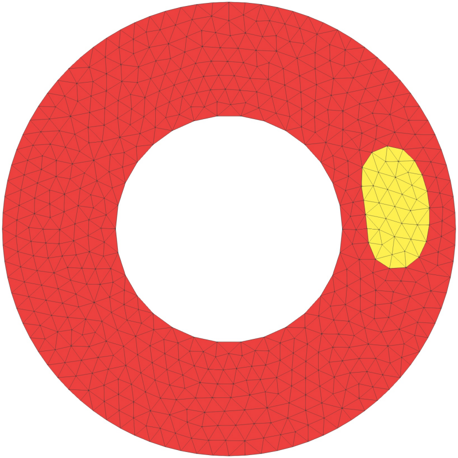

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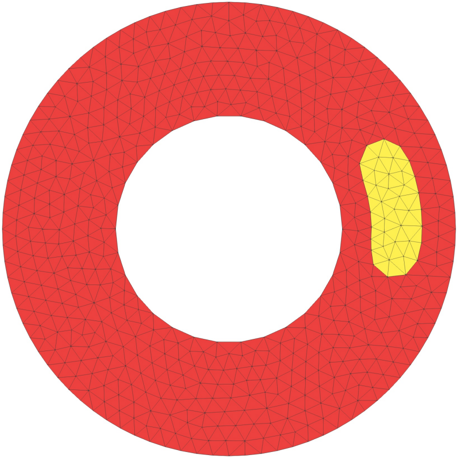

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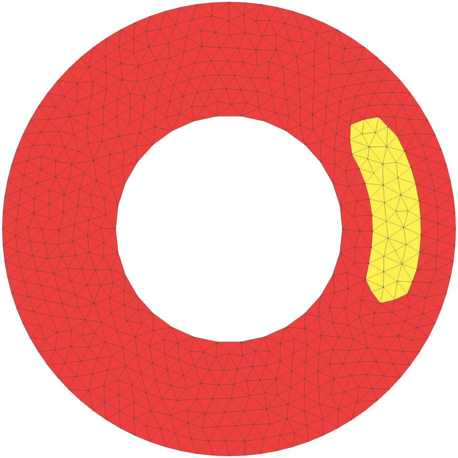

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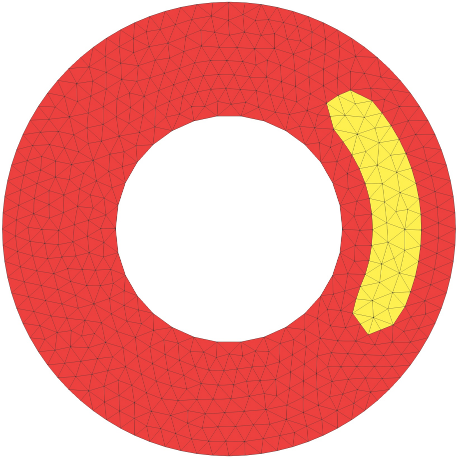

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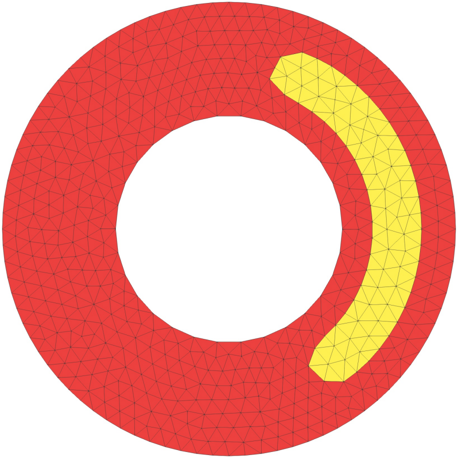

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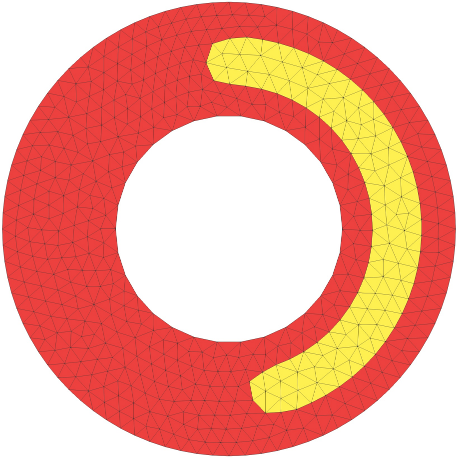

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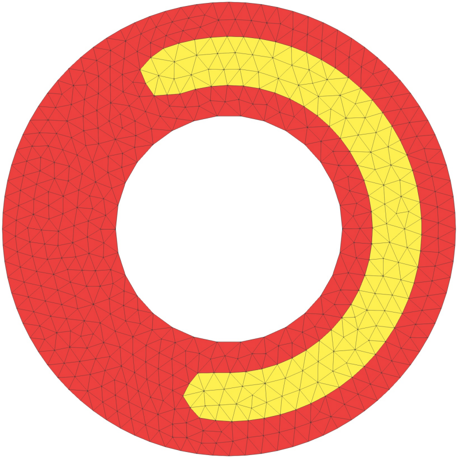

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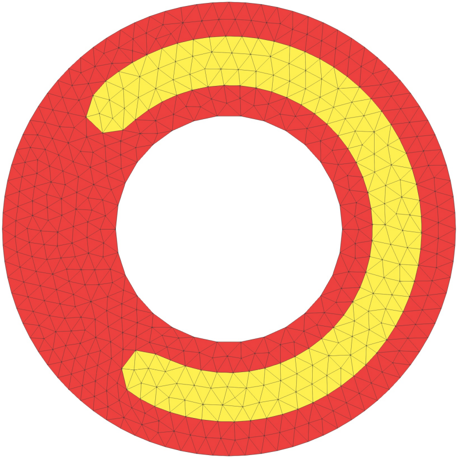

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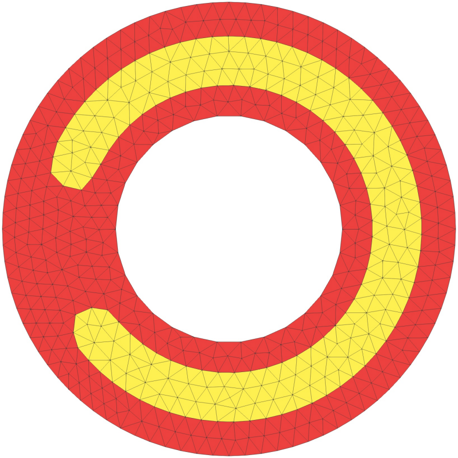

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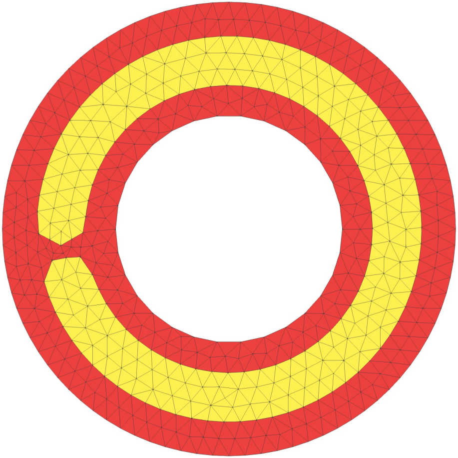

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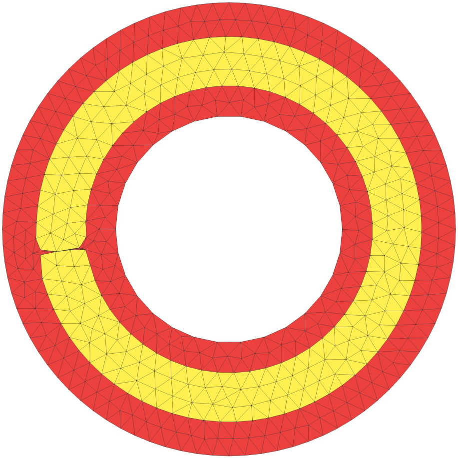

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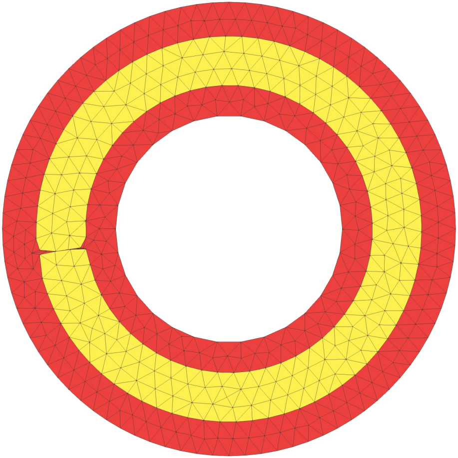

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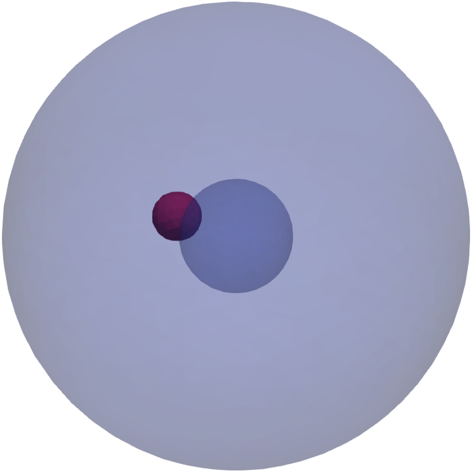

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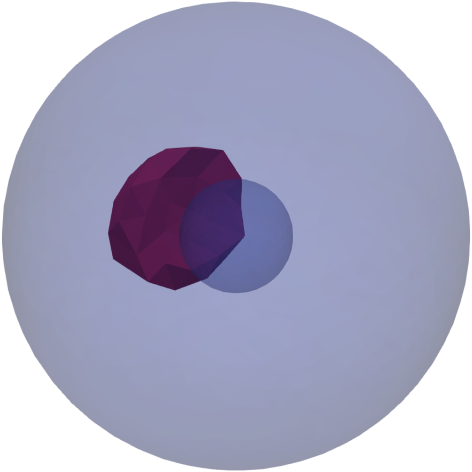

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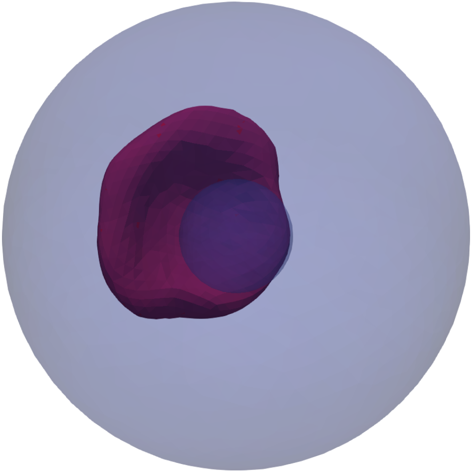

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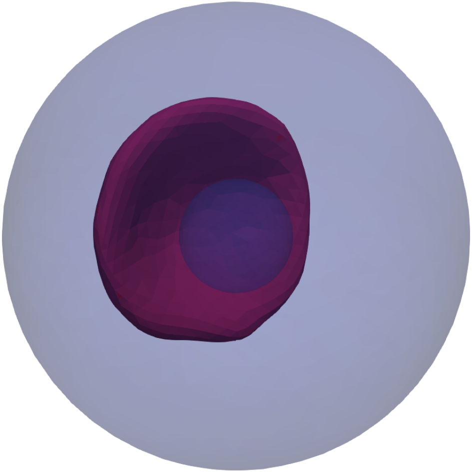

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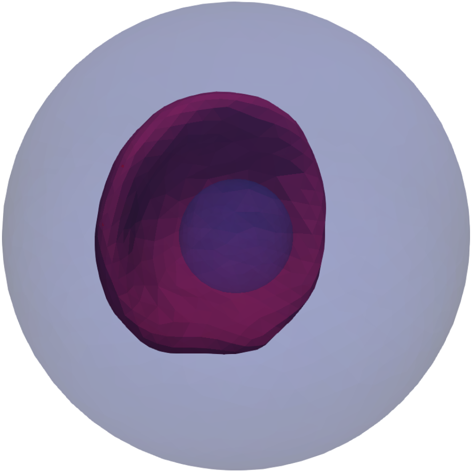

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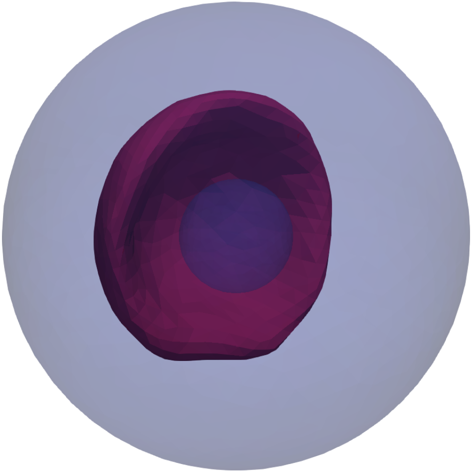

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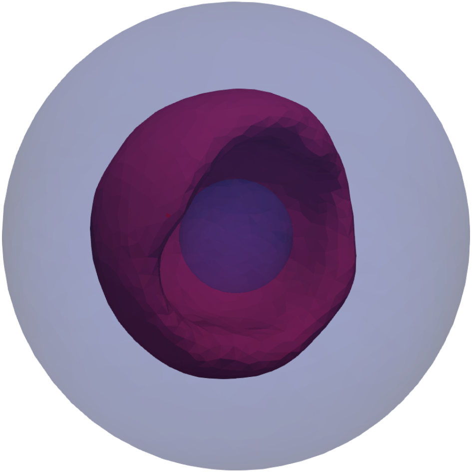

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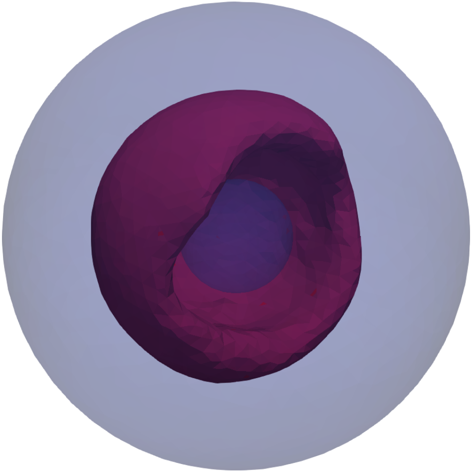

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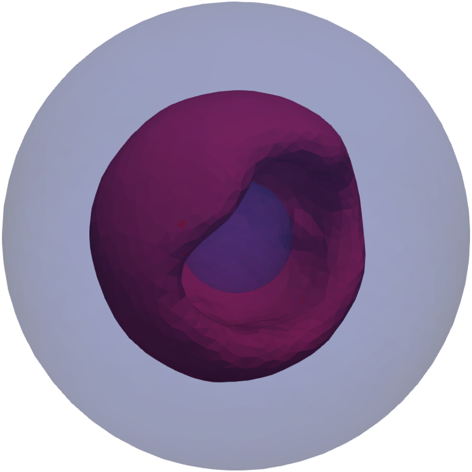

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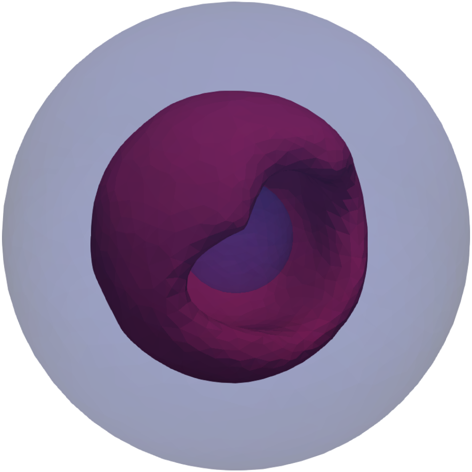

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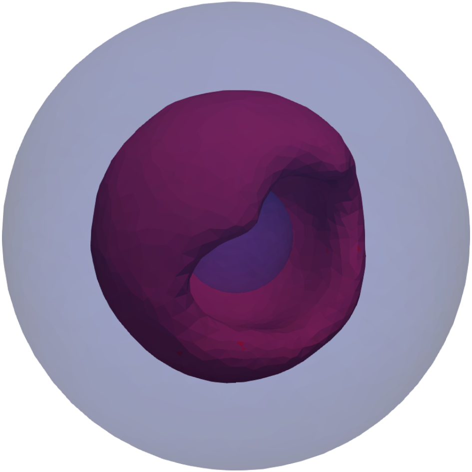

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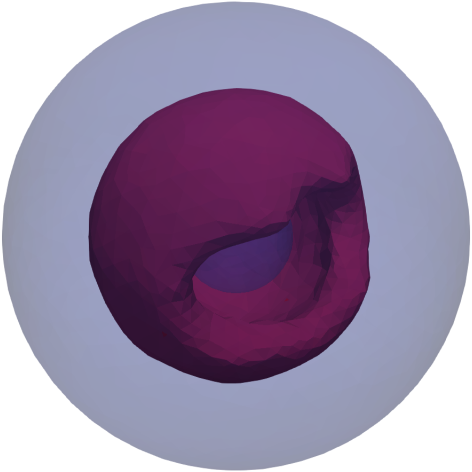

The end

Thank you for your attention!