# On a mixed boundary problem for scalar conservation laws

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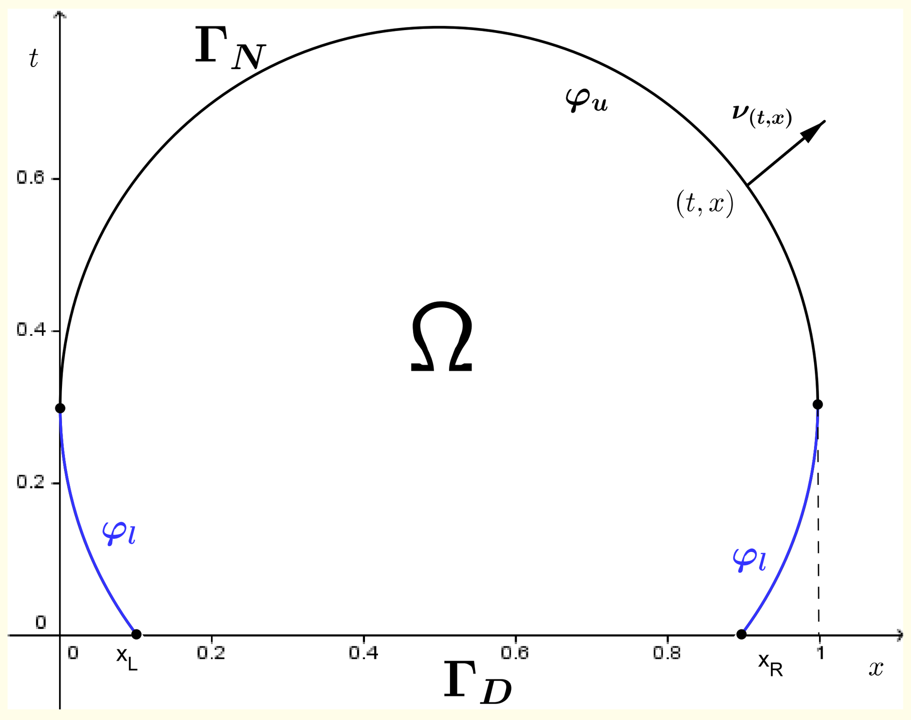
## Statement of the problem

We examine the following mixed boundary problem on an open bounded domain $\Omega \subseteq [0,\infty\rangle \times \mathbf{R}$ with boundary of class $C^{1,1}$:

$$\partial_t u + \partial_x(f(t,x,u)) = 0 \text{ in } \Omega \quad (1)$$
$$\nabla_{(t,x)} u \cdot \nu = 0 \text{ on } \Gamma_N \quad (2)$$
$$u(0,.) = u^0(.) \in L^\infty(\mathbf{R}) \text{ on } \Gamma_D, \quad (3)$$

where $\Gamma_N$ and $\Gamma_D \subset \{t=0\}$ are partitions of $\partial\Omega$ of strictly positive (Hausdorff) measure and $\nu$ is the outer unit normal vector on $\Gamma_N$. We also assume that $f \in C^2(\Omega \times \mathbf{R})$.




An example of the domain of the problem we investigate. Note that the transition from $\Gamma_D$ to $\Gamma_N$ at points $x_L$ and $x_R$ is assumed to be smooth enough.

## Motivation

Initial-boundary value problems for scalar conservation laws were first considered in [1]. Applications of such problems are usually in the traffic flow [5] or filtration and sedimentation models [2].

In [3] initial-boundary problem for (1) with zero-flux boundary conditions was considered. It is called the Neumann problem for (1). In this situation the total mass is conserved, but there can be difference in the inflow and outflow through a boundary point.

In the problem we consider, we require that the inflow is equal to the outflow in the normal direction of every point $(t,x)$ of the boundary $\Gamma_N$.

**Motivation:** possible application on traffic flow models.

Namely, if we consider (1) as a model of cars density on the road, the Neumann boundary conditions mean that there are no density change through the boundary at each direction (i.e. the boundary is permeable in the sense that the quantity of cars reaching the boundary in the direction of a normal on $\partial\Omega$ corresponds to the quantity of cars leaving the domain $\Omega$ in the same direction).

## Approximation and the main result

**Question:** What would be proper solution concept for (1), (2), (3)?

At this moment, we are not able to introduce appropriate definition (this will be a subject of further investigation). Instead, we shall consider an elliptic approximation to the problem:

$$\begin{aligned} \partial_t u_n + \partial_x(f(t,x,u_n)) &= \frac{1}{n}\triangle_{(t,x)} u_n \text{ in } \Omega \\ \nabla_{(t,x)} u_n \cdot \nu &= 0 \text{ on } \Gamma_N \\ u_n(0,.) &= u_n^0(.) \text{ on } \Gamma_D, \end{aligned} \quad (4)$$

where $(u_n^0)$ is a bounded sequence of functions converging strongly in $L^1_{loc}(\mathbf{R})$ toward $u_0$.

It can be shown that $u_n \in H^1_D(\Omega) \cap H^{\frac{3}{2}-\varepsilon}(\Omega)$, where $\varepsilon > 0$ and $H^1_D(\Omega)$ is the set of $H^1(\Omega)$ functions which are zero on $\Gamma_D$.

**Theorem.** *Assume that the sequence $(u_n)$ of solutions to (4) satisfies*

- *(i) $\frac{1}{n}\int_{\Gamma_D} |\partial_t u_n(0,x)| dx \le C < \infty$;*
- *(ii) the sequence $(u_n)$ is uniformly bounded by a constant $M$.*

*Then the weak $L^2(\Omega)$-limit of $(u_n)$ along a subsequence satisfies the equation (1) in $\Omega$.* ∎

## Sketch of the proof I

The proof employs standard methods of compensated compactness (div-rot lemma) and the theory of Young measures.

Denote by $u$ a weak $L^2(\Omega)$-limit of $(u_n)$ along some subsequence. The main difficulty is to identify $f(\cdot,u)$ as a weak limit of $f(\cdot,u_n)$.

**Step 1:** using the assumptions, obtain the strong convergence:

$$\partial_t u_n + \partial_x(f(t,x,u_n)) \longrightarrow 0 \quad \text{in} \quad H^{-1}(\Omega) \quad (5)$$

**Step 2:** take a convex $\Phi : \mathbf{R} \to \mathbf{R}$ such that $\Phi'$ is bounded and define $\Psi(y) = \int_0^y f'_\lambda(t,x,\lambda)\Phi'(\lambda)d\lambda$;

show that $\left(\partial_t(\Phi(u_n))\right)_n$ is precompact in $H^1(\Omega)$ and that $\left(\partial_x(\Psi(u_n))\right)_n$ is bounded in $\mathcal{M}_b(\Omega)$;

## Sketch of the proof II

**Step 3:** use a particular entropy-entropy flux pair $(\Phi,\Psi)$:

$$\begin{aligned} \Phi(\lambda) &= |\lambda - u(t,x)|, \\ \Psi(\lambda) &= \operatorname{sgn}(\lambda - u(t,x))(f(t,x,\lambda) - f(t,x,u(t,x))), \end{aligned}$$

and Murat's lemma to show that

$$\left(\partial_t(\Phi(u_n)) + \partial_x(\Psi(u_n))\right)_n \text{ is precompact in } H^{-1}(\Omega) \quad (6)$$

**Step 4:** use the div-rot lemma (div and rot with respect to variables $(t,x)$) on the vector fields $v_k = (u_n, f(u_n))$ and $w_k = (\Psi(u_n), -\Phi(u_n))$ and Young measure associated with the weak $L^2(\Omega)$-convergence $u_n \rightharpoonup u$, to show that $f(\cdot,u)$ is a weak limit of $f(\cdot,u_n)$;

now we conclude that $u$ is the weak solution of (1) in $\Omega$;

**Q.E.D.**

## Numerical example

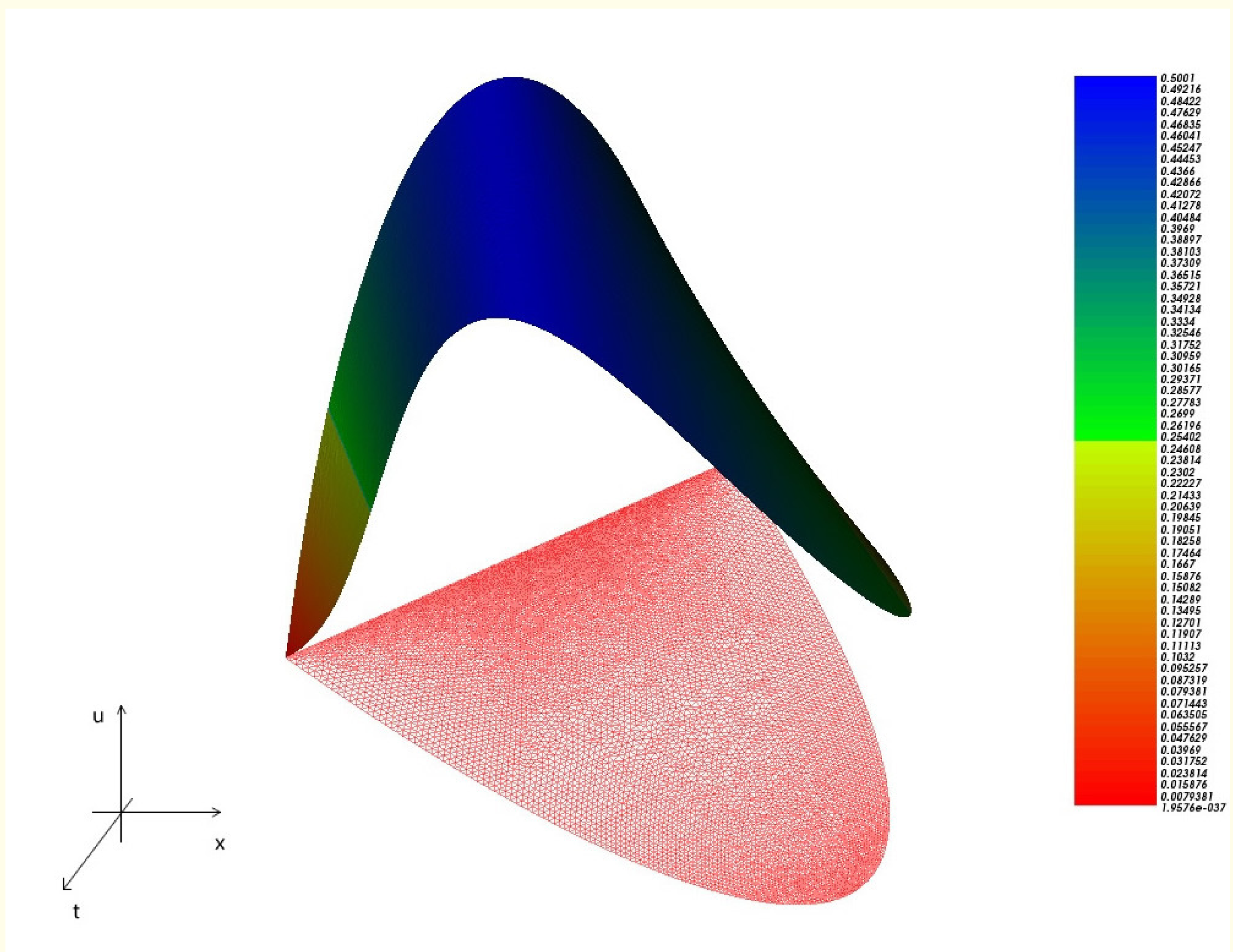
Let $\Omega = \{(t,x) \in \mathbf{R}^2 : 0 \le x \le 1,\ 0 \le t \le -4x(x-1)\}$. We focus on solving the regularized Burgers equation:

$$\begin{aligned} \partial_t u + \partial_x\left(u^2/2\right) &= \epsilon\Delta_{(t,x)} u \quad \text{in } \Omega, \\ \nabla_{(t,x)} u \cdot \nu &= 0 \quad \text{on } \Gamma_N, \\ u(0,x) &= u_D \quad \text{on } \Gamma_D, \end{aligned}$$

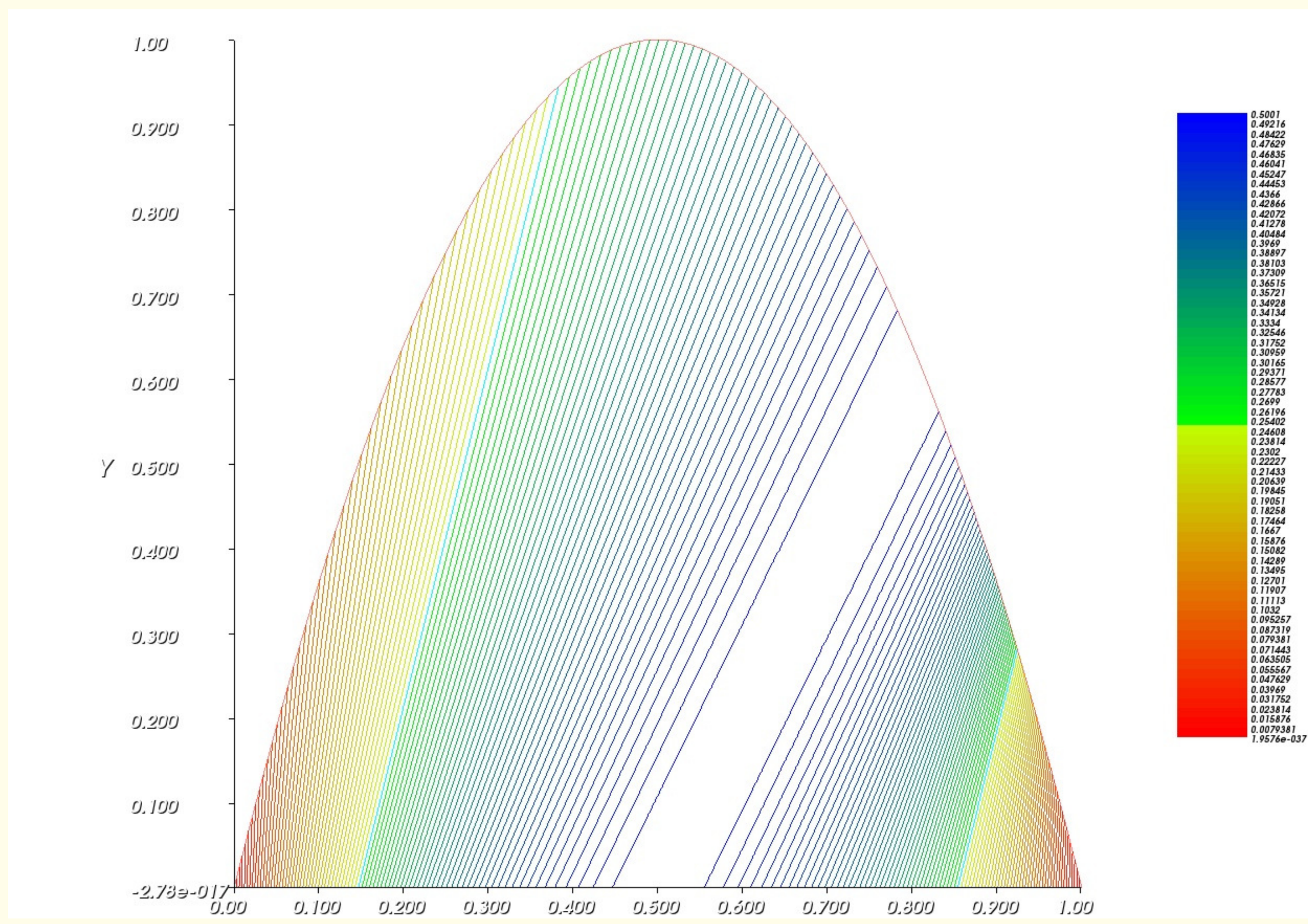
where $u_D = -2x(x-1)$ and $\Gamma_D = \{(t,x) \in \partial\Omega : t = 0\}$. The standard fixed point arguments (Picard iterations) is: For given initial guess $u_0$, construct a sequence of solutions $u_n \in \{v \in H^1(\Omega) : v|_{\Gamma_D} = u_D\}$ of

$$(\forall \psi \in H^1_D(\Omega)) \qquad \int_\Omega (\partial_t u_{n+1} + u_n \partial_x u_{n+1})\psi dt dx + \\ + \epsilon \int_\Omega \nabla_{(t,x)} u_{n+1} \cdot \nabla_{(t,x)} \psi dt dx = 0.$$

## Numerical solution of elliptic approximation to Burgers equation with $N = 160$ and $\epsilon = 1/160^2$




Solution of the equation with triangulation of the domain (represented in red).




Iso–values of the solution.

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