**Introduction and some tools**

**Div-rot lemma and Quadratic theorem**

**Lemma.** Assume that $\Omega$ is open and bounded subset of $\mathbb{R}^d$, and that it holds $w_n \rightharpoonup w$ in $L^2(\mathbb{R}^d; \mathbb{R}^d)$, $w_n \rightharpoonup_\ast w$ in $L^2(\mathbb{R}^d)$, and $\nabla w_n \rightharpoonup_\ast \nabla w$ in $L^2(\mathbb{R}^d)$.

Then $w_n \rightharpoonup w$ in the sense of distributions.

**Theorem.** Assume that $\Omega \subset \mathbb{R}^d$ is open and that $\mathcal{A} \subset \mathbb{R}^d$ is defined by

$$\mathcal{A} = \left\{ x \in \mathbb{R}^d : (x, y) \in \Omega \times \mathbb{R}^d \right\},$$

and that $Q$ is a real quadratic form on $\mathbb{R}^d$, which is nonnegative on $\mathcal{A}$, i.e.

$$(y, \mathcal{A}y) \geq 0, \quad \forall (y, x) \in Q(\mathcal{A}) = 0.$$

Furthermore, assume that the sequence of functions $(u_n)$ satisfies $u_n \rightharpoonup u$ weakly in $L^2(\mathbb{R}^d)$.

Then every subsequence of $(u_n)$ which converges in distributions to its limit $u$ satisfies

$$Q(u) \leq Q(u_n).$$

**H-distributions**

H-distributions were introduced by N. Antonic and D. Mitrović (2011) as an extension of H-measures to the $L^p - L^q$ content. Existing applications are related to the velocity averaging and $L^p - L^q$ compactness by compensation.

**Theorem.** If $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^d)$ and $u_n \rightharpoonup_\ast u$ in $L^2(\mathbb{R}^d)$ for some $r \in (1, \infty)$, and $p, q \geq 2$, then there exist subsequences $(u_{n_k})$ and $(u_{n_j})$ and a complex valued distribution $\mu \in \mathcal{D}^\prime(\mathbb{R}^d)$, such that, for every $\phi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} u_{n_k}(x) \phi(x) \, dx = \int_{\mathbb{R}^d} \mu(x) \phi(x) \, dx,$$

for $\Omega \subset \mathbb{R}^d$, satisfies conditions of the existence theorem, $(u_n)$ and $(v_n)$ form a pure and the corresponding H-distribution is compact.

**H-measures**

H-measures are mathematical objects introduced by L. Tartar, which were motivated by possible applications in homogenization, and independently by P. Gérard, who was motivated by problems in kinetic theory.

**Theorem.** If $u_n \rightharpoonup u$ and $u_n \rightharpoonup_\ast u$ in $L^2(\mathbb{R}^d)$, then there exist subsequences $(u_{n_k})$ and a complex valued Radon measure $\mu \in \mathcal{D}^\prime(\mathbb{R}^d)$, and for each $\phi \in C_0^\infty(\mathbb{R}^d)$ and $\epsilon \in (0, \infty)$ one has

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} u_{n_k}(x) \phi(x) \, dx = \int_{\mathbb{R}^d} \mu(x) \phi(x) \, dx + \epsilon \int_{\mathbb{R}^d} |\nabla \phi(x)| \, dx,$$

for $\Omega \subset \mathbb{R}^d$, satisfies conditions of the existence theorem, $(u_n)$ and $(v_n)$ form a pure and the corresponding H-distribution is compact.

**One-scale H-measures**

**Theorem.** Complexity $R^d \setminus \{0\}$ by adding two spheres (around the origin, $S_0$, and in the infinity, $S_\infty$).

**Distributions of anisotropic order**

Let $\Omega$ and $S_0$ be open sets in $\mathbb{R}^d$ and $\mathbb{R}^d$ $(C_0^\infty)$-manifolds of dimensions $d'$ and $d$ and $\Omega \subset S_0 \times \mathbb{R}^d$ an open set.

By $C^{\infty}(\mathbb{R}^d)$ we denote the space of functions $f$ on $\mathbb{R}^d$ such that for any $n \in \mathbb{N}$ and $R \in (0, \infty)$, we have $f \in C^{\infty}(\mathbb{R}^d)$.

A distribution of order $\alpha$ in $x$ and order $\beta$ in $y$ is any linear functional on $C^{\infty}(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $D^{\alpha, \beta}(\Omega)$.

**Some properties of $L^p$ - $L^q$ variant of compensation**

**Lemma.** For a sequence $(u_n)$ in $L^p(\mathbb{R}^d)$, $p \in (1, \infty)$, the following are equivalent:

1. $u_n \rightharpoonup u$ in $L^p(\mathbb{R}^d)$.
2. For every sequence $(v_n)$ satisfying conditions of the existence theorem, $(u_n)$ and $(v_n)$ form a pure and the corresponding H-distribution is compact.

**Example.** $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^d)$, $u_n \rightharpoonup_\ast u$ in $L^2(\mathbb{R}^d)$, for some $r \in (1, \infty)$, and $p, q \geq 2$. Then there exist subsequences $(u_{n_k})$, $(u_{n_j})$, and a complex valued distribution $\mu \in \mathcal{D}^\prime(\mathbb{R}^d)$, such that, for every $\phi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} u_{n_k}(x) \phi(x) \, dx = \int_{\mathbb{R}^d} \mu(x) \phi(x) \, dx,$$

for $\Omega \subset \mathbb{R}^d$, satisfies conditions of the existence theorem, $(u_n)$ and $(v_n)$ form a pure and the corresponding H-distribution is compact.

**Application**

Now, let us consider the following non-linear parabolic type equation

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = 0,$$

where $u_0 = u(0, \cdot)$ is an open subset of $\mathbb{R}^d$. We assume that $u \in L^2(\mathbb{R}^d)$, $f(u, x) \in L^2(\mathbb{R}^d)$, $1 < p < 2$, and $A \in L^2(0, \infty; L^2(\mathbb{R}^d)), \lambda \in L^2(0, \infty; L^2(\mathbb{R}^d)), \nabla \phi \in L^2(\mathbb{R}^d)$, $\beta$, where $\lambda(x, \cdot)$ is a matrix valued function on $\mathbb{R}^d$.

Furthermore, assume that $f$ is a Carathéodory function and non-decreasing with respect to the third variable.

**References**


