Microlocal defect functionals and applications

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Introduction and some tools

Div-rot lemma and Quadratic theorem

Lemma. *Assume that Ω is open and bounded subset of $\mathbf{R}^3$, and that it holds:*

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } \mathrm{L}^2(\Omega;\mathbf{R}^3),\ \mathbf{v}_n \rightharpoonup \mathbf{v} \text{ in } \mathrm{L}^2(\Omega;\mathbf{R}^3),$$

$$\mathrm{rot}\,\mathbf{u}_n \text{ bounded in } \mathrm{L}^2(\Omega;\mathbf{R}^3),\ \mathrm{div}\,\mathbf{v}_n \text{ bounded in } \mathrm{L}^2(\Omega).$$

Then

$$\mathbf{u}_n \cdot \mathbf{v}_n \rightharpoonup \mathbf{u}\cdot\mathbf{v}$$

in the sense of distributions.

Theorem. *Assume that $\Omega \subseteq \mathbf{R}^d$ is open and that $\Lambda \subseteq \mathbf{R}^r$ is defined by*

$$\Lambda := \Big\{\boldsymbol{\lambda} \in \mathbf{R}^r : (\exists \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}) \quad \sum_{k=1}^{d} \xi_k \mathbf{A}^k \boldsymbol{\lambda} = 0\Big\},$$

and that Q is a real quadratic form on $\mathbf{R}^r$, which is nonnegative on Λ, i.e.

$$(\forall \boldsymbol{\lambda} \in \Lambda) \quad Q(\boldsymbol{\lambda}) \geqslant 0\,.$$

Furthermore, assume that the sequence of functions $(\mathbf{u}_n)$ satisfies

$$\mathbf{u}_n \longrightarrow \mathbf{u} \quad \text{weakly in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{R}^r)\,,$$

$$\Big(\sum_k \mathbf{A}^k \partial_k \mathbf{u}_n\Big) \quad \text{relatively compact in} \quad \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega;\mathbf{R}^q)\,.$$

Then every subsequence of $(Q\circ\mathbf{u}_n)$ which converges in distributions to it's limit L, satisfies

$$L \geqslant Q\circ\mathbf{u}$$

in the sense of distributions. ∎

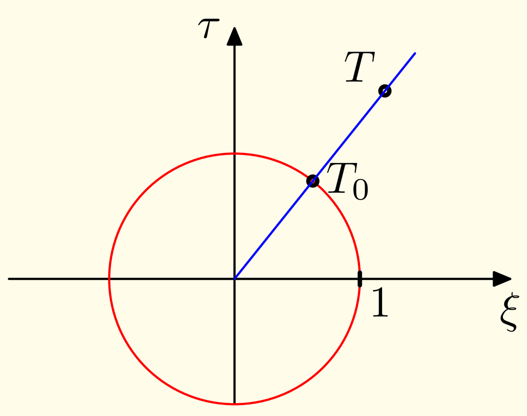
H-measures

H-measures are mathematical objects introduced by L. Tartar, who was motivated by possible applications in homogenisation, and independently by P. Gerard, who was motivated by problems in kinetic theory.

Theorem. *If $u_n \rightharpoonup 0$ and $v_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, then there exist their subsequences and a complex valued Radon measure μ on $\mathbf{R}^d \times \mathrm{S}^{d-1}$, such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ one has*

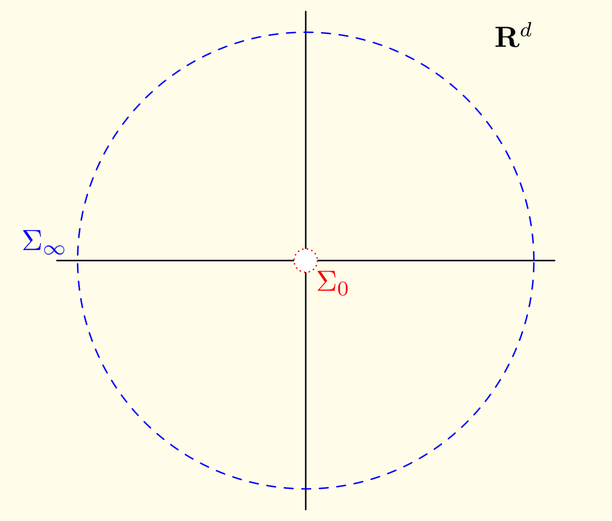
$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}}\overline{\widehat{\varphi_2 v_{n'}}}(\psi\circ\pi)d\boldsymbol{\xi} = \langle \mu, \varphi_1\overline{\varphi_2}\boxtimes\psi\rangle,$$

where $\pi : \mathbf{R}^d\setminus\{0\} \longrightarrow \mathrm{S}^{d-1}$ is the projection along rays. ∎



One-scale H-measures

Compactify $\mathbf{R}^d\setminus\{0\}$ by adding two spheres (around the origin, Σ_0, and in the infinity, Σ_∞):



$$\Sigma_0 := \{0^{\boldsymbol{\xi}_0} : \boldsymbol{\xi}_0 \in \mathrm{S}^{d-1}\}, \qquad \Sigma_\infty := \{\infty^{\boldsymbol{\xi}_0} : \boldsymbol{\xi}_0 \in \mathrm{S}^{d}1\}$$

$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d\setminus\{0\}\cup\Sigma_0\cup\Sigma_\infty$$

Theorem. *If $\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} \mathsf{u}$, and $\varepsilon_n \to 0$, then there exist $(\mathsf{u}_{n'})$ and $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\Omega\times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that $(\forall\,\varphi_1,\varphi_2 \in \mathrm{C}_c(\Omega))(\forall\,\psi\in\mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d)))$*

$$\lim_{n'}\int_{\mathbf{R}^d} \widehat{\varphi_1\mathsf{u}_{n'}}(\boldsymbol{\xi})\otimes\overline{\widehat{\varphi_2\mathsf{u}_{n'}}}(\boldsymbol{\xi})\psi(\varepsilon_{n'}\boldsymbol{\xi})\,d\boldsymbol{\xi} = \langle\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2\boxtimes\psi\rangle\,.$$

A distribution of order zero $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ we call the 1-scale H-measure with characteristic length *(ε_n) corresponding to the (sub)sequence (u_n).* ∎

H-distributions

H-distributions were introduced by N. Antonić and D. Mitrović (2011) as an extension of H-measures to the $\mathrm{L}^p-\mathrm{L}^q$ context.
Existing applications are related to the velocity averaging and $\mathrm{L}^p-\mathrm{L}^q$ compactness by compensation.

Theorem. *If $u_n \longrightarrow 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ and $v_n \xrightarrow{*} v$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ for some $p\in\langle 1,\infty\rangle$ and $q \geq p'$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a complex valued distribution $\mu\in\mathcal{D}'(\mathbf{R}^d\times\mathrm{S}^{d-1})$, such that, for every $\varphi_1,\varphi_2\in\mathrm{C}^\infty_c(\mathbf{R}^d)$ and $\psi\in\mathrm{C}^\kappa(\mathrm{S}^{d-1})$, for $\kappa=[d/2]+1$, one has:*

$$\lim_{n'\to\infty}\int_{\mathbf{R}^d}\mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}d\mathbf{x} =$$

$$\lim_{n'\to\infty}\int_{\mathbf{R}^d}(\varphi_1 u_{n'})(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})}d\mathbf{x} = \langle\mu,\varphi_1\overline{\varphi_2}\boxtimes\psi\rangle,$$

where $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is the Fourier multiplier operator with symbol $\psi\in\mathrm{C}^\kappa(\mathrm{S}^{d-1})$. ∎

Distributions of anisotropic order

Let X and Y be open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$ (or C^∞ manifolds of dimenions d and r) and $\Omega\subseteq X\times Y$ an open set.
By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha}\in\mathbf{N}^d_0$ and $\boldsymbol{\beta}\in\mathbf{N}^r_0$, if $|\boldsymbol{\alpha}|\leq l$ and $|\boldsymbol{\beta}|\leq m$, $\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}f = \partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\partial^{\boldsymbol{\beta}}_{\mathbf{y}}f\in\mathrm{C}(\Omega)$.
$\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$p^{l,m}_{K_n}(f) := \max_{|\boldsymbol{\alpha}|\leq l,|\boldsymbol{\beta}|\leq m}\|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\|_{\mathrm{L}^\infty(K_n)}\,,$$

where $K_n\subseteq\Omega$ are compacts, such that $\Omega=\cup_{n\in\mathbf{N}}K_n$ and $K_n\subseteq \mathrm{Int}K_{n+1}$,
Consider the space

$$\mathrm{C}^{l,m}_c(\Omega) := \bigcup_{n\in\mathbf{N}}\mathrm{C}^{l,m}_{K_n}(\Omega)\,,$$

and equip it by the topology of *strict inductive limit*.

Definition. A *distribution of order l in $\mathbf{x}$ and order m in $\mathbf{y}$* is any linear functional on $\mathrm{C}^{l,m}_c(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

Conjecture

Conjecture. *Let X,Y be C^∞ manifolds and let u be a linear functional on $\mathrm{C}^{l,m}_c(X\times Y)$. If $u\in\mathcal{D}'(X\times Y)$ and satisfies*
$(\forall K\in\mathcal{K}(X))(\forall L\in\mathcal{K}(Y)(\exists C>0)(\forall\varphi\in\mathrm{C}^\infty_K(X))(\forall\psi\in\mathrm{C}^\infty_L(Y))$

$$|\langle u,\varphi\boxtimes\psi\rangle|\leq Cp^l_K(\varphi)p^m_L(\psi),$$

then u can be uniquely extended to a continuous functional on $\mathrm{C}^{l,m}_c(X\times Y)$ (i.e. it can be considered as an element of $\mathcal{D}'_{l,m}(X\times Y)$). ∎

If the conjecture were true, then the H-distribution μ from the preceeding theorem belongs to the space $\mathcal{D}'_{0,\kappa}(\mathbf{R}^d\times\mathrm{S}^{d-1})$, i.e. it is a distribution of order 0 in $\mathbf{x}$ and of order not more than κ in $\boldsymbol{\xi}$.

Indeed, we already have $\mu\in\mathcal{D}'(\mathbf{R}^d\times\mathrm{S}^{d-1})$ and the following bound with $\varphi:=\varphi_1\overline{\varphi_2}$:

$$|\langle\mu,\varphi\boxtimes\psi\rangle|\leq C\|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}\|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)}\,,$$

where C does not depend on φ and ψ.

Some properties and $\mathrm{L}^p-\mathrm{L}^q$ variant of compactness by compensation

Strong convergence, concentrations and defect measures

Lemma. *For a sequence (u_n) in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, $p\in\langle 1,\infty\rangle$, the following are equivalent*

- *$u_n\to 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$,*
- *for every sequence (v_n) satisfing conditions of the existence theorem, (u_n) and (v_n) form a pure pair and the corresponding H-distribution is zero.*

∎

Concentration example: Take $p\in\langle 1,\infty\rangle$. For $u\in\mathrm{L}^p_c(\mathbf{R}^d)$, define a sequence $u_n(\mathbf{x}) = n^{\frac{d}{p}}u(n(\mathbf{x}-\mathbf{z}))$ for some $\mathbf{z}\in\mathbf{R}^d$. A simple change of variables shows that $\|u_n\|_{\mathrm{L}^p(\mathbf{R}^d)} = \|u\|_{\mathrm{L}^p(\mathbf{R}^d)}$ and that it weakly converges to 0 in $\mathrm{L}^p(\mathbf{R}^d)$.

The H-distribution corresponding to the whole sequences (u_n) and $(|u_n|^{p-2}u_n)$ is given by $\delta_{\mathbf{z}}\boxtimes\nu$, where ν is a distribution on $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ defined for $\psi\in\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by

$$\langle\nu,\psi\rangle = \int_{\mathbf{R}^d}u(\mathbf{x})\overline{\mathcal{A}_{\bar\psi}(|u|^{p-2}u)(\mathbf{x})}d\mathbf{x}.$$

Connection with defect measures: Let (u_n) be a sequence weakly converging to 0 in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$. Then the sequence $(|u_n|^p)$ is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$, so $|u_n|^p\xrightarrow{*}\nu$ in $\mathcal{D}'(\mathbf{R}^d)$ (after passing to a subsequence).
Since all terms of $(|u_n|^p)$ are non-negative (in terms of distributions), the limit ν is a non-negative distributions, hence (unbounded) Radon measure.

Let μ be any H-distribution corresponding to the above chosen subsequence of (u_n) and $(\Phi_p(u_n))$. Taking ψ to be equal to one and test functions φ_1, φ_2 such that φ_2 is equal to one on the support of φ_1, we get the following connection between μ and ν:

$$\langle\mu,\varphi_1\boxtimes 1\rangle = \lim_n\int_{\mathbf{R}^d}\varphi_1|u_n|^p d\mathbf{x} = \langle\nu,\varphi_1\rangle.$$

Compactness by compensation result

Introduce the set

$$\Lambda_{\mathcal{D}} = \Big\{\boldsymbol{\mu}\in\ \mathrm{L}^{\bar{s}}(\mathbf{R}^d;(\mathrm{C}^d(\mathrm{P}))')^r : \Big(\sum_{k=1}^n(2\pi i\xi_k)^{\alpha_k}\mathbf{A}^k\Big)\boldsymbol{\mu} = \mathbf{0}_m\Big\},$$

where the given equality is understood in the sense of $\mathrm{L}^{\bar{s}}(\mathbf{R}^d;(\mathrm{C}^d(\mathrm{P}))')^m$.

Let us assume that coefficients of the bilinear form $q(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\eta}) = \mathbf{Q}(\mathbf{x})\boldsymbol{\lambda}\cdot\boldsymbol{\eta}$ on $\mathbf{C}^r$ belong to space $\mathrm{L}^t_{\mathrm{loc}}(\mathbf{R}^d)$, where $1/t+1/p+1/q<1$.

Definition. We say that set $\Lambda_{\mathcal{D}}$, bilinear form q and matrix $\boldsymbol{\mu} = [\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_r]$, $\boldsymbol{\mu}_j\in\mathrm{L}^{\bar{s}}(\mathbf{R}^d;(\mathrm{C}^d(\mathrm{P}))')^r$ satisfy the strong consistency condition if $(\forall j\in\{1,\ldots,r\})$ $\boldsymbol{\mu}_j\in\Lambda_{\mathcal{D}}$, and it holds

$$\langle\phi\mathbf{Q}\otimes 1,\boldsymbol{\mu}\rangle\geq\mathbf{0},\quad \phi\in\mathrm{L}^{\bar{s}}(\mathbf{R}^d;\mathbf{R}^+_0).$$

Theorem. *Assume that sequences $(\mathbf{u}_n)$ and $(\mathbf{v}_n)$ are bounded in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)$ and $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^r)$, respectively, and converge toward $\mathbf{u}$ and $\mathbf{v}$ in the sense of distributions. Assume that (1) holds and that*

$$q(\mathbf{x};\mathbf{u}_n,\mathbf{v}_n)\rightharpoonup\omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d).$$

If the set $\Lambda_{\mathcal{D}}$, the bilinear form q, and matrix H-distribution $\boldsymbol{\mu}$, corresponding to subsequences of $(\mathbf{u}_n-\mathbf{u})$ and $(\mathbf{v}_n-\mathbf{v})$, satisfy the strong consistency condition, then

$$q(\mathbf{x};\mathbf{u},\mathbf{v})\leq\omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d).$$

∎

A variant of H-distributions

We need multiplier operators with symbols defined on a manifold P determined by an d-tuple $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d)\in\mathbf{R}^d_+$ where $\alpha_k\in\mathbf{N}$ or $\alpha_k\geq d$:

$$\mathrm{P} = \Big\{\boldsymbol{\xi}\in\mathbf{R}^d : \sum_{k=1}^d|\xi_k|^{2\alpha_k} = 1\Big\},$$

In order to associate an L^p Fourier multiplier to a function defined on P, we extend it to $\mathbf{R}^d\setminus\{0\}$ by means of the projection π_{P}.
We need the following variant of H-distributions.

Theorem. *Let (u_n) be a bounded sequence in $\mathrm{L}^p(\mathbf{R}^d)$, $p>1$, and let (v_n) be a bounded sequence of uniformly compactly supported functions in $\mathrm{L}^q(\mathbf{R}^d)$, $1/q+1/p<1$, weakly converging to 0 in the sense of distributions. Then, after passing to a subsequence (not relabelled), for any $\bar{s}\in(1,\frac{pq}{p+q})$ there exists a continuous bilinear functional B on $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d)\otimes\mathrm{C}^d(\mathrm{P})$ such that for every $\varphi\in\mathrm{L}^{\bar{s}'}(\mathbf{R}^d)$ and $\psi\in\mathrm{C}^d(\mathrm{P})$, it holds*

$$B(\varphi,\psi) = \lim_{n\to\infty}\int_{\mathbf{R}^d}\varphi(\mathbf{x})u_n(\mathbf{x})\big(\mathcal{A}_{\psi_{\mathrm{P}}}v_n\big)(\mathbf{x})d\mathbf{x}\,,$$

*where $\mathcal{A}_{\psi_{\mathrm{P}}}$ is the Fourier multiplier operator on $\mathbf{R}^d$ associated to $\psi\circ\pi_{\mathrm{P}}$.
The bilinear functional B can be continuously extended as a linear functional on $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d;\mathrm{C}^d(\mathrm{P}))$.* ∎

Localisation principle

For $\alpha\in\mathbf{R}^+$, we define $\partial^\alpha_{x_k}$ to be a pseudodifferential operator with a polyhomogeneous symbol $(2\pi i\xi_k)^\alpha$, i.e.

$$\partial^\alpha_{x_k}u = ((2\pi i\xi_k)^\alpha\hat{u}(\boldsymbol{\xi}))^\vee.$$

In the sequel, we shall assume that sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ are uniformly compactly supported. This assumption can be removed if the orders of derivatives $(\alpha_1,\ldots,\alpha_d)$ are natural numbers.

Lemma. *Assume that sequences $(\mathbf{u}_n)$ and $(\mathbf{v}_n)$ are bounded in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)$ and $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^r)$, respectively, and converge toward $\mathbf{0}$ and $\mathbf{v}$ in the sense of distributions. Furthermore, assume that sequence $(\mathbf{u}_n)$ satisfies:*

$$\mathbf{G}_n := \sum_{k=1}^d\partial^{\alpha_k}_k(\mathbf{A}^k\mathbf{u}_n)\ \to\mathbf{0} \text{ in } \mathrm{W}^{-\alpha_1,\ldots,-\alpha_d,p}(\Omega;\mathbf{R}^m), \tag{1}$$

*where either $\alpha_k\in\mathbf{N}$, $k=1,\ldots,d$ or $\alpha_k>d$, $k=1,\ldots,d$, and elements of matrices $\mathbf{A}^k$ belong to $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d)$, $\bar{s}\in(1,\frac{pq}{p+q})$.
Finally, by $\boldsymbol{\mu}$ denote a matrix H-distribution corresponding to subsequences of $(\mathbf{u}_n)$ and $(\mathbf{v}_n-\mathbf{v})$. Then the following relation holds*

$$\Big(\sum_{k=1}^d(2\pi i\xi_k)^{\alpha_k}\mathbf{A}^k\Big)\boldsymbol{\mu} = \mathbf{0}.$$

∎

Application

Now, let us consider the following non-linear parabolic type equation

$$L(u) = \partial_t u - \mathrm{div\,div}\,(g(t,\mathbf{x},u)\mathbf{A}(t,\mathbf{x})),$$

on $(0,\infty)\times\Omega$, where Ω is an open subset of $\mathbf{R}^d$. We assume that

$$u\in\mathrm{L}^p((0,\infty)\times\Omega),\ \ g(t,\mathbf{x},u(t,\mathbf{x}))\in\mathrm{L}^q((0,\infty)\times\Omega),\ \ 1<p,q,$$
$$\mathbf{A}\in\mathrm{L}^s_{\mathrm{loc}}((0,\infty)\times\Omega)^{d\times d},\ \ \text{where } 1/p+1/q+1/s<1,$$

and that the matrix $\mathbf{A}$ is strictly positive definite, i.e.

$$\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi}>0,\ \ \boldsymbol{\xi}\in\mathbf{R}^d\setminus\{\mathbf{0}\},\ \ (a.e.(t,\mathbf{x})\in(0,\infty)\times\Omega).$$

Furthermore, assume that g is a Carathèodory function and non-decreasing with respect to the third variable.

Theorem. *Assume that sequences*

- *(u_r) and $g(\cdot,u_r)$ are such that $u_r, g(u_r)\in\mathrm{L}^2(\mathbf{R}^+\times\mathbf{R}^d)$ for every $r\in\mathbf{N}$;*
- *that they are bounded in $\mathrm{L}^p(\mathbf{R}^+\times\mathbf{R}^d)$, $p\in(1,2]$, and $\mathrm{L}^q(\mathbf{R}^+\times\mathbf{R}^d)$, $q>2$, respectively, where $1/p+1/q<1$;*
- *$u_r\rightharpoonup u$ and, for some, $f\in\mathrm{W}^{-1,-2;p}(\mathbf{R}^+\times\mathbf{R}^d)$, the sequence*

$$L(u_r) = f_r\to f \quad \text{strongly in } \mathrm{W}^{-1,-2;p}(\mathbf{R}^+\times\mathbf{R}^d).$$

Under the assumptions given above, it holds

$$L(u) = f \ \text{ in } \ \mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^d).$$

∎

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