On the Dirichlet-Neumann boundary problem for scalar conservation laws

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Problem statement

- \( \Omega \subseteq [0, \infty) \times \mathbb{R} \) open bounded domain
- boundary \( \partial \Omega = \Gamma_N \cup \Gamma_D \) of class \( C^{0,1} \), where \( \Gamma_D \subset \{ t = 0 \} \)
- consider the following mixed boundary problem:

\[
\begin{align*}
\frac{\partial t}{\partial t} u + \frac{\partial x}{\partial x} (f(t, x, u)) &= 0 \text{ in } \Omega \\
\nabla_{(t,x)} u \cdot \nu &= 0 \text{ on } \Gamma_N \\
u(0, .) &= u^0(.) \in L^\infty(\mathbb{R}) \text{ on } \Gamma_D,
\end{align*}
\]

- \( f(t, x, \lambda) \) is a Caratheodory type function i.e. it is of bounded variation with respect to the variables \((t, x)\) and differentiable with respect to the third variable \( \lambda \).
An example of domain $\Omega \subseteq [0, \infty) \times \mathbb{R}$
Additional assumptions on $f$

Take $p \in \langle 2, \infty \rangle$ fixed.

**A1:**  \((\forall \Lambda \subset \mathbb{R} \text{ compact})(\forall K \subset \Omega \text{ compact})(\exists C_1 = C_1(K, \Lambda) > 0)(\forall \xi \in \Lambda)\)

$$\left\| \chi_K \int_0^{\xi} f(t, x, \lambda) d\lambda \right\|_{L^p(\Omega)} < C_1,$$

**A2:**  \((\forall \Lambda \subset \mathbb{R} \text{ compact})(\forall K \subset \Omega \text{ compact})(\exists C_2 = C_2(K, \Lambda) > 0)(\forall \xi \in \Lambda)\)

$$\left\| \chi_K \int_0^{\xi} f'_x(t, x, \lambda) d\lambda \right\|_{L^1(\Omega)} < C_2,$$

**A3:**  \((\forall \Lambda \subset \mathbb{R} \text{ compact})(\forall K \subset \Omega \text{ compact})(\exists C_3 = C_3(K, \Lambda) > 0)(\forall \lambda \in \Lambda)\)

$$\left\| \chi_K f(t, x, \lambda) \right\|_{L^p(\Omega)} < C_3.$$
Assumptions A1 and A3, due to the boundedness of $\Omega$, imply that for every $\Lambda \subset \mathbb{R}$ compact and every $\varphi \in C_c(\Omega)$, the following holds for positive constants $C_{1,p,K,\Lambda}$ and $C_{3,p,K,\Lambda}$ with $K = \text{supp} \varphi$:

C1: $(\forall \xi \in \Lambda) \quad \left\| \varphi(t, x) \int_{0}^{\xi} f(t, x, \lambda) d\lambda \right\|_{L^1(\Omega)} < C_{1,p,K,\Lambda} \| \varphi \|_{L^\infty(\Omega)}$

C3: $(\forall \lambda \in \Lambda) \quad \left\| \varphi(t, x) f(t, x, \lambda) \right\|_{L^1(\Omega)} < C_{3,p,K,\Lambda} \| \varphi \|_{L^\infty(\Omega)}$.
Approximation\(^1\) of the problem

\[
\partial_t u_n + \partial_x (f_n(t, x, u_n)) = \frac{1}{n} \triangle_{(t, x)} u_n \text{ in } \Omega
\]

\[
\nabla_{(t, x)} u_n \cdot \nu = 0 \text{ on } \Gamma_N
\]

\[
u_n(0, \cdot) = u_n^0(\cdot) \text{ on } \Gamma_D,
\]

\( f_n(t, x, \lambda) = f(\cdot, \cdot, \lambda) \star n^2 \omega(nt, nx) \) is a regularization of the flux \( f \) via the standard non-negative mollifier \( \omega \in C^\infty_c((−1, 1)^2) \),

\( (u_n^0) \) is a bounded sequence of functions converging strongly in \( L^1_{loc}(\mathbb{R}) \) toward \( u_0 \).

Problem: what is the appropriate solution concept?

Concept of solution

Multiplying equation

\[ \partial_t u_n + \partial_x (f_n(t, x, u_n)) = (1/n) \triangle (t, x) u_n \]

by \( \text{sgn}(u_n(t, x) - \lambda) \), we get:

\[ \partial_t |u_n - \lambda| + \partial_x (\text{sgn}(u_n - \lambda)(f_n(u_n) - f_n(\lambda))) \leq \]

\[ \leq \frac{1}{n} \triangle (t, x) |u_n - \lambda| - \text{sgn}(u_n - \lambda) f'_{n,x}(t, x, \lambda) \text{ in } \Omega. \]

Multiply by \( \varphi \in C^2(\Omega) \) supported away from \( \{t = 0\} \) and integrate over \( \Omega \). After taking into account (2), we get:

\[ - \int_{\Omega} (|u_n - \lambda| \partial_t \varphi + \text{sgn}(u_n - \lambda)(f_n(u_n) - f_n(\lambda)) \partial_x \varphi) \, dx \, dt + \]

\[ + \int_{\partial \Omega} \left( |u_n - \lambda|, \text{sgn}(u_n - \lambda)(f_n(u_n) - f_n(\lambda)) \right) \cdot \nu \, \varphi \, ds \leq \]

\[ \leq \frac{1}{n} \int_{\Omega} \nabla (t, x) |u_n - \lambda| \cdot \nabla (t, x) \varphi \, dx \, dt - \int_{\Omega} \varphi \text{sgn}(u_n - \lambda) f'_{n,x}(t, x, \lambda) \, d\lambda \, dx \, dt. \]
Concept of solution - continued

Using the idea from the recent article by Andreianov & Mitrović\(^2\), we introduce the following definition:

**Definition**

The function \( u \in L^2(\Omega) \) is called a solution to (1), (2), (3) if there exists a function \( p \in L^1(\Gamma_N) \) such that for every \( \varphi \in C_c(\Omega \setminus \Gamma_D) \) the following holds:

- \[
\int_{\Omega} (|u - \lambda| \partial_t \varphi + \text{sgn}(u - \lambda)(f(t, x, u) - f(t, x, \lambda)) \partial_x \varphi) \, dx dt - \int_{\partial\Omega} \left( |p - \lambda|, \text{sgn}(p - \lambda)(f(t, x, p) - f(t, x, \lambda)) \right) \cdot \nu \varphi \, ds \geq \int_{\Omega} \varphi \text{sgn}(u - \lambda) f'_x(t, x, \lambda) \, d\lambda \, dx dt. \] (6)

- Initial data are satisfied in the strong sense i.e. for almost every \( x \in \Gamma_D \) it holds

\[
\lim_{t \to 0} |u(t, x) - u_0(x)| = 0.
\]

The main result

**Theorem**
Assume that the sequence \((u_n)\) of solutions to (4) is uniformly bounded by a constant \(M\). If the flux \(f\) satisfies the assumptions A1, A2 and A3, then the weak \(L^2(\Omega)\)-limit of \((u_n)\) along a subsequence satisfies the equation (1) in \(\Omega\).

**Outline (of the proof):**

- \[
\partial_t u_n + \partial_x (f(t, x, u_n)) \to 0 \quad \text{in } H^{-1}_{loc}(\Omega)
\]

- for all entropy-entropy flux pairs \((\Phi(\lambda), \Psi_n(t, x, \lambda))\):
  \[
  \partial_t(\Phi(u_n)) + \partial_x(\Psi_n(t, x, u_n)) \text{ is precompact in } H^{-1}_{loc}(\Omega)
  \]

- for all \(k \in \mathbb{R}\):
  \[
  \partial_t|u_n - k| + \partial_x(\text{sgn}(u_n - k)(f(t, x, u_n) - f(t, x, k))) \text{ is precompact in } H^{-1}_{loc}(\Omega)
  \]
Case when $f \in \mathcal{C}^1$

A corollary of the proof of the theorem and Panov’s result\(^3\) in the case when the flux is continuously differentiable with respect to all variables is the fact that the limiting function $u$ satisfies the Kruzhkov admissibility conditions. However, we do not have a working solution concept for (1), (3), (2) so we cannot say anything about uniqueness.

**Corollary**

Assume that the flux $f \in \mathcal{C}^1(\Omega \times (-M, M))$. The distributional limit $u$ of the sequence $(u_n)$ of solutions to (4) satisfies for every entropy-entropy flux pair $(\Phi, \Psi)$

$$
\partial_t(\Phi(u)) + \partial_x(\Psi(t, x, u)) \leq -\int_0^u f_x'(t, x, \lambda) \Phi''(\lambda) d\lambda \quad \text{in} \quad \mathcal{D}'(\Omega).
$$

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Lighthill-Whitham-Richards model for traffic flow

\[ \partial_t \rho + \partial_x (\rho v(\rho)) = 0, \]

where the velocity is assumed to have linear dependence upon density of the cars

\[ v(\rho) = v_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right), \quad 0 \leq \rho \leq \rho_{\text{max}}. \]

Let \( L \) and \( \tau \) be a typical length and time, respectively, such that \( v_{\text{max}} = L/\tau \). Introducing new variables

\[ \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{x}{L}, \quad u = 1 - \frac{2\rho}{\rho_{\text{max}}}, \]

we obtain the inviscid Burgers equation

\[ \partial_t \rho + \partial_x \left[ \rho \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) \right] = -\frac{\rho_{\text{max}}}{2\tau} \partial_{\bar{t}} u - \frac{\rho_{\text{max}}}{2\tau} \partial_{\bar{x}} \left( \frac{u^2}{2} \right) = 0. \]
Examples

Let \( \Omega = \{(t, x) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq t \leq -4x(x - 1)\} \).
We focus on solving the (regularized) Burgers equation

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = \epsilon \Delta_{(t,x)} u \quad \text{in } \Omega,
\]
\[
\nabla_{(t,x)} u \cdot \nu = 0 \quad \text{on } \Gamma_N,
\]
\[
u(0, x) = u_D \quad \text{on } \Gamma_D,
\]

where \( \Gamma_D = \{(t, x) \in \partial \Omega : t = 0\} \) and \( \Gamma_N = \partial \Omega \setminus \Gamma_D \).
Let \( V_D(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = u_D\} \) and \( H^1_D(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\} \).

We use the following numerical scheme:

For given initial guess \( u_0 \), construct sequence \( u_n \in V_D, n \geq 1 \), that are solutions of

\[
\int_{\Omega} \left( \frac{\partial u_n}{\partial t} + u_{n-1} \frac{\partial}{\partial x} u_n \right) \psi dt dx + \epsilon \int_{\Omega} \nabla_{(t,x)} u_n \cdot \nabla_{(t,x)} \psi dt dx = 0, \quad \forall \psi \in H^1_D(\Omega).
\]

(7)
Two scenarios: in the first one $\epsilon = 1/N$ and in the second one $\epsilon = 1/N^2$ with $u_D = -2x(x-1)$ in both.

We performed two convergence tests, where referent solution $u_R$ has been computed on $N \times N = 640^2$ grid.

<table>
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<th>$N = 1/\epsilon$</th>
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<th>$N = 1/\sqrt{\epsilon}$</th>
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Example 1 - $N = 160$ and $\epsilon = 1/160^2$
Example 1 - $N = 160$ and $\epsilon = 1/160^2$, iso-values of the solution
$u_D = H(0.5 - x)$, where $H$ is Heaviside function
Example 3

$u_D = H(x - 0.5)$, where $H$ is Heaviside function