Non-stationary Friedrichs systems

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Joint work with Marko Erceg

WeConMApp

Motivation

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- unified treatment of equations and systems of different type.

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$$\mathcal{L}\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

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Boundary conditions and the trace operator

Weak solution of a Friedrichs system belongs only to the *graph space*

$$W := \{\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) : \mathcal{L}\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)\},$$

which is a Hilbert space with $\langle \cdot \mid \cdot \rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} + \langle \mathcal{L}\cdot \mid \mathcal{L}\cdot \rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}$.

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$$\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^{d} \nu_k \mathbf{A}_k \in \mathrm{L}^\infty(\partial\Omega; \mathrm{M}_r(\mathbf{R})) ,$$

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Then

$$\mathsf{u} \mapsto \mathbf{A}_{\boldsymbol{\nu}}\mathsf{u}$$

defines the trace operator $W \longrightarrow \mathrm{H}^{-\frac{1}{2}}(\partial\Omega; \mathbf{R}^r)$.

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Contribution: K. O. Friedrichs, C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...

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Numerics:

A. Ern, J.-L. Guermond, SIAM JNA, 2006, 2006, 2008

T. Bui-Thanh, L. Demkowicz, O. Ghattas, SIAM JNA, 2013

Assumptions

L - a real Hilbert space ($L' \equiv L$), $\mathcal{D} \subseteq L$ a dense subspace, and $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \longrightarrow L$ linear unbounded operators satisfying

(T1) $$(\forall\, \varphi, \psi \in \mathcal{D}) \quad \langle\, \mathcal{L}\varphi \mid \psi \,\rangle_L = \langle\, \varphi \mid \tilde{\mathcal{L}}\psi \,\rangle_L\,;$$

(T2) $$(\exists\, c > 0)(\forall\, \varphi \in \mathcal{D}) \quad \|(\mathcal{L} + \tilde{\mathcal{L}})\varphi\|_L \leq c\|\varphi\|_L\,;$$

(T3) $$(\exists\, \mu_0 > 0)(\forall\, \varphi \in \mathcal{D}) \quad \langle\, (\mathcal{L} + \tilde{\mathcal{L}})\varphi \mid \varphi \,\rangle_L \geq 2\mu_0\|\varphi\|_L^2\,.$$

An example: The classical Friedrichs operator

Let $\mathcal{D} := \mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)$, $L = \mathrm{L}^2(\Omega; \mathbf{R}^r)$ and $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \longrightarrow L$ be defined by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}\,,$$

$$\tilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k^\top \mathsf{u}) + (\mathbf{C}^\top + \sum_{k=1}^{d} \partial_k \mathbf{A}_k^\top)\mathsf{u}\,,$$

where $\mathbf{A}_k$ and $\mathbf{C}$ are as before (they satisfy (F1)–(F2)).

Then $\mathcal{L}$ and $\tilde{\mathcal{L}}$ satisfy (T1)–(T3)

Extensions

$(\mathcal{D}, \langle \cdot \mid \cdot \rangle_{\mathcal{L}})$ is an inner product space, with

$$\langle \cdot \mid \cdot \rangle_{\mathcal{L}} := \langle \cdot \mid \cdot \rangle_L + \langle \mathcal{L} \cdot \mid \mathcal{L} \cdot \rangle_L \, ;$$

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Further extensions ... $\mathcal{L} := \tilde{\mathcal{L}}^*$, $\tilde{\mathcal{L}} := \mathcal{L}^*$, ... $\mathcal{L}, \tilde{\mathcal{L}} \in \mathcal{L}(L, W_0')$ ... (T)

Posing the problem

Lemma

The graph space

$$W := \{u \in L : \mathcal{L}u \in L\} = \{u \in L : \tilde{\mathcal{L}}u \in L\}$$

is a Hilbert space with respect to $\langle \cdot \mid \cdot \rangle_{\mathcal{L}}$.

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Problem: for given $f \in L$ find $u \in W$ such that $\mathcal{L}u = f$.

Find sufficient conditions on $V \leqslant W$ such that $\mathcal{L}_{|_V} : V \longrightarrow L$ is an isomorphism.

Boundary operator

Boundary operator $D \in \mathcal{L}(W, W')$:

$$ {}_{W'}\langle\, Du, v \,\rangle_W := \langle\, \mathcal{L}u \mid v \,\rangle_L - \langle\, u \mid \tilde{\mathcal{L}}v \,\rangle_L \,, \qquad u, v \in W \,. $$

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If $\mathcal{L}$ is the classical Friedrichs operator, then for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$:

$$_{W'}\langle\, D\mathsf{u}, \mathsf{v} \,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) \,.$$

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Let V and $\tilde{V}$ be subspaces of W that satisfy

(V1)
$$ \begin{aligned} &(\forall\, u \in V) \qquad {}_{W'}\langle\, Du, u\,\rangle_W \geq 0\,, \\ &(\forall\, v \in \tilde{V}) \qquad {}_{W'}\langle\, Dv, v\,\rangle_W \leq 0\,, \end{aligned} $$

(V2)
$$ V = D(\tilde{V})^0\,, \qquad \tilde{V} = D(V)^0\,. $$

Well-posedness theorem

Theorem

Under assumptions (T1)–(T3) and (V1)–(V2), the operators $\mathcal{L}_{|_V} : V \longrightarrow L$ *and* $\tilde{\mathcal{L}}_{|_{\tilde{V}}} : \tilde{V} \longrightarrow L$ *are isomorphisms.*

Theorem

(**Banach–Nečas–Babuška**) *Let* V *and* L *be two Banach paces,* L' *dual of* L *and* $\mathcal{L} \in \mathcal{L}(V; L)$. *Then the following statements are equivalent:*
a) $\mathcal{L}$ *is a bijection;*
b) It holds:

$$(\exists\, \alpha > 0)(\forall\, u \in V) \qquad \|\mathcal{L}u\|_L \geq \alpha \|u\|_V \,;$$

$$(\forall\, v \in L') \qquad \Big((\forall\, u \in V) \quad {}_{L'}\langle\, v, \mathcal{L}u \,\rangle_L = 0 \Big) \quad \Longrightarrow \quad v = 0 \,.$$

An example – stationary diffusion equation

N. Antonić, K. B., M. Vrdoljak, NA-RWA, 2014

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New unknown vector function taking values in $\mathbf{R}^{d+1}$:

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Then the starting equation can be written as a first-order system

$$\begin{cases} \nabla u^u + \mathbf{A}^{-1}\mathsf{u}^{\boldsymbol{\sigma}} = 0 \\ \mathrm{div}\,\mathsf{u}^{\boldsymbol{\sigma}} + cu^u = f \end{cases} ,$$

An example – stationary diffusion equation (cont.)

which is a Friedrichs system with the choice of

$$\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R})\,, \qquad \mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ 0 & c \end{bmatrix}.$$

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Dirichlet, Neumann and Robin boundary conditions are imposed by the following choice of V and $\widetilde{V}$:

$$\begin{aligned} V_D = \widetilde{V}_D &:= \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)\,, \\ V_N = \widetilde{V}_N &:= \{(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}^{\boldsymbol{\sigma}} = 0\}\,, \\ V_R &:= \{(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}^{\boldsymbol{\sigma}} = a {u^u}_{|\Gamma}\}\,, \\ \widetilde{V}_R &:= \{(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}^{\boldsymbol{\sigma}} = -a {u^u}_{|\Gamma}\}\,. \end{aligned}$$

An example – heat equation

N. Antonić, K. B., M. Vrdoljak, JMAA, 2013

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Dirichlet boundary condition and zero initial condition:

$$\begin{aligned} V &= \Big\{ \mathsf{u} \in W : u^u \in \mathrm{L}^2(0, T; \mathrm{H}^1_0(\Omega)), \quad u^u(\cdot, 0) = 0 \text{ a.e. on } \Omega \Big\}, \\ \widetilde{V} &= \Big\{ \mathsf{v} \in W : v^u \in \mathrm{L}^2(0, T; \mathrm{H}^1_0(\Omega)), \quad v^u(\cdot, T) = 0 \text{ a.e. on } \Omega \Big\}. \end{aligned}$$

Homogenisation theory for (classical) Friedrichs systems

K. B., M. Vrdoljak, CPAA, 2014

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- notions of H- and G-convergence

Homogenisation theory for (classical) Friedrichs systems

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- notions of H- and G-convergence
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Homogenisation theory for (classical) Friedrichs systems

K. B., M. Vrdoljak, CPAA, 2014

- notions of H- and G-convergence
- compactness result
- contains homogenisation theory for the stationary diffusion equation and the heat equation

Non-stationary problem

L - real Hilbert space, as before $(L' \equiv L)$, $T > 0$
We consider an abstract Cauchy problem in L:

(P) $$\begin{cases} \mathsf{u}'(t) + \mathcal{L}\mathsf{u}(t) = \mathsf{f}(t) \\ \mathsf{u}(0) = \mathsf{u}_0 \end{cases},$$

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where

- $\mathsf{f} : \langle 0, T\rangle \longrightarrow L$, $\mathsf{u}_0 \in L$ are given,
- $\mathcal{L}$ (not depending on t) satisfies (T1), (T2) and

  (T3′) $$(\forall \varphi \in \mathcal{D}) \quad \langle\, (\mathcal{L} + \tilde{\mathcal{L}})\varphi \mid \varphi\,\rangle_L \geq 0\,,$$

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Numerics:

E. Burman, A. Ern, M. A. Fernandez, SIAM JNA, 2010

D. A. Di Pietro, A. Ern, 2012

Semigroup setting

A priori estimate:

$$(\forall\, t \in [0,T]) \qquad \|\mathsf{u}(t)\|_L^2 \leqslant e^t \left(\|\mathsf{u}_0\|_L^2 + \int_0^t \|\mathsf{f}(s)\|_L^2 \right).$$

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Let $\mathcal{A} : V \subseteq L \longrightarrow L$, $\mathcal{A} := -\mathcal{L}_{|_V}$

Then (P) becomes:

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Theorem

The operator $\mathcal{A}$ is an infinitensimal generator of a C_0-semigroup on L.

Existence and uniqueness result

Corollary

Let $\mathcal{L}$ be an operator that satisfies (T1)–(T2) and (T3)′, let V be a subspace of its graph space satisfying (V1)–(V2), and $\mathrm{f} \in \mathrm{L}^1(\langle 0, T\rangle; L)$.

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- *Then for every $\mathsf{u}_0 \in L$ the problem (P) has the unique weak solution $\mathsf{u} \in \mathrm{C}([0, T]; L)$ given with*

$$\mathsf{u}(t) = \mathcal{T}(t)\mathsf{u}_0 + \int_0^t \mathcal{T}(t-s)\mathsf{f}(s)ds\,, \qquad t \in [0, T]\,,$$

  where $(\mathcal{T}(t))_{t \geqslant 0}$ is the semigroup generated by $\mathcal{A}$.

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  where $(\mathcal{T}(t))_{t\geqslant 0}$ *is the semigroup generated by* $\mathcal{A}$.

- *If additionally* $\mathsf{f} \in \mathrm{C}([0, T]; L) \cap \Big(\mathrm{W}^{1,1}(\langle 0, T\rangle; L) \cup \mathrm{L}^1(\langle 0, T\rangle; V)\Big)$ *with V equipped with the graph norm and* $\mathsf{u}_0 \in V$, *then the above weak solution is the classical solution of (P) on* $[0, T\rangle$.

Weak solution

Theorem

Let $\mathsf{u}_0 \in L$, $\mathsf{f} \in \mathrm{L}^1(\langle 0, T\rangle; L)$ *and let*

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be the weak solution of (P).

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be the weak solution of (P).

Then $\mathsf{u}', \mathcal{L}\mathsf{u}, \mathsf{f} \in \mathrm{L}^1(\langle 0, T\rangle; W_0')$ *and*

$$\mathsf{u}' + \mathcal{L}\mathsf{u} = \mathsf{f}\,,$$

in $\mathrm{L}^1(\langle 0, T\rangle; W_0')$.

Bound on solution

From

$$\mathsf{u}(t) = \mathcal{T}(t)\mathsf{u}_0 + \int_0^t \mathcal{T}(t-s)\mathsf{f}(s)ds\,, \qquad t \in [0,T]\,,$$

we get:

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A priori estimate was:

$$(\forall\, t \in [0,T]) \qquad \|\mathsf{u}(t)\|_L^2 \leqslant e^t\left(\|\mathsf{u}_0\|_L^2 + \int_0^t \|\mathsf{f}(s)\|_L^2\right).$$

Non-stationary Maxwell system 1/5

Let $\Omega \subseteq \mathbf{R}^3$ be open and bounded with a Lipshitz boundary Γ, $\mu, \varepsilon \in \mathrm{W}^{1,\infty}(\Omega)$ positive and *away from zero*, $\mathbf{\Sigma}_{ij} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_3(\mathbf{R}))$, $i, j \in \{1, 2\}$, and $\mathsf{f}_1, \mathsf{f}_2 \in \mathrm{L}^1(\langle 0, T\rangle; \mathrm{L}^2(\Omega; \mathbf{R}^3))$.

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We consider a generalized non-stationary Maxwell system

$$(\mathrm{MS}) \qquad \begin{cases} \mu\partial_t \mathsf{H} + \operatorname{rot} \mathsf{E} + \mathbf{\Sigma}_{11}\mathsf{H} + \mathbf{\Sigma}_{12}\mathsf{E} = \mathsf{f}_1 \\ \varepsilon\partial_t \mathsf{E} - \operatorname{rot} \mathsf{H} + \mathbf{\Sigma}_{21}\mathsf{H} + \mathbf{\Sigma}_{22}\mathsf{E} = \mathsf{f}_2 \end{cases} \quad \text{in } \langle 0, T\rangle \times \Omega\,,$$

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Change of variable

$$\mathsf{u} := \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\mu}\mathsf{H} \\ \sqrt{\varepsilon}\mathsf{E} \end{bmatrix}, \quad c := \frac{1}{\sqrt{\mu\varepsilon}} \in \mathrm{W}^{1,\infty}(\Omega)\,,$$

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turns (MS) to the Friedrichs system

$$\partial_t \mathsf{u} + \mathcal{L}\mathsf{u} = \mathsf{F}\,,$$

Non-stationary Maxwell system 2/5

with

$$\mathbf{A}_1 := c\begin{bmatrix} & \mathbf{0} & & 0 & 0 & 0 \\ & & & 0 & 0 & -1 \\ & & & 0 & 1 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 1 & & \mathbf{0} & \\ 0 & -1 & 0 & & & \end{bmatrix}, \qquad \mathbf{A}_2 := c\begin{bmatrix} & \mathbf{0} & & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ & & & -1 & 0 & 0 \\ 0 & 0 & -1 & & & \\ 0 & 0 & 0 & & \mathbf{0} & \\ 1 & 0 & 0 & & & \end{bmatrix},$$

$$\mathbf{A}_3 := c\begin{bmatrix} & & & 0 & -1 & 0 \\ & \mathbf{0} & & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & \mathbf{0} & \\ 0 & 0 & 0 & & & \end{bmatrix}, \qquad \mathsf{F} = \begin{bmatrix} \frac{1}{\sqrt{\mu}}\mathsf{f}_1 \\ \frac{1}{\sqrt{\varepsilon}}\mathsf{f}_2 \end{bmatrix}, \qquad \mathbf{C} := \ldots .$$

Non-stationary Maxwell system 2/5

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(F1) and (F2) are satisfied (with change $\mathsf{v} := e^{-\lambda t}\mathsf{u}$ for large $\lambda > 0$, if needed)

Non-stationary Maxwell system 3/5

The spaces involved:

$$\begin{aligned} L &= \mathrm{L}^2(\Omega;\mathbf{R}^3)\times \mathrm{L}^2(\Omega;\mathbf{R}^3)\,, \\ W &= \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)\times \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)\,, \\ W_0 &= \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3)\times \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3) = \mathsf{Cl}_W\,\mathrm{C}^\infty_c(\Omega;\mathbf{R}^6)\,, \end{aligned}$$

where $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)$ is the graph space of the rot operator.

Non-stationary Maxwell system 3/5

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The boundary condition

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\boldsymbol{\nu}\times\mathsf{E}_{|_\Gamma}=0
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corresponds to the following choice of spaces $V,\widetilde{V}\subseteq W$:

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$$
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&= \{\mathsf{u} \in W : \boldsymbol{\nu} \times \mathsf{E} = 0\}\\
&= \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3) \times \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3)\,.
\end{aligned}
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Non-stationary Maxwell system 4/5

Theorem

Let $\mathsf{E}_0 \in \mathrm{L}^2_{\mathrm{rot},0}(\Omega; \mathbf{R}^3), \mathsf{H}_0 \in \mathrm{L}^2_{\mathrm{rot}}(\Omega; \mathbf{R}^3)$ *and let* $\mathsf{f}_1, \mathsf{f}_2 \in \mathrm{C}([0,T]; \mathrm{L}^2(\Omega; \mathbf{R}^3))$ *satisfy at least one of the following conditions:*

- $\mathsf{f}_1, \mathsf{f}_2 \in \mathrm{W}^{1,1}(\langle 0, T\rangle; \mathrm{L}^2(\Omega; \mathbf{R}^3))$;
- $\mathsf{f}_1 \in \mathrm{L}^1(\langle 0, T\rangle; \mathrm{L}^2_{\mathrm{rot}}(\Omega; \mathbf{R}^3))$, $\mathsf{f}_2 \in \mathrm{L}^1(\langle 0, T\rangle; \mathrm{L}^2_{\mathrm{rot},0}(\Omega; \mathbf{R}^3))$.

Non-stationary Maxwell system 4/5

Theorem

Let $\mathsf{E}_0 \in \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3), \mathsf{H}_0 \in \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)$ *and let* $\mathsf{f}_1, \mathsf{f}_2 \in \mathrm{C}([0,T];\mathrm{L}^2(\Omega;\mathbf{R}^3))$ *satisfy at least one of the following conditions:*

- $\mathsf{f}_1, \mathsf{f}_2 \in \mathrm{W}^{1,1}(\langle 0,T\rangle;\mathrm{L}^2(\Omega;\mathbf{R}^3))$;
- $\mathsf{f}_1 \in \mathrm{L}^1(\langle 0,T\rangle;\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3))$, $\mathsf{f}_2 \in \mathrm{L}^1(\langle 0,T\rangle;\mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3))$.

Then the abstract initial-boundary value problem

$$\begin{cases} \mu\mathsf{H}' + \mathsf{rot}\,\mathsf{E} + \mathbf{\Sigma}_{11}H + \mathbf{\Sigma}_{12}E = \mathsf{f}_1 \\ \varepsilon\mathsf{E}' - \mathsf{rot}\,\mathsf{H} + \mathbf{\Sigma}_{21}H + \mathbf{\Sigma}_{22}E = \mathsf{f}_2 \\ \mathsf{E}(0) = \mathsf{E}_0 \\ \mathsf{H}(0) = \mathsf{H}_0 \\ \boldsymbol{\nu}\times\mathsf{E}_{|_\Gamma} = 0 \end{cases},$$

Non-stationary Maxwell system 5/5

Theorem

...has unique classical solution given by

$$\begin{bmatrix} \mathsf{H} \\ \mathsf{E} \end{bmatrix}(t) = \begin{bmatrix} \frac{1}{\sqrt{\mu}}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{\varepsilon}}\mathbf{I} \end{bmatrix} \mathcal{T}(t) \begin{bmatrix} \sqrt{\mu}\mathsf{H}_0 \\ \sqrt{\varepsilon}\mathsf{E}_0 \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{\mu}}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{\varepsilon}}\mathbf{I} \end{bmatrix} \int_0^t \mathcal{T}(t-s) \begin{bmatrix} \frac{1}{\sqrt{\mu}}\mathsf{f}_1(s) \\ \frac{1}{\sqrt{\varepsilon}}\mathsf{f}_2(s) \end{bmatrix} ds\,, \quad t \in [0,T]\,,$$

where $(\mathcal{T}(t))_{t\geqslant 0}$ is the contraction C_0-semigroup generated by $-\mathcal{L}$.

Other examples

- Symmetric hyperbolic system

$$\begin{cases} \partial_t \mathsf{u} + \sum_{k=1}^{d} \partial_k (\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} = \mathsf{f} \quad \text{in } \langle 0, T\rangle \times \mathbf{R}^d \\ \mathsf{u}(0,\cdot) = \mathsf{u}_0 \end{cases},$$

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- Non-stationary div-grad problem

$$\begin{cases} \partial_t \mathsf{q} + \nabla p = \mathsf{f}_1 \quad \text{in } \langle 0, T\rangle \times \Omega\,, \quad \Omega \subseteq \mathbf{R}^d\,, \\ \frac{1}{c_0^2}\partial_t p + \operatorname{div} \mathsf{q} = f_2 \quad \text{in } \langle 0, T\rangle \times \Omega\,, \\ p_{|_{\partial\Omega}} = 0\,, \quad p(0) = p_0\,, \quad \mathsf{q}(0) = \mathsf{q}_0 \end{cases}$$

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- Wave equation

$$\begin{cases} \partial_{tt} u - c^2 \triangle u = f & \text{in } \langle 0, T\rangle \times \mathbf{R}^d \\ u(0,\cdot) = u_0\,, \quad \partial_t u(0,\cdot) = u_0^1 \end{cases}.$$

Friedrichs systems in a complex Hilbert space

Let L be a complex Hilbert space, $L' \equiv L$ its antidual, $\mathcal{D} \subseteq L$, $\mathcal{L}, \tilde{\mathcal{L}} : L \longrightarrow L$ linear operators that satisfy (T1)–(T3) (or T3' instead of T3).

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and again (F1)–(F2) imply (T1)–(T3).

Application to Dirac system 1/2

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where $\mathsf{u} : [0, T\rangle \times \mathbf{R}^3 \longrightarrow \mathbf{C}^4$ is an unknown function, $\mathsf{u}_0 : \mathbf{R}^3 \longrightarrow \mathbf{C}^4$, $\mathsf{f} : [0, T\rangle \times \mathbf{R}^3 \longrightarrow \mathbf{C}^4$ are given, and

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$$\mathbf{A}_k := \begin{bmatrix} \mathbf{0} & \boldsymbol{\sigma}_k \\ \boldsymbol{\sigma}_k & \mathbf{0} \end{bmatrix} , k \in 1..3 , \qquad \mathbf{C} := \begin{bmatrix} c_1\mathbf{I} & \mathbf{0} \\ \mathbf{0} & c_2\mathbf{I} \end{bmatrix} ,$$

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where

$$\boldsymbol{\sigma}_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

are Pauli matrices, and $c_1, c_2 \in \mathrm{L}^\infty(\mathbf{R}^3; \mathbf{C})$. . . . (F1)–(F2)

Application to Dirac system 2/2

Theorem

Let $\mathsf{u}_0 \in W$ *and let* $\mathsf{f} \in \mathrm{C}([0,T];\mathrm{L}^2(\mathbf{R}^3;\mathbf{C}^4))$ *satisfies at least one of the following conditions:*

- $\mathsf{f} \in \mathrm{W}^{1,1}(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^3;\mathbf{C}^4))$;
- $\mathsf{f} \in \mathrm{L}^1(\langle 0,T\rangle;W)$.

Then the abstract Cauchy problem

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has unique classical solution given with

$$\mathsf{u}(t) = \mathcal{T}(t)\mathsf{u}_0 + \int_0^t \mathcal{T}(t-s)\mathsf{f}(s)ds\,, \qquad t \in [0,T]\,,$$

where $(\mathcal{T}(t))_{t\geqslant 0}$ *is the contraction* C_0*-semigroup generated by* $-\mathcal{L}$.

Time-dependent coefficients

The operator $\mathcal{L}$ depends on t (i.e. the matrix coefficients $\mathbf{A}_k$ and $\mathbf{C}$ depend on t if $\mathcal{L}$ is a classical Friedrichs operator):

$$\begin{cases} \mathsf{u}'(t) + \mathcal{L}(t)\mathsf{u}(t) = \mathsf{f}(t) \\ \qquad\qquad\quad \mathsf{u}(0) = \mathsf{u}_0 \end{cases} .$$

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- Semigroup theory can treat time-dependent case, but conditions that ensure existence/uniqueness result are rather complicated to verify...

Semilinear problem

Consider

$$\begin{cases} \mathsf{u}'(t) + \mathcal{L}\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) \\ \mathsf{u}(0) = \mathsf{u}_0 \end{cases},$$

where $\mathsf{f} : [0, T\rangle \times L \longrightarrow L$.

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- semigroup theory gives existence and uniqueness of solution
- it requires (locally) Lipshitz continuity of f in variable u
- if $L = \mathrm{L}^2$ it is not *appropriate* assumption, as power functions do not satisfy it; $L = \mathrm{L}^\infty$ is better...

Banach space setting

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Let L be a reflexive complex Banach space, L' its antidual, $\mathcal{D} \subseteq L$, $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \longrightarrow L'$ linear operators that satisfy *a modified versions* of (T1)–(T3)

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(T1) $$(\forall\, \varphi, \psi \in \mathcal{D}) \quad {}_{L'}\langle\, \mathcal{L}\varphi, \psi\,\rangle_L = \overline{{}_{L'}\langle\, \tilde{\mathcal{L}}\psi, \phi\,\rangle_L}\,.$$

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Problems:

- in the classical case (F1)–(F2) need not to imply (T2) and (T3): instead of (T3) we get

$$ {}_{\mathrm{L}^{p'}}\langle\, (\mathcal{L} + \tilde{\mathcal{L}})\varphi, \varphi \,\rangle_{\mathrm{L}^p} \geq \|\varphi\|_{\mathrm{L}^2}$$

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- for semigroup treatment of non-stationary case we need to have $\mathcal{L} : \mathcal{D} \subseteq L \longrightarrow L$

Other. . .

- regularity of the solution

Other...

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- application to new examples

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M. Hochbruck, T. Pažur, A. Schulz, E. Thawinan, C. Wieners, *Efficient time integration for discontinuous Galerkin approximations of linear wave equations*, *Technical report*, Karlsruhe Institute of Technology, 2013