One-scale H-measures and variants

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Introduction

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$$u_n(\mathbf{x}) := e^{2\pi i n x} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,,$$

but

$$|u_n(\mathbf{x})| = 1 \quad \Longrightarrow \quad u_n \not\to 0 \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})\,.$$

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It is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\Omega) \hookrightarrow \mathcal{M}(\Omega) = (\mathrm{C}_c(\Omega))'$, so

$$|u_{n'}|^2 \xrightharpoonup{*} \nu\,.$$

ν is called the defect measure.

Of course, we have

$$u_{n'} \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \nu = 0\,.$$

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If the defect measure is not trivial we need another objects to determine all the properties of the sequence:

- H-measures
- semiclassical measures
- ...

Outline

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$\Omega \subseteq \mathbf{R}^d$ open.

Theorem (Tartar, 1990)

If $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$*, then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_H \in \mathcal{M}(\Omega \times \mathrm{S}^{d-1}; \mathrm{M}_r(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$

$$\lim_{n'} \int_{\mathbf{R}^d} \Big(\widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi})\Big) \psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}_H, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle .$$

(Unbounded) Radon measure $\boldsymbol{\mu}_H$ *we call the H-measure corresponding to the (sub)sequence* (u_n)*.*

Notation:
$\mathbf{x} = (x^1, x^2, \dots, x^d) \in \Omega$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbf{R}^d$
$\hat{\mathsf{u}}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} d\mathbf{x}$
$\mathsf{a} \cdot \mathsf{b} = \sum_{i=1}^d a^i \bar{b}^i$ $(\mathsf{a}, \mathsf{b} \in \mathbf{C}^r)$
$(\mathsf{a} \otimes \mathsf{b})\mathsf{v} = (\mathsf{v} \cdot \mathsf{b})\mathsf{a} \quad \Longrightarrow \quad [\mathsf{a} \otimes \mathsf{b}]_{ij} = a^i \bar{b}^j$
$\langle \cdot, \cdot \rangle$ sesquilinear dual product; $\langle \mathbf{A}, \varphi \rangle := [A^{ij}, \varphi]_{ij}$
$\mathcal{M}(X) = (\mathrm{C}_c(X))'$

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Corollary

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_H = \mathbf{0} \,.$$

Theorem (Gérard, 1991)

If $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_{n'})} \in \mathcal{M}(\Omega \times \mathbf{R}^d; \mathrm{M}_r(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$

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(Unbounded) measure $\boldsymbol{\mu}_{sc}^{(\omega_{n'})}$ *we call the semiclassical measure with characteristic length* $(\omega_{n'})$ *corresponding to the (sub)sequence* $(\mathsf{u}_{n'})$.

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$$\mathsf{u}_n\xrightarrow{\mathrm{L}^2_{\mathrm{loc}}}0\iff\boldsymbol{\mu}_{sc}^{(\omega_n)}=\mathbf{0}\quad\&\quad(\mathsf{u}_n)\ \textit{is}\ (\omega_n)-\textit{oscillatory}\,.$$

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Example 1: Oscillations - one characteristic length

$\alpha > 0$, $\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}$,

$$u_n(\mathbf{x}) := e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,,\ n \to \infty\,,$$

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If $\mathsf{u}_n \longrightarrow \mathsf{u}$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ *is* (ω_n)*-oscillatory and* $\operatorname{tr}\boldsymbol{\mu}_{sc}^{(\omega_n)}(\Omega\times\{0\}) = 0$, *then*

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If $\mathsf{u}_n \longrightarrow \mathsf{u}$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *is* (ω_n)*-oscillatory and* (ω_n)*-concentrating, then*

$$\langle \boldsymbol{\mu}_H, \varphi \boxtimes \psi \rangle = \left\langle \boldsymbol{\mu}_{sc}^{(\omega_n)}, \varphi \boxtimes \psi\Big(\frac{\cdot}{|\cdot|}\Big) \right\rangle .$$

For an arbitrary bounded sequence (u_n) in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ is there a characteristic length $\omega_n \to 0^+$ such that (u_n) is

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$0 < \alpha < \beta$, $\mathsf{k}, \mathsf{s} \in \mathbf{Z}^d \setminus \{0\}$,

$$u_n(\mathbf{x}) := e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,,\ \ n \to \infty$$

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$$\mu_H = \lambda \boxtimes \left(\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\frac{\mathsf{s}}{|\mathsf{s}|}}\right)$$

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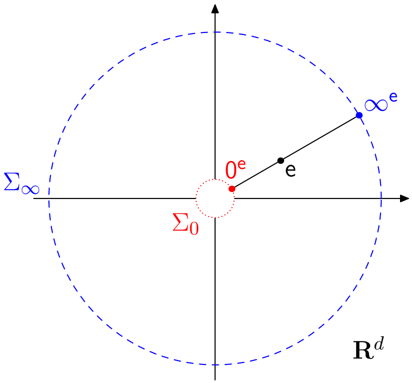
$$\mu_{sc}^{(\omega_n)} = \lambda \boxtimes \begin{cases} 2\delta_0 & , \quad \lim_n n^\beta \omega_n = 0 \\ (\delta_{c\mathsf{s}} + \delta_0) & , \quad \lim_n n^\beta \omega_n = c \in \langle 0, \infty \rangle \\ \delta_0 & , \quad \lim_n n^\beta \omega_n = \infty \;\&\; \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty \rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

Outline

$\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ is a compactification of $\mathbf{R}^d_*$ homeomorphic to a spherical layer (i.e. an annulus in $\mathbf{R}^2$):



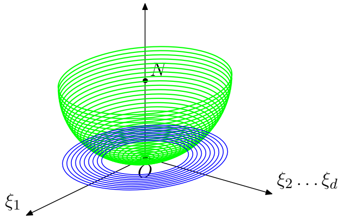
Precise description of $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ 1/3

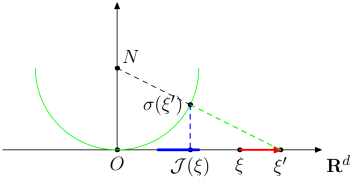
For fixed $r_0 > 0$ let us define $r_1 = \frac{r_0}{\sqrt{r_0^2+1}}$, and denote by

$$A[0, r_1, 1] := \left\{ \boldsymbol{\zeta} \in \mathbf{R}^d : r_1 \leqslant |\boldsymbol{\zeta}| \leqslant 1 \right\}$$

closed d-dimensional spherical layer equipped with the standard topology (inherited from $\mathbf{R}^d$). In addition let us define $A(0, r_1, 1) := \mathrm{Int}\, A[0, r_1, 1]$, and by $A_0[0, r_1, 1] := \mathrm{S}^{d-1}(0; r_1)$ and $A_\infty[0, r_1, 1] := \mathrm{S}^{d-1}$ we denote boundary spheres.

We want to construct a homeomorphism $\mathcal{J} : \mathbf{R}^d_* \longrightarrow A(0, r_1, 1)$.





From the previous construction we get that $\mathcal{J} : \mathbf{R}^d_* \longrightarrow A(0, r_1, 1)$ is given by

$$\mathcal{J}(\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}}{\sqrt{|\boldsymbol{\xi}|^2 + \left(\frac{|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|+r_0}\right)^2}} = \frac{|\boldsymbol{\xi}| + r_0}{|\boldsymbol{\xi}| K(\boldsymbol{\xi})}\boldsymbol{\xi}\,,$$

where $K(\boldsymbol{\xi}) = K(|\boldsymbol{\xi}|) := \sqrt{1 + (|\boldsymbol{\xi}| + r_0)^2}$.
$\boldsymbol{\xi}$ and $\mathcal{J}(\boldsymbol{\xi})$ lie on the same line:

$$\frac{\mathcal{J}(\boldsymbol{\xi})}{|\mathcal{J}(\boldsymbol{\xi})|} = \frac{\frac{|\boldsymbol{\xi}|+r_0}{|\boldsymbol{\xi}|K(\boldsymbol{\xi})}\boldsymbol{\xi}}{\frac{|\boldsymbol{\xi}|+r_0}{|\boldsymbol{\xi}|K(\boldsymbol{\xi})}|\boldsymbol{\xi}|} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\,.$$

$\mathcal{J}$ is homeomorphism and its inverse $\mathcal{J}^{-1} : A(0, r_1, 1) \longrightarrow \mathbf{R}^d_*$ is given by

$$\mathcal{J}^{-1}(\boldsymbol{\zeta}) = \frac{|\boldsymbol{\zeta}| - r_0\sqrt{1 - |\boldsymbol{\zeta}|^2}}{|\boldsymbol{\zeta}|\sqrt{1 - |\boldsymbol{\zeta}|^2}}\boldsymbol{\zeta} = \boldsymbol{\zeta}(1 - |\boldsymbol{\zeta}|^2)^{-\frac{1}{2}} - r_0\boldsymbol{\zeta}|\boldsymbol{\zeta}|^{-1}\,,$$

resulting that $(A[0, r_1, 1], \mathcal{J})$ is a compactification of $\mathbf{R}^d_*$.

Now we define

$$\Sigma_0 := \{0^{\mathsf{e}} : \mathsf{e} \in \mathrm{S}^{d-1}\} \qquad \text{and} \qquad \Sigma_\infty := \{\infty^{\mathsf{e}} : \mathsf{e} \in \mathrm{S}^{d-1}\},$$

and $\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d_* \cup \Sigma_0 \cup \Sigma_\infty$.
Let us extend $\mathcal{J}$ to the whole $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ by $\mathcal{J}(0^{\mathsf{e}}) := r_1\mathsf{e}$ and $\mathcal{J}(\infty^{\mathsf{e}}) = \mathsf{e}$, which gives $\mathcal{J}^{\rightarrow}(\Sigma_0) = A_0[0, r_1, 1]$ and $\mathcal{J}^{\rightarrow}(\Sigma_\infty) = A_\infty[0, r_1, 1]$.

$d_*(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) := |\mathcal{J}(\boldsymbol{\xi}_1) - \mathcal{J}(\boldsymbol{\xi}_2)|$ is a metric on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$, so $(\mathrm{K}_{0,\infty}(\mathbf{R}^d), d_*)$ is a metric space isomorphic to $A[0, r_1, 1]$.

$$\lim_{|\boldsymbol{\xi}|\to 0}\left|\mathcal{J}(\boldsymbol{\xi}) - \mathcal{J}(0^{\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}})\right| = 0\ , \quad \lim_{|\boldsymbol{\xi}|\to\infty}\left|\mathcal{J}(\boldsymbol{\xi}) - \mathcal{J}(\infty^{\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}})\right| = 0\,,$$

$$\lim_{|\boldsymbol{\zeta}|\to r_1} |\mathcal{J}^{-1}(\boldsymbol{\zeta})| = 0\ , \quad \lim_{|\boldsymbol{\zeta}|\to 1} |\mathcal{J}^{-1}(\boldsymbol{\zeta})| = +\infty\,.$$

Lemma

For $\psi : \mathrm{K}_{0,\infty}(\mathbf{R}^d) \longrightarrow \mathbf{C}$ *the following is equivalent:*

a) $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$,

b) $(\exists\, \tilde{\psi} \in \mathrm{C}(A[0, r_1, 1]))\ \psi = \tilde{\psi} \circ \mathcal{J}$,

c) $\psi|_{\mathbf{R}^d_*} \in \mathrm{C}(\mathbf{R}^d_*)$, *and*

$$\lim_{|\boldsymbol{\xi}|\to 0} |\psi(\boldsymbol{\xi}) - \psi(0^{\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}})| = 0 \qquad \textit{and} \qquad \lim_{|\boldsymbol{\xi}|\to\infty} |\psi(\boldsymbol{\xi}) - \psi(\infty^{\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}})| = 0\,.$$

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For $\psi \in \mathrm{C}(\mathbf{R}^d_*)$ we have $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ iff there exist $\psi_0, \psi_\infty \in \mathrm{C}(\mathrm{S}^{d-1})$ such that

$$\psi(\boldsymbol{\xi}) - \psi_0\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \to 0\,, \quad |\boldsymbol{\xi}| \to 0\,,$$
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In particular, $\psi - \psi_0(\frac{\cdot}{|\cdot|}) \in \mathrm{C}_{ub}(\mathbf{R}^d)$ (uniformly continuous bounded functions).

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In particular, $\psi - \psi_0(\frac{\cdot}{|\cdot|}) \in \mathrm{C}_{ub}(\mathbf{R}^d)$ (uniformly continuous bounded functions).

Lemma

i) $\mathrm{C}_0(\mathbf{R}^d) \hookrightarrow \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *and*

ii) $\{\psi \circ \boldsymbol{\pi} \ :\ \psi \in \mathrm{C}(\mathrm{S}^{d-1})\} \hookrightarrow \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

Theorem (Tartar, 2009)

If $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d); \mathrm{M}_r(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$\lim_{n'} \int_{\mathbf{R}^d} \Big((\widehat{\varphi_1 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \otimes (\widehat{\varphi_2 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \Big) \psi(\omega_{n'}\boldsymbol{\xi})\, d\boldsymbol{\xi} = \Big\langle \boldsymbol{\mu}^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \Big\rangle \ .$$

(Unbounded) Radon measure $\boldsymbol{\mu}^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ *we call the one-scale H-measure with characteristic length* $(\omega_{n'})$ *corresponding to the (sub)sequence* $(\mathsf{u}_{n'})$.

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$$\lim_{n'}\int_{\mathbf{R}^d}\Big((\widehat{\varphi_1\mathsf{u}_{n'}})(\boldsymbol{\xi})\otimes(\widehat{\varphi_2\mathsf{u}_{n'}})(\boldsymbol{\xi})\Big)\psi(\omega_{n'}\boldsymbol{\xi})\,d\boldsymbol{\xi} = \left\langle \boldsymbol{\mu}^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}},\varphi_1\bar{\varphi}_2\boxtimes\psi\right\rangle\,.$$

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The original proof:

- $\mathsf{v}_n(\mathbf{x},x^{d+1}) := \mathsf{u}_n(\mathbf{x})e^{\frac{2\pi i x^{d+1}}{\omega_n}} \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega\times\mathbf{R};\mathbf{C}^r)$
- $\boldsymbol{\nu}_H\in\mathcal{M}(\Omega\times\mathbf{R}\times\mathrm{S}^d;\mathrm{M}_r(\mathbf{C}))$
- $\boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}$ is obtained from $\boldsymbol{\nu}_H$ (suitable projection in x^{d+1} and ξ_{d+1})

Alternative proof (Antonić, E., Lazar)

- Cantor diagonal procedure (separability)
- commutation lemma

Lemma

Let $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, *and denote* $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. *Then the commutator can be expressed as a sum*

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K \,,$$

where K *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$, *while* $\tilde{C}_n \longrightarrow 0$ *in the operator norm on* $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$.

- variant of the kernel lemma

Lemma

Let X *and* Y *be two Hausdorff second countable topological manifolds (with or without a boundary), and let* B *be a non-negative continuous bilinear form on* $\mathrm{C}_c(X) \times \mathrm{C}_c(Y)$. *Then there exists a Radon measure* $\mu \in \mathcal{M}(X \times Y)$ *such that*

$$(\forall\, f \in \mathrm{C}_c(X))(\forall\, g \in \mathrm{C}_c(Y)) \quad B(f, g) = \langle \mu, f \boxtimes g \rangle \,.$$

Furthermore, the above remains valid if we replace C_c *by* C_0, *and* $\mathcal{M}$ *by* $\mathcal{M}_b$ *(the space of bounded Radon measures, i.e. the dual of Banach space* C_0*).*

Some properties of $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$

Theorem

a)	$\boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$,	$\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \geqslant \mathbf{0}$	
c)	$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0$	$\iff$	$\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}$
d)	$\mathrm{tr}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0$	$\iff$	(u_n) *is* (ω_n) − *oscillatory*

Theorem

$\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$, $\psi \in \mathrm{C}_0(\mathbf{R}^d)$, $\tilde\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\omega_n \to 0^+$,

$$\text{a)} \quad \langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle = \langle \boldsymbol{\mu}^{(\omega_n)}_{sc}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle ,$$

$$\text{b)} \quad \langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi \circ \boldsymbol{\pi}\rangle = \langle \boldsymbol{\mu}_H, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi\rangle ,$$

where $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

$$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}},$$

$$\mu_H = \lambda \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$$

$$\mu_{sc}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_0 & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty \rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

$$\mu_{\mathrm{K}_{0,\infty}}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_{0^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty \rangle \\ \delta_{\infty^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

Example 2 - revisited

$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}}$, $v_n(\mathbf{x}) = e^{2\pi i n^\beta \mathsf{s}\cdot\mathbf{x}}$,
associated objects to $(u_n + v_n)$:

$$\mu_H = \lambda \boxtimes \left(\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\frac{\mathsf{s}}{|\mathsf{s}|}}\right)$$

$$\mu_{sc}^{(\omega_n)} = \lambda \boxtimes \begin{cases} 2\delta_0 & , \quad \lim_n n^\beta \omega_n = 0 \\ (\delta_0 + \delta_{c\mathsf{s}}) & , \quad \lim_n n^\beta \omega_n = c \in \langle 0, \infty\rangle \\ \delta_0 & , \quad \lim_n n^\beta \omega_n = \infty \ \&\ \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

$$\mu_{\mathrm{K}_{0,\infty}}^{(\omega_n)} = \lambda \boxtimes \begin{cases} (\delta_{0\,\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{0\,\frac{\mathsf{s}}{|\mathsf{s}|}}) & , \quad \lim_n n^\beta \omega_n = 0 \\ (\delta_{0\,\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{c\mathsf{s}}) & , \quad \lim_n n^\beta \omega_n = c \in \langle 0, \infty\rangle \\ (\delta_{0\,\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\infty\,\frac{\mathsf{s}}{|\mathsf{s}|}}) & , \quad \lim_n n^\beta \omega_n = \infty \ \&\ \lim_n n^\alpha \omega_n = 0 \\ (\delta_{c\mathsf{k}} + \delta_{\infty\,\frac{\mathsf{s}}{|\mathsf{s}|}}) & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ (\delta_{\infty\,\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\infty\,\frac{\mathsf{s}}{|\mathsf{s}|}}) & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,, \qquad (*)$$

where

- $l \in 0..m$
- $\varepsilon_n > 0$ bounded
- $\mathbf{A}_n^{\boldsymbol{\alpha}} \to \mathbf{A}^{\boldsymbol{\alpha}}$ in $\mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (**)$$

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

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For $l = 0$ the condition on (f_n) is equivalent to

$$(\forall \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \|\varphi \mathsf{f}_n\|_{\mathrm{H}^{-m}_{\varepsilon_n}} \to 0\,,$$

where $\|\mathsf{u}\|^2_{\mathrm{H}^s_h} = \int_{\mathbf{R}^d} (1 + 2\pi |h\boldsymbol{\xi}|^2)^s |\hat{\mathsf{u}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}$ is the semiclassical norm of $\mathsf{u} \in \mathrm{H}^s(\Omega; \mathbf{R}^d)$.

Localisation principle - theorem

$$(*) \quad \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n$$

$$(**) \quad (\forall\,\varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$$

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Theorem (Tartar, 2009)

Under previous assumptions and $l=1$, $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{(\varepsilon_n)}$ *associated to* (u_n) *satisfies*

$$\mathrm{supp}\,(\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top) \subseteq \Omega\times\Sigma_0\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

$(*)\quad \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n$

$(**)\quad (\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega))\qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$

Theorem (Antonić, E., Lazar, 2015)

Under previous assumptions, $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{(\varepsilon_n)}$ *associated to* (u_n) *satisfies*

$$\mathbf{p}_1\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

where

$$\mathbf{p}_1(\mathbf{x},\boldsymbol{\xi}) := \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

$(*) \quad \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n$

$(**) \quad (\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$

Theorem

For $\omega_n \to 0^+$ *such that* $c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0,\infty]$, *corresponding one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ω_n) *satisfies*

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

where

$$\mathbf{p}_c(\mathbf{x},\boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c=0 \\ \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m}(2\pi i c)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c\in\langle 0,\infty\rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c=\infty \end{cases}$$

Moreover, if there exists $\varepsilon_0>0$ *such that* $\varepsilon_n>\varepsilon_0$, $n\in\mathbf{N}$, *we can take*

$$\mathbf{p}_\infty(\mathbf{x},\boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Theorem

$\infty > \varepsilon_\infty \geqslant \varepsilon_n \geqslant \varepsilon_0 > 0$, $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$,

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_q}(r))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ *in* $\mathrm{C}(\Omega; \mathrm{M_q}(r))$, *and* $\mathsf{f}_n \longrightarrow 0$ *in* $\mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^q)$.
Then the associated H-measure $\boldsymbol{\mu}_H$ *satisfies*

$$\mathbf{p}_{pr} \boldsymbol{\mu}_H = \mathbf{0} \,.$$

Theorem

$\infty > \varepsilon_\infty \geqslant \varepsilon_n \geqslant \varepsilon_0 > 0$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$,

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n\,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega;\mathrm{M_q}(r))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ in $\mathrm{C}(\Omega;\mathrm{M_q}(r))$, and $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{H}^{-m}_{\mathrm{loc}}(\Omega;\mathbf{C}^q)$.
Then the associated H-measure $\boldsymbol{\mu}_H$ satisfies

$$\mathbf{p}_{pr}\boldsymbol{\mu}_H = \mathbf{0}\,.$$

Sketch of the proof:

- If (ε_n) is bounded from below and above by positive constants, $(**)$ is equivalent to the strong convergence to zero in $\mathrm{H}^{-m}_{\mathrm{loc}}(\Omega;\mathbf{C}^q)$.
- $\boldsymbol{\mu}_H$ and $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ coincide on the space of homogeneous functions of the zero order (in $\boldsymbol{\xi}$).
- $\mathbf{p}_{pr}$ is homogeneous of the zero order in $\boldsymbol{\xi}$.

Theorem

$\varepsilon_n > 0$ bounded, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$,

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega;\mathrm{M}_{\mathrm{q}}(r))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ in $\mathrm{C}(\Omega;\mathrm{M}_{\mathrm{q}}(r))$, and $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega;\mathbf{C}^q)$ satisfies $(**)$.
Then the associated semiclassical measure $\boldsymbol{\mu}_{sc}^{(\omega_n)}$ satisfies

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi})\left(\boldsymbol{\mu}_{sc}^{(\omega_n)}\right)^{\top} = \mathbf{0}\,,$$

where $c := \lim_n \frac{\varepsilon_n}{\omega_n}$ and

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = 0 \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i c)^{|\boldsymbol{\alpha}|} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c \in \langle 0,\infty\rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = \infty \end{cases}$$

Proof (only the case $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle$)

$$\psi \in \mathcal{S}(\mathbf{R}^d) \quad \Longrightarrow \quad \boldsymbol{\xi} \mapsto (|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m)\psi(\boldsymbol{\xi}) \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$$

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$$\begin{aligned}
\mathbf{0} &= \left\langle \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i c)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}} \boldsymbol{\mu}^{\top}_{\mathrm{K}_{0,\infty}}, \varphi \boxtimes (|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m)\psi \right\rangle \\
&= \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left\langle \mathbf{A}^{\boldsymbol{\alpha}} \boldsymbol{\mu}^{\top}_{\mathrm{K}_{0,\infty}}, \overline{(2\pi i c)^{|\boldsymbol{\alpha}|}} \varphi \boxtimes \boldsymbol{\xi}^{\boldsymbol{\alpha}} \psi \right\rangle \\
&= \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left\langle \mathbf{A}^{\boldsymbol{\alpha}} \boldsymbol{\mu}^{\top}_{sc}, \overline{(2\pi i c)^{|\boldsymbol{\alpha}|}} \varphi \boxtimes \boldsymbol{\xi}^{\boldsymbol{\alpha}} \psi \right\rangle = \left\langle \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i c)^{|\boldsymbol{\alpha}|} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}} \boldsymbol{\mu}^{\top}_{sc}, \varphi \boxtimes \psi \right\rangle ,
\end{aligned}$$

where in the third equality the fact that $\boldsymbol{\xi}^{\boldsymbol{\alpha}} \psi \in \mathcal{S}(\mathbf{R}^d)$ was used.

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

$$\begin{cases} u_n^1 + \varepsilon_n \partial_{x_1}(a_1 u_n^1) = f_n^1 \\ u_n^2 + \varepsilon_n \partial_{x_2}(a_2 u_n^2) = f_n^2 \end{cases},$$

where $\varepsilon_n \to 0^+$, $\mathsf{f}_n := (f_n^1,\ f_n^2) \in \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

$$(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \|\varphi \mathsf{f}_n\|_{\mathrm{H}^{-1}_{\varepsilon_n}} \to 0\,,$$

while $a_1, a_2 \in \mathrm{C}(\Omega; \mathbf{R})$, $a_1, a_2 \neq 0$ everywhere.

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while $a_1, a_2 \in \mathrm{C}(\Omega; \mathbf{R})$, $a_1, a_2 \neq 0$ everywhere.

By the localisation principle for one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) (i.e. $c = 1$) associated to (u_n) we get the relation

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2\pi i \xi_1}{1+|\boldsymbol{\xi}|} \begin{bmatrix} a_1(\mathbf{x}) & 0 \\ 0 & 0 \end{bmatrix} + \frac{2\pi i \xi_2}{1+|\boldsymbol{\xi}|} \begin{bmatrix} 0 & 0 \\ 0 & a_2(\mathbf{x}) \end{bmatrix} \right) \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0}\,,$$

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whose $(1,1)$ component reads

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} + i \frac{2\pi \xi_1}{1+|\boldsymbol{\xi}|} a_1(\mathbf{x}) \right) \mu_{\mathrm{K}_{0,\infty}}^{11} = 0\,,$$

hence

$$\frac{1}{1+|\boldsymbol{\xi}|} \mu_{\mathrm{K}_{0,\infty}}^{11} = 0\,, \qquad \frac{\xi_1}{1+|\boldsymbol{\xi}|} \mu_{\mathrm{K}_{0,\infty}}^{11} = 0$$

Example 3: equations with characteristic length (1/2)

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

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$$(\forall\, \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \|\varphi \mathsf{f}_n\|_{\mathrm{H}^{-1}_{\varepsilon_n}} \to 0\,,$$

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whose $(1,1)$ component reads

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} + i \frac{2\pi \xi_1}{1+|\boldsymbol{\xi}|} a_1(\mathbf{x}) \right) \mu^{11}_{\mathrm{K}_{0,\infty}} = 0\,,$$

hence

$$\mathrm{supp}\, \mu^{11}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times \Sigma_\infty\,, \qquad \frac{\xi_1}{1+|\boldsymbol{\xi}|} \mu^{11}_{\mathrm{K}_{0,\infty}} = 0$$

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

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whose $(1,1)$ component reads

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$$\operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times \Sigma_\infty\,, \quad \operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times (\Sigma_0 \cup \{\xi_1 = 0\})$$

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where $\varepsilon_n \to 0^+$, $\mathsf{f}_n := (f_n^1,\ f_n^2) \in \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

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while $a_1, a_2 \in \mathrm{C}(\Omega; \mathbf{R})$, $a_1, a_2 \neq 0$ everywhere.

By the localisation principle for one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) (i.e. $c = 1$) associated to (u_n) we get the relation

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whose $(1,1)$ component reads

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hence

$$\operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times \{\infty^{(0,-1)}, \infty^{(0,1)}\}$$

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

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while $a_1, a_2 \in \mathrm{C}(\Omega; \mathbf{R})$, $a_1, a_2 \neq 0$ everywhere.

By the localisation principle for one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) (i.e. $c = 1$) associated to (u_n) we get the relation

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$$\operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times \{\infty^{(0,-1)}, \infty^{(0,1)}\}$$

Analogously, from the $(2,2)$ component we get

$$\operatorname{supp}\mu^{22}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times \{\infty^{(-1,0)}, \infty^{(1,0)}\},$$

hence $\operatorname{supp}\mu^{11}_{\mathrm{K}_{0,\infty}} \cap \operatorname{supp}\mu^{22}_{\mathrm{K}_{0,\infty}} = \emptyset$ which implies $\mu^{12}_{\mathrm{K}_{0,\infty}} = \mu^{21}_{\mathrm{K}_{0,\infty}} = 0$.

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The very definition of one-scale H-measures gives $u^1_n \bar{u^2_n} \xrightarrow{*} 0$.

This approach can be systematically generalised by introducing a variant of compensated compactness suitable for problems with characteristic length.

Let $\mathsf{u}_n \longrightarrow \mathsf{u}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ satisfy

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ in $\mathrm{C}(\Omega; \mathrm{M}_{\mathrm{q}\times\mathrm{r}}(\mathbf{C}))$, let $\varepsilon_n \to 0^+$, and $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^q)$ be such that for any $\varphi \in \mathrm{C}^\infty_c(\Omega)$

$$\frac{\widehat{\varphi \mathsf{f}_n}}{1 + k_n}$$

is precompact in $\mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^q)$. Furthermore, let $Q(\mathbf{x}; \boldsymbol{\lambda}) := \mathbf{Q}(\mathbf{x})\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}$, where $\mathbf{Q} \in \mathrm{C}(\Omega; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$, $\mathbf{Q}^* = \mathbf{Q}$, is such that $Q(\cdot; \mathsf{u}_n) \xrightarrow{*} \nu$ in $\mathcal{M}(\Omega)$.
Then we have

a) $(\exists\, c \in [0, \infty])(\forall\, (\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)\mathbf{R}^d)(\forall\, \boldsymbol{\lambda} \in \Lambda_{c;\mathbf{x},\boldsymbol{\xi}})\ Q(\mathbf{x}; \boldsymbol{\lambda}) \geqslant 0 \implies \nu \geqslant Q(\cdot, \mathsf{u})$,

b) $(\exists\, c \in [0, \infty])(\forall\, (\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)\mathbf{R}^d)(\forall\, \boldsymbol{\lambda} \in \Lambda_{c;\mathbf{x},\boldsymbol{\xi}})\ Q(\mathbf{x}; \boldsymbol{\lambda}) = 0 \implies \nu = Q(\cdot, \mathsf{u})$,

where

$$\Lambda_{c;\mathbf{x},\boldsymbol{\xi}} := \{\boldsymbol{\lambda} \in \mathbf{C}^r : \mathbf{p}_c(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\lambda} = 0\} \,,$$

and $\mathbf{p}_c$ is given as before.

Outline

$\Omega \subseteq \mathbf{R}^d$ open

Theorem

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ *and* $\omega_n \to 0^+$, *then there exist* $(u_{n'})$, $(v_{n'})$ *and* $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 v_{n'}}(\boldsymbol{\xi})} \psi(\omega_{n'} \boldsymbol{\xi}) \, d\boldsymbol{\xi} = \langle \mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \, .$$

The measure $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ *is called the one-scale H-measure with characteristic length* $(\omega_{n'})$ *associated to the (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

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$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_n}(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}\, d\mathbf{x} = \langle \mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,.$$

The measure $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ *is called the one-scale H-measure with characteristic length* $(\omega_{n'})$ *associated to the (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$

$\Omega \subseteq \mathbf{R}^d$ open

Theorem

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ *in* $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega)$ *and* $\omega_n \to 0^+$, *then there exist* $(u_{n'})$, $(v_{n'})$ *and* $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\Omega)$ *and* $\psi \in E$

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_n}(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}\, d\mathbf{x} = \langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,.$$

The distribution $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ *is called the one-scale H-distribution with characteristic length* $(\omega_{n'})$ *associated to the (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$

$\Omega \subseteq \mathbf{R}^d$ open, $p \in \langle 1, \infty \rangle$, $\frac{1}{p} + \frac{1}{p'} = 1$

Theorem

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ *in* $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega)$ *and* $\omega_n \to 0^+$, *then there exist* $(u_{n'})$, $(v_{n'})$ *and* $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\Omega)$ *and* $\psi \in E$

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_n}(\varphi_1 u_{n'})(\mathbf{x}) \overline{(\varphi_2 v_{n'})(\mathbf{x})} \, d\mathbf{x} = \langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,.$$

The distribution $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ *is called the one-scale H-distribution with characteristic length* $(\omega_{n'})$ *associated to the (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$

Determine E such that

- $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is continuous
- The First commutation lemma is valid

For $\kappa \in \mathbf{N}_0 \cup \{\infty\}$ let us define

$$\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) := \left\{\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) : \psi^* := \psi \circ \mathcal{J}^{-1} \in \mathrm{C}^\kappa(A[0, r_1, 1])\right\}.$$

It is not hard to check that $\mathrm{C}^0(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ coincide.

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$\mathrm{C}^\kappa(A[0, r_1, 1])$ Banach algebra $\implies$ $\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ Banach algebra

$A[0, r_1, 1]$ compact $\implies$ $\mathrm{C}^\kappa(A[0, r_1, 1])$ separable
$\implies$ $\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ separable

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$$\mathrm{C}^\kappa(A[0, r_1, 1]) \quad \text{Banach algebra} \quad \Longrightarrow \quad \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) \quad \text{Banach algebra}$$

$$\begin{aligned} A[0, r_1, 1] \quad \text{compact} \quad &\Longrightarrow \quad \mathrm{C}^\kappa(A[0, r_1, 1]) \quad \text{separable} \\ &\Longrightarrow \quad \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) \quad \text{separable} \end{aligned}$$

Is $\mathcal{A}_\psi = (\psi \hat{\cdot})^\vee : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ continuous?

Theorem (Hörmander-Mihlin)

If for $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ *there exists* $C > 0$ *such that*

$$(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d_*)(\forall\, \boldsymbol{\alpha} \in \mathbf{N}^d_0,\ |\boldsymbol{\alpha}| \leqslant \kappa) \qquad |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})| \leqslant \frac{C}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}}\,,$$

where $\kappa = \lfloor \frac{d}{2} \rfloor + 1$, *then* ψ *is a Fourier multiplier. Moreover, we have*

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_d \max\Big\{p, \frac{1}{p-1}\Big\} C\,.$$

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We shall use *Faá di Bruno formula*: for sufficiently smooth functions $\mathrm{g} : \mathbf{R}^d \longrightarrow \mathbf{R}^r$ and $f : \mathbf{R}^r \longrightarrow \mathbf{R}$ we have

$$\partial^{\boldsymbol{\alpha}}(f \circ \mathrm{g})(\boldsymbol{\xi}) = |\boldsymbol{\alpha}|! \sum_{1 \leqslant |\boldsymbol{\beta}| \leqslant |\boldsymbol{\alpha}|\,,\ \boldsymbol{\beta} \in \mathbf{N}^r_0} C(\boldsymbol{\beta}, \boldsymbol{\alpha})\,,$$

where

$$C(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \frac{(\partial^{\boldsymbol{\beta}} f)(\mathrm{g}(\boldsymbol{\xi}))}{\boldsymbol{\beta}!} \sum_{\substack{\sum_{i=1}^r \boldsymbol{\alpha}_i = \boldsymbol{\alpha}\,, \\ \boldsymbol{\alpha}_i \in \mathbf{N}^d_0}} \prod_{j=1}^r \sum_{\substack{\sum_{i=1}^{\beta_j} \boldsymbol{\gamma}_i = \boldsymbol{\alpha}_j\,, \\ \boldsymbol{\gamma}_i \in \mathbf{N}^d_0 \setminus \{0\}}} \prod_{s=1}^{\beta_j} \frac{\partial^{\boldsymbol{\gamma}_s} g_j(\boldsymbol{\xi})}{\boldsymbol{\gamma}_s!}\,.$$

Lemma

For every $j \in 1..d$ *and* $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ *we have*

$$\partial^{\boldsymbol{\alpha}}(\mathcal{J}_j)(\boldsymbol{\xi}) = P_{\boldsymbol{\alpha}}\Big(\boldsymbol{\xi}, \frac{1}{|\boldsymbol{\xi}|}\Big) K(\boldsymbol{\xi})^{-1-2|\boldsymbol{\alpha}|} , \quad \boldsymbol{\xi} \in \mathbf{R}_*^d ,$$

where $P_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \eta)$ *is a polynomial of degree less or equal to* $|\boldsymbol{\alpha}| + 1$ *in* $\boldsymbol{\xi}$ *and* $2|\boldsymbol{\alpha}| + 1$ *in* η*, in addition that in the expression* $\lambda^{|\boldsymbol{\alpha}|} P_{\boldsymbol{\alpha}}\Big(\lambda, \ldots, \lambda, \frac{1}{\lambda}\Big)$ *we do not have terms of the negative order. Precisely, polynomial* $P_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \eta)$ *has only terms of the form* $C\boldsymbol{\xi}^{\boldsymbol{\beta}}\eta^k$ *where* $|\boldsymbol{\beta}| + |\boldsymbol{\alpha}| \geqslant k$*.*

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For every $j \in 1..d$ *and* $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ *we have*

$$|\partial^{\boldsymbol{\alpha}}(\mathcal{J}_j)(\boldsymbol{\xi})| \leqslant \frac{C_{\boldsymbol{\alpha},d}}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}} , \quad \boldsymbol{\xi} \in \mathbf{R}_*^d .$$

Theorem

Let $\kappa \in \mathbf{N}_0$*. For every* $\psi \in \mathrm{C}^{\kappa}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *and* $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ *such that* $|\boldsymbol{\alpha}| \leqslant \kappa$ *we have*

$$|\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})| \leqslant C_{\kappa,d} \frac{\|\psi\|_{\mathrm{C}^{\kappa}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}} , \quad \boldsymbol{\xi} \in \mathbf{R}_*^d .$$

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Therefore, for $\kappa \geqslant \lfloor \frac{d}{2} \rfloor + 1$ and $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we have

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_{d,p} \|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))} .$$

Theorem

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Lemma

i) $\mathcal{S}(\mathbf{R}^d) \hookrightarrow \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *and*
ii) $\{\psi \circ \boldsymbol{\pi} \;:\; \psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})\} \hookrightarrow \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

Commutation lemma

$B_\varphi u := \varphi u$, $\mathcal{A}_\psi u := (\psi \hat{u})^\vee$.

Lemma

Let $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\kappa \geqslant \lfloor \frac{d}{2} \rfloor + 1$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, *and denote* $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. *Then the commutator can be expressed as a sum*

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K \,,$$

where for any $p \in \langle 1, \infty \rangle$ *we have that* K *is a compact operator on* $\mathrm{L}^p(\mathbf{R}^d)$, *while* $\tilde{C}_n \longrightarrow 0$ *in the operator norm on* $\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))$.

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Dem.

$$\mathcal{A}_{\psi_n} = \underbrace{\mathcal{A}_{\psi_n - \psi_0 \circ \boldsymbol{\pi}}}_{\tilde{C}_n} + \underbrace{\mathcal{A}_{\psi_0 \circ \boldsymbol{\pi}}}_{K} \,,$$

where $\boldsymbol{\pi}(\boldsymbol{\xi}) := \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$ and

$$\psi(\boldsymbol{\xi}) - (\psi_0 \circ \boldsymbol{\pi})(\boldsymbol{\xi}) \longrightarrow 0 \,, \quad |\boldsymbol{\xi}| \to 0 \,.$$

Let $r \in \langle 1, \infty \rangle$ and $\theta \in \langle 0, 1 \rangle$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r}$.

Proof of Comm. Lemma: $\tilde{C}_n := \mathcal{A}_{\psi_n - \psi_0 \circ \boldsymbol{\pi}}$

$$\psi_n - \psi_0 \circ \boldsymbol{\pi} \in \mathrm{C}^{\kappa}(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) \quad \Longrightarrow \quad \tilde{C}_n \text{ bounded on } \mathrm{L}^r(\mathbf{R}^d)$$

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Lemma (Tartar, 2009)

Let $\psi \in \mathrm{C}_{ub}(\mathbf{R}^d)$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, *and denote* $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. *Then the commutator* $C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = B_\varphi \mathcal{A}_{\psi_n} - \mathcal{A}_{\psi_n} B_\varphi$ *tends to zero in the operator norm on* $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$.

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By the Riesz-Thorin interpolation theorem we have

$$\|\tilde{C}_n\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant \|\tilde{C}_n\|^{\theta}_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))} \|\tilde{C}_n\|^{1-\theta}_{\mathcal{L}(\mathrm{L}^r(\mathbf{R}^d))} ,$$

implying $\tilde{C}_n \longrightarrow 0$ in the operator norm on $\mathrm{L}^p(\mathbf{R}^d)$.

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For $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ and $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$ the commutator $C := [B_\varphi, \mathcal{A}_\psi]$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$.

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Lemma (Antonić, Mišur, Mitrović, 2016)

Let A be compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty\rangle \setminus \{2\}$. Then A is also compact on $\mathrm{L}^p(\mathbf{R}^d)$, for any p between 2 and r (i.e. such that $1/p = \theta/2 + (1-\theta)/r$, for some $\theta \in \langle 0, 1\rangle$).

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r} \quad \Longrightarrow \quad K \text{ compact on } \mathrm{L}^p(\mathbf{R}^d)$$

Theorem

If $u_n \longrightarrow 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$ *and* (v_n) *is bounded in* $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$, *for some* $p \in \langle 1, \infty\rangle$ *and* $q \geqslant p'$, *and* $\omega_n \to 0^+$, *then there exist subsequences* $(u_{n'})$, $(v_{n'})$ *and a complex distribution of finite order* $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *where* $\kappa = \lfloor \frac{d}{2} \rfloor + 1$, *we have*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\overline{\varphi_2 v_{n'}}\, d\mathbf{x} = \lim_{n'} \int_{\mathbf{R}^d} \varphi_1 u_{n'} \overline{\mathcal{A}_{\bar{\psi}_{n'}}(\varphi_2 v_{n'})}\, d\mathbf{x}$$

$$= \left\langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \right\rangle ,$$

where $\psi_n := \psi(\omega_n \cdot)$. *The distribution* $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ *we call one-scale H-distribution (with characteristic length* $(\omega_{n'})$*) associated to (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

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$$\int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\overline{\varphi_2 v_{n'}}\, d\mathbf{x} = \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \right\rangle .$$

K_m compacts such that $K_m \subseteq \operatorname{Int} K_{m+1}$ and $\bigcup_m K_m = \Omega$.

For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ such that $\operatorname{supp}\varphi_1, \operatorname{supp}\varphi_2 \subseteq K_m$, we have

$$|\,\langle \varphi_2 v_n, \mathcal{A}_{\psi_n}(\varphi_1 u_n)\rangle\,| \leqslant C_{m,d}\|\varphi_1\|_{\mathrm{L}^\infty(K_m)}\|\varphi_2\|_{\mathrm{L}^\infty(K_m)}\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}\,.$$

The existence of one-scale H-distributions: proof 1/2

For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ such that $\operatorname{supp}\varphi_1, \operatorname{supp}\varphi_2 \subseteq K_m$, we have

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By the Cantor diagonal procedure (we have separability) ... we get trilinear form L:

$$L(\varphi_1, \varphi_2, \psi) = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\right\rangle\,.$$

For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ such that $\operatorname{supp}\varphi_1, \operatorname{supp}\varphi_2 \subseteq K_m$, we have

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$$L(\varphi_1, \varphi_2, \psi) = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\right\rangle\,.$$

L depends only on the product $\varphi_1\bar{\varphi}_2$: $\zeta_i \in \mathrm{C}_c(\Omega)$ such that $\zeta_i \equiv 1$ on $\operatorname{supp}\varphi_i$, $i = 1, 2$,

$$\begin{aligned}
\lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\right\rangle &= \lim_{n'} \left\langle \varphi_2 v_{n'}, \varphi_1\mathcal{A}_{\psi_{n'}}(\zeta_1 u_n)\right\rangle \\
&= \lim_{n'} \left\langle \bar{\varphi}_1\varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\zeta_1 u_n)\right\rangle \\
&= \lim_{n'} \left\langle \zeta_1\zeta_2 v_{n'}, \varphi_1\bar{\varphi}_2\mathcal{A}_{\psi_{n'}}(\zeta_1 u_n)\right\rangle \\
&= \lim_{n'} \left\langle \zeta_1\zeta_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1\bar{\varphi}_2 u_n)\right\rangle\,,
\end{aligned}$$

$$\Longrightarrow \quad L(\varphi_1, \varphi_2, \psi) = L(\varphi_1\bar{\varphi}_2, \zeta_1\zeta_2, \psi)\,.$$

The existence of one-scale H-distributions: proof 2/2

For $\varphi \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we define

$$\mathcal{L}(\varphi, \psi) := L(\varphi, \zeta, \psi)\,,$$

where $\zeta \equiv 1$ on $\operatorname{supp}\varphi$.
$\mathcal{L}$ is continuous bilinear form on $\mathrm{C}_c(\Omega) \times \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

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Theorem

Let $\Omega \subseteq \mathbf{R}^d$ be open, and let B be a continuous bilinear form on $\mathrm{C}^\infty_c(\Omega) \times \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$. Then there exists a unique distribution $\nu \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that

$$(\forall\, f \in \mathrm{C}^\infty_c(\Omega))(\forall\, g \in \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))) \quad B(f, g) = \langle \nu, f \boxtimes g \rangle\ .$$

Moreover, if B is continuous on $\mathrm{C}^k_c(\Omega) \times \mathrm{C}^l(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ for some $k, l \in \mathbf{N}_0$, ν is of a finite order $q \leqslant k + l + 2d + 1$.

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Therefore, we have that there exists $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'_{\kappa+2d+1}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that

$$\begin{aligned}
\left\langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \psi \right\rangle =& \mathcal{L}(\varphi_1\bar\varphi_2, \psi)\\
=& L(\varphi_1\bar\varphi_2, \zeta_1\zeta_2, \psi)\\
=& L(\varphi_1, \varphi_2, \psi) = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \right\rangle
\end{aligned}$$

Localisation principle: assumptions

$$\mathrm{H}^{s,p}(\mathbf{R}^d) := \Big\{ u \in \mathcal{S}' : \mathcal{A}_{(1+|\boldsymbol{\xi}|^2)^{\frac{s}{2}}} u \in \mathrm{L}^p(\mathbf{R}^d) \Big\}$$

$$\mathrm{H}^{s,p}_{\mathrm{loc}}(\Omega) := \Big\{ u \in \mathcal{D}' : (\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \; \varphi u \in \mathrm{H}^{s,p}(\mathbf{R}^d) \Big\}$$

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Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $p \in \langle 1, \infty \rangle$, and

$$\sum_{0 \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,, \qquad (\star)$$

where

- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}^\infty(\Omega; \mathrm{M}_{\mathrm{q}\times\mathrm{r}}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m,p}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall\, \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \mathcal{A}_{(1+|\varepsilon_n \boldsymbol{\xi}|^2)^{-\frac{m}{2}}} (\varphi \mathsf{f}_n) \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^q)\,. \qquad (\star\star)$$

Localisation principle: assumptions

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$$(1 + |\boldsymbol{\xi}|^2)^{-\frac{m}{2}} \text{ is a Fourier multiplier} \quad \Longrightarrow \quad \left(\mathsf{f}_n \xrightarrow{\mathrm{L}^p_{\mathrm{loc}}} 0 \quad \Longrightarrow \quad (\star\star) \right)$$

$$\left| \partial^{\boldsymbol{\alpha}} \left(\left(\frac{1 + |\varepsilon_n \boldsymbol{\xi}|^2}{1 + |\boldsymbol{\xi}|^2} \right)^{\frac{m}{2}} \right) \right| \leqslant \frac{2^\kappa}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}} \quad \Longrightarrow \quad \left((\star\star) \quad \Longrightarrow \quad \mathsf{f}_n \xrightarrow{\mathrm{H}^{-m,p}_{\mathrm{loc}}} 0 \right)$$

Theorem

Under previous assumptions let (v_n) *be a bounded sequence in* $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$*. Then one-scale H-distribution* $\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}}$ *associated to (sub)sequences* (v_n) *and* (u_n) *with characteristic length* (ε_n) *satisfies:*

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) = \sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m}(2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{(1+|\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,,$$

while q *is order of* $\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}}$.

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while q *is order of* $\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}}$.

Dem. Multiplying $(\star)$ by $\varphi \in \mathrm{C}^{\infty}_c(\Omega)$ and using the Leibniz rule we get

$$\sum_{0 \leqslant |\boldsymbol{\alpha}| \leqslant m} \sum_{0 \leqslant \boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} (-1)^{|\boldsymbol{\beta}|} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha} - \boldsymbol{\beta}} \Big((\partial_{\boldsymbol{\beta}} \varphi) \mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n \Big) = \varphi \mathsf{f}_n \,.$$

Lemma

Let (ε_n) be a sequence in $\mathbf{R}^+$ bounded from above and let (f_n) be a sequence of vector valued functions such that for some $k \in 0..m$ it converges strongly to zero in $\mathrm{H}^{-k,p}(\mathbf{R}^d;\mathbf{C}^q)$. Then $(\varepsilon_n^k \mathsf{f}_n)$ satisfies $(\star\star)$.

$$\boldsymbol{\beta} \neq 0 \quad \Longrightarrow \quad \varepsilon_n^{|\boldsymbol{\alpha}|}\partial_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\Big((\partial_{\boldsymbol{\beta}}\varphi)\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n\Big) \text{ satisfies } (\star\star)$$

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Thus, we have

$$\sum_{0 \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \varphi \mathsf{u}_n) = \tilde{\mathsf{f}}_n \,,$$

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where $(\tilde{\mathsf{f}}_n)$ satisfies $(\star\star)$.

Lemma

For $m \in \mathbf{N}$ and $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ such that $m \geqslant 2q + |\boldsymbol{\alpha}| + 2$ we have $\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{(1+|\boldsymbol{\xi}|^2)^{\frac{m}{2}}} \in \mathrm{C}^q(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

$$(\forall\, |\boldsymbol{\alpha}| \leqslant m) \quad \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{(1+|\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}} \in \mathrm{C}^q(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$$

Localisation principle: proof 2/2

Applying $\mathcal{A}_{\psi_n^{m+2q+2,0}}$ we get

$$\sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m} \mathcal{A}_{(2\pi i)^{|\boldsymbol{\alpha}|}\psi_n^{m+2q+2,\boldsymbol{\alpha}}}(\varphi\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^p(\mathbf{R}^d;\mathbf{C}^q)\,,$$

where $\psi_n^{m+2q+2,\boldsymbol{\alpha}} := \frac{(\varepsilon_n\boldsymbol{\xi})^{\boldsymbol{\alpha}}}{(1+|\varepsilon_n\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}}$.

Localisation principle: proof 2/2

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where $\psi_n^{m+2q+2,\boldsymbol{\alpha}} := \frac{(\varepsilon_n\boldsymbol{\xi})^{\boldsymbol{\alpha}}}{(1+|\varepsilon_n\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}}$.

After applying $\mathcal{A}_{\psi(\varepsilon_n\cdot)}$, for $\psi\in \mathrm{C}^q(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, to the above sum, forming a tensor product with $\varphi_1\mathsf{v}_n$, for $\varphi_1\in \mathrm{C}_c^\infty(\Omega)$, and taking the complex conjugation, for the (i,j) component of the above sum we get

$$\begin{aligned} 0 &= \sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m}\sum_{s=1}^d \overline{\lim_n \int_{\mathbf{R}^d} \mathcal{A}_{(2\pi i)^{|\boldsymbol{\alpha}|}\psi_n\psi_n^{m+2q+2,\boldsymbol{\alpha}}}(\varphi A^{\boldsymbol{\alpha}}_{js} u^s_n)\overline{\varphi_1 v^k_n}\, d\mathbf{x}} \\ &= \sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m}\sum_{s=1}^d \Big\langle (2\pi i)^{|\boldsymbol{\alpha}|}\psi^{m+2q+2,\boldsymbol{\alpha}} A^{\boldsymbol{\alpha}}_{js}\nu^{ks}_{\mathrm{K}_{0,\infty}}, \bar\varphi\varphi_1\boxtimes\bar\psi\Big\rangle \\ &= \Bigg\langle \sum_{0\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{(1+|\boldsymbol{\xi}|^2)^{\frac{m}{2}+q+1}}[\mathbf{A}^{\boldsymbol{\alpha}}\boldsymbol{\nu}^\top_{\mathrm{K}_{0,\infty}}]_{jk}, \bar\varphi\varphi_1\boxtimes\bar\psi\Bigg\rangle\,. \end{aligned}$$

Outline

Example 4: oscillations - two characteristic length

$0 < \alpha < \beta$, $\mathsf{k}, \mathsf{s} \in \mathbf{Z}^d \setminus \{0\}$,

$$u_n(\mathbf{x}) := e^{2\pi i (n^\alpha \mathsf{s} + n^\beta \mathsf{k}) \cdot \mathbf{x}} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,, \ n \to \infty$$

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$$u_n(\mathbf{x}) := e^{2\pi i (n^\alpha \mathsf{s} + n^\beta \mathsf{k})\cdot \mathbf{x}} \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0\,, \; n \to \infty$$

$$\mu_H = \lambda(\mathbf{x}) \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi})$$

$$\mu^{(\omega_n)}_{\mathrm{K}_{0,\infty}} = \lambda(\mathbf{x}) \boxtimes \begin{cases} \delta_{0\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \omega_n = 0 \\ \delta_{c\mathsf{k}}(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \omega_n = c \in \langle 0, \infty \rangle \\ \delta_{\infty\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \omega_n = \infty \end{cases}$$

Lower order term n^α and corresponding direction of oscillations s we cannot resemble in any case.
Therefore, we need some new methods and/or tools.

In [T3] Tartar introduced multi-scale objects, called multi-scale H-measures.
$\omega_n^1, \dots, \omega_n^l \to 0^+$, $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$, $\psi \in \mathrm{C}_0(\mathbf{R}^{ld})$:

$$\lim_{n'} \int_{\mathbf{R}^d} \Big(\widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi})\Big) \psi(\omega_{n'}^1 \boldsymbol{\xi}, \dots, \omega_{n'}^l \boldsymbol{\xi}) \, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}^{(\omega_{n'}^1), \dots, (\omega_{n'}^l)}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,.$$

Our approach: instead of $\psi(\omega_{n'}^1 \boldsymbol{\xi}, \dots, \omega_{n'}^l \boldsymbol{\xi})$ work with $\psi(\omega_n^1 \xi_1, \dots, \omega_n^d \xi_d)$.

For example, starting from parabolic H-measure construct parabolic one-scale H-measure (an object with two scales in the ratio 1:2).

$$\lim_{n'} \int_{\mathbf{R}^{d+1}} \widehat{\varphi_1 \mathsf{u}_{n'}}(\tau, \boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\tau, \boldsymbol{\xi}) \psi(\varepsilon_{n'}^2 \tau, \varepsilon_{n'} \boldsymbol{\xi}) \, d\tau d\boldsymbol{\xi} = \langle \boldsymbol{\nu}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,.$$

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