The propagation principle for fractional H-measures

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Joint work with Ivan Ivec

Let $\Omega \subseteq \mathbf{R}^d$ open and $u_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ satisfies

$$\mathsf{b} \cdot \nabla u_n + c u_n = f_n \,,$$

where $\mathsf{b} \in \mathrm{C}^1(\Omega; \mathbf{R}^d)$, $c \in \mathrm{C}(\Omega)$, and $f_n \longrightarrow 0$ in $\mathrm{H}^{-1}_{\mathrm{loc}}(\Omega)$.

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Theorem (Tartar, 1990)

If $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu} \in \mathcal{M}(\Omega \times \mathrm{S}^{d-1}; \mathrm{M}_r(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *we have*

$$\lim_{n'} \int_{\mathbf{R}^d} \Big(\widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \Big) \psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,.$$

Measure $\boldsymbol{\mu}$ *we call the H-measure corresponding to the (sub)sequence* (u_n).

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$\mu \sim u_n$
What we can tell about (the support) of μ?

Theorem (Tartar, 1990)

Let $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, *and let for a given* $m \in \mathbf{N}$

$$\sum_{|\boldsymbol{\alpha}| \leqslant m} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) \longrightarrow 0 \quad \text{strongly in} \quad \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^q)\,,$$

where $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M}_{q\times r}(\mathbf{C}))$, *then for the associated H-measure* $\boldsymbol{\mu}$ *we have*

$$\mathbf{p}_{pr} \boldsymbol{\mu}^{\top} = \mathbf{0}\,,$$

where

$$\mathbf{p}_{pr}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}| = m} (2\pi i)^m \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\ , \quad (\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathrm{S}^{d-1}\,.$$

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$$\mathsf{div}\,(\mathsf{b}u_n) + (c - \mathsf{div}\,\mathsf{b})u_n = f_n \quad\Longrightarrow\quad \underbrace{(\boldsymbol{\xi}\cdot\mathsf{b})}_{p}\,\mu = 0$$

Propagation principle for $\mathsf{b}\cdot\nabla u_n + c u_n = f_n$

If in addition we assume:

- $f_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ ($\boldsymbol{\nu} \sim (u_n, f_n)$, thus $\mu = \nu^{11}$)
- $\mathsf{b} \in X^1(\mathbf{R}^d) := \left\{ b \in \mathcal{S}' : k\hat{b} \in \mathrm{L}^1(\mathbf{R}^d) \right\}$, where $k(\boldsymbol{\xi}) := (1 + |2\pi\boldsymbol{\xi}|^2)^{\frac{1}{2}}$

then we have

Theorem (Tartar, 1990)

$$(\forall\, \Phi \in \mathrm{C}^1_c(\Omega \times \mathrm{S}^{d-1})) \qquad \langle \mu, \{\Phi, p\}\rangle + \langle(-\mathrm{div}\,\mathsf{b} + 2\mathrm{Re}\, c), \Phi\rangle = \langle 2\mathrm{Re}\,\nu^{12}, \Phi\rangle$$

Poisson bracket: $\{\psi, \varphi\} := \nabla^{\boldsymbol{\xi}}\psi \cdot \nabla_{\mathbf{x}}\varphi - \nabla_{\mathbf{x}}\psi \cdot \nabla^{\boldsymbol{\xi}}\varphi$

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Conclusion: Oscillations and concentration effects propagate along bicharacteristic rays defined by

$$\begin{cases} \dfrac{d\mathbf{x}}{ds} &= \nabla^{\boldsymbol{\xi}} p \\ \dfrac{d\boldsymbol{\xi}}{ds} &= -\nabla_{\mathbf{x}} p \end{cases} .$$

Second commutation lemma

Fourier multiplier: $\mathcal{A}_\psi u := (\psi \hat{u})^\vee$; for $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ we have $\mathcal{A}_\psi \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$
Operator of multiplication: $B_\varphi u := \varphi u$; for $\varphi \in \mathrm{L}^\infty(\mathbf{R}^d)$ we have
$B_\varphi \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$
Commutator: $C := [\mathcal{A}_\psi, B_\varphi] = \mathcal{A}_\psi B_\varphi - B_\varphi \mathcal{A}_\psi$

$$X^m(\mathbf{R}^d) := \left\{ u \in \mathcal{S}' : k^m \hat{u} \in \mathrm{L}^1(\mathbf{R}^d) \right\} \quad , \quad k(\boldsymbol{\xi}) := (1 + |2\pi\boldsymbol{\xi}|^2)^{\frac{1}{2}}$$

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Let $\psi \in \mathrm{C}^1(\mathrm{S}^{d-1})$ *and* $\varphi \in X^1(\mathbf{R}^d)$, *then* C *is continuous from* $\mathrm{L}^2(\mathbf{R}^d)$ *to* $\mathrm{H}^1(\mathbf{R}^d)$, *and up to a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$ *we have*

$$\partial_j C = \mathcal{A}_{\xi_j \nabla^{\boldsymbol{\xi}} \tilde{\psi}} B_{\nabla_{\mathbf{x}} \varphi} \,,$$

where $\tilde{\psi}(\boldsymbol{\xi}) := \psi(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})$.

Fractional H-measures

- surface: $\mathrm{S}^{d-1} := \left\{\boldsymbol{\xi} \in \mathbf{R}^d : \xi_1^2 + \xi_2^2 + \cdots + \xi_d^2 = 1\right\}$,
- curves: $\mathbf{R}^+ \ni s \mapsto s\boldsymbol{\eta} \in \mathbf{R}^d \setminus \{0\}$ $(\boldsymbol{\eta} \in \mathrm{S}^{d-1})$
- projection: $\boldsymbol{\pi} : \mathbf{R}^d \setminus \{0\} \longrightarrow \mathrm{S}^{d-1}$,

$$\boldsymbol{\pi}(\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$$

Theorem

If $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu} \in \mathcal{M}(\Omega \times \mathrm{S}^{d-1}; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *we have*

$$\lim_{n'} \int_{\mathbf{R}^d} \left(\widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi})\right)(\psi \circ \boldsymbol{\pi})(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle .$$

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- surface: $\mathrm{S}^{d-1} := \left\{\boldsymbol{\xi} \in \mathbf{R}^d : \frac{\xi_1^2}{\alpha_1} + \frac{\xi_2^2}{\alpha_2} + \cdots + \frac{\xi_d^2}{\alpha_d} = \frac{1}{\alpha_{\min}}\right\}$, $\boldsymbol{\alpha} \in \langle 0, 1]^d$
- curves: $\mathbf{R}^+ \ni s \mapsto \mathrm{diag}\{s^{\frac{1}{\alpha_1}}, \ldots, s^{\frac{1}{\alpha_d}}\}\boldsymbol{\eta} \in \mathbf{R}^d \setminus \{0\}$ $(\boldsymbol{\eta} \in Q)$
- projection: $\boldsymbol{\pi}_Q : \mathbf{R}^d \setminus \{0\} \longrightarrow Q$,

$$\boldsymbol{\pi}_Q(\boldsymbol{\xi}) = \left(\frac{\xi_1}{s(\boldsymbol{\xi})^{\frac{1}{\alpha_1}}}, \ldots, \frac{\xi_d}{s(\boldsymbol{\xi})^{\frac{1}{\alpha_d}}}\right),$$

where s is implicitly given.

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$\alpha_1 = \cdots = \alpha_d = 1 \quad \Longrightarrow \quad$ H-measure

$\alpha_1 = \frac{1}{2}, \alpha_2 = \cdots = \alpha_d = 1 \quad \Longrightarrow \quad$ parabolic H-measure [Antonić, Lazar, '07]

Example (oscillations)

Let $\mathsf{k} \in \mathbf{R}^d \setminus \{0\}$ and let us define

$$u_n(\mathbf{x}) := e^{2\pi i \mathsf{k}\cdot(n^2x^1, nx^2, \dots, nx^d)} \longrightarrow 0 \text{ in } \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)\,.$$

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H-measure:

$$\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}) = \lambda(\mathbf{x})\delta_{(1,0,\dots,0)}(\boldsymbol{\xi})$$

Fractional H-measure (with $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \cdots = \alpha_d = 1$):

$$\boldsymbol{\mu}_Q(\mathbf{x}, \boldsymbol{\xi}) = \lambda(\mathbf{x})\delta_{\boldsymbol{\pi}_Q(\mathsf{k})}$$

Second commutation lemma (generalisation)

Fourier multiplier: $\mathcal{A}_\psi u := (\psi \hat{u})^\vee$; for $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ we have $\mathcal{A}_\psi \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$
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$$X^m(\mathbf{R}^d) := \left\{ u \in \mathcal{S}' : k^m \hat{u} \in \mathrm{L}^1(\mathbf{R}^d) \right\} \quad , \quad k(\boldsymbol{\xi}) := (1 + |2\pi\boldsymbol{\xi}|^2)^{\frac{1}{2}}$$

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Let $\psi \in \mathrm{C}^1(\mathrm{S}^{d-1})$ and $\varphi \in X^1(\mathbf{R}^d)$, then C is continuous from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{H}^1(\mathbf{R}^d)$, and up to a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$ we have

$$\partial_j C = \mathcal{A}_{\xi_j \nabla_{\boldsymbol{\xi}} \tilde{\psi}} B_{\nabla_{\mathbf{x}} \varphi} \,,$$

where $\tilde{\psi} := \psi \circ \boldsymbol{\pi}$.

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$$X^{m\boldsymbol{\alpha}}(\mathbf{R}^d) := \left\{u \in \mathcal{S}' : k^m_{\boldsymbol{\alpha}}\hat{u} \in \mathrm{L}^1(\mathbf{R}^d)\right\} \quad , \quad k_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) := 1 + |\xi_1|^{\alpha_1} + \cdots + |\xi_d|^{\alpha_d}$$

$$\mathrm{H}^{s\boldsymbol{\alpha}}(\mathbf{R}^d) := \left\{u \in \mathcal{S}' : k^s_{\boldsymbol{\alpha}}\hat{u} \in \mathrm{L}^2(\mathbf{R}^d)\right\}$$

For $m \in 0..d$ we assume $\alpha_1, \dots, \alpha_m \in \langle 0, 1\rangle$, $\alpha_{m+1} = \cdots = \alpha_d = 1$. We also use $\mathbf{x} = (\bar{\mathbf{x}}, \mathbf{x}')$, $\bar{\mathbf{x}} = (x^1, \dots, x^m)$, $\mathbf{x}' = (x^{m+1}, \dots, x^d)$.

Theorem

Let $\psi \in \mathrm{C}^1(\mathrm{S}^{d-1})$ and $\varphi \in X^{\boldsymbol{\alpha}}(\mathbf{R}^d)$, then C is continuous from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{H}^{\boldsymbol{\alpha}}(\mathbf{R}^d)$, and up to a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$ we have

$$\partial_j^{\alpha_j} C = \mathcal{A}_{\frac{(2\pi i \xi_j)^{\alpha_j}}{2\pi i} \nabla_{\boldsymbol{\xi}'} \tilde{\psi}} B_{\nabla_{\mathbf{x}'}\phi} \,.$$

where $\tilde{\psi} := \psi \circ \boldsymbol{\pi}_Q$.

Example 1/3

Let us consider

$$i\partial_t u^n + (a u^n_{xx})_{xx} = f^n \text{ in } \mathbf{R}\times\mathbf{R}\,,$$

where $a \in X^{(\frac{1}{4},1)}(\mathbf{R}^2)$ is real and $f_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^2)$. In addition, let us assume $u^n_{xx} \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^2)$.

We study a fractional H-measure with $\boldsymbol{\alpha} = (\frac{1}{4}, 1)$, i.e. on the

$$Q \ldots \quad \tau^2 + \frac{\xi^2}{4} = 1\,.$$

We want to derive a transport equation for the corresponding fractional H-measure μ associated to (u^n_{xx}).

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For $\psi \in \mathrm{C}^1(Q)$ and $\varphi \in \mathrm{C}^1_c(\mathbf{R}^2)$ we first apply $B_\varphi \mathcal{A}_\psi$ on the equation above, and then take the scalar product in $\mathrm{L}^2(\mathbf{R}^2)$ by u^n_x:

$$\langle\, i\phi P_\psi u_t \mid u_x \,\rangle + \langle\, \phi P_\psi (a(x) u_{xx})_{xx} \mid u_x \,\rangle = \langle\, \phi P_\psi f \mid u_x \,\rangle.$$

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After some more calculation (mostly using partial integration), on the limit we get

$$4\langle \mu, a\partial_x\varphi\psi\rangle - \langle \mu, \varphi\partial_x a\psi\rangle - \lim_n \langle\, \varphi\partial_x[\mathcal{A}_\psi, B_a]u^n_{xx} \mid u^n_{xx} \,\rangle = 0\,.$$

By the Second commutation lemma we have

$$\begin{aligned}\lim_n \langle\, \varphi\partial_x[\mathcal{A}_\psi, B_a]u^n_{xx} \mid u^n_{xx} \,\rangle &= \lim_n \langle\, \varphi\xi\partial^\xi\tilde{\psi}\partial_x a u^n_{xx} \mid u^n_{xx} \,\rangle \\ &= \lim_n \langle\, \varphi(\xi\partial^\xi\tilde{\psi}) \circ \boldsymbol{\pi}_Q \partial_x a u^n_{xx} \mid u^n_{xx} \,\rangle \\ &= \langle \mu, \varphi\partial_x a\xi\partial^\xi\tilde{\psi} \rangle \,,\end{aligned}$$

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so finally we obtain

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Now we want to rewrite the above equality in terms of the principle symbol $p(t,x;\tau,\xi) := 2\pi\tau - 16\pi^4 \xi^4 a$. Taking $\Psi := \varphi\psi$ we have

$$\left\langle \frac{\mu}{\xi^3}, \{\Psi, p\} \right\rangle + \left\langle \frac{\mu}{\xi^4}, \Psi \partial_x p \right\rangle \quad (\{\Psi, p\} = \partial^\xi \Psi \partial_x p - \partial_x \Psi \partial^\xi p) \,.$$

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$$\left\langle \frac{\mu}{\xi^3}, \{\Psi, p\}\right\rangle + \left\langle \frac{\mu}{\xi^4}, \Psi\partial_x p\right\rangle \quad (\{\Psi, p\} = \partial^\xi\Psi\partial_x p - \partial_x\Psi\partial^\xi p)\,.$$

Substituting ψ by $\xi^3\psi$ we get

$$\langle \mu, \{\Psi, p\}\rangle + \left\langle \mu, \Psi\frac{3\alpha^2(5-\alpha^2)}{16(\alpha^2-1)}\xi\partial_x p\right\rangle = 0\,,$$

where $\alpha = (1 - \frac{3}{16}\xi^2)^{-\frac{1}{2}}$.

Example 2/3

$$\partial_x \mu \left(\partial^\xi p - \Big(\frac{\alpha^2}{16} + \frac{\alpha^2}{4} + \frac{3\alpha^2(5-\alpha^2)}{16(\alpha^2-1)} \Big) p\xi \right) - \nabla^{\tau,\xi}\mu \cdot \left(\begin{bmatrix} 0 \\ \partial_x p \end{bmatrix} - \Big(\begin{bmatrix} 0 \\ \partial_x p \end{bmatrix} \cdot \mathsf{n} \Big) \mathsf{n} \right) = 0\,,$$

where $\mathsf{n} = \alpha[\tau \ \xi/4]$ is the outwardly directed unit normal vector to Q.