# Friedrichs operators as dual pairs

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Joint work with N. Antonić and A. Michelangeli

# Abstract Friedrichs operators

$(L, \langle \cdot \mid \cdot \rangle)$ complex Hilbert space ($L' \equiv L$), $\|\cdot\| := \sqrt{\langle \cdot \mid \cdot \rangle}$
$\mathcal{D} \subseteq L$ dense subspace

## Definition

Let $T, \widetilde{T} : \mathcal{D} \to L$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

(T1) $$(\forall\, \phi, \psi \in \mathcal{D}) \qquad \langle\, T\phi \mid \psi \,\rangle = \langle\, \phi \mid \widetilde{T}\psi \,\rangle\,;$$

(T2) $$(\exists\, c > 0)(\forall\, \phi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\phi\| \leqslant c\|\phi\|\,;$$

(T3) $$(\exists\, \mu_0 > 0)(\forall\, \phi \in \mathcal{D}) \qquad \langle\, (T + \widetilde{T})\phi \mid \phi \,\rangle \geqslant \mu_0 \|\phi\|^2\,.$$

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems*, Comm. Partial Diff. Eq. **32** (2007) 317–341.

Assumptions:
$d, r \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary;
$\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega)^{r\times r}$, $k \in \{1, \dots, d\}$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega)^{r\times r}$ satisfying (a.e. on $\Omega$):

(F1) $$\mathbf{A}_k = \mathbf{A}_k^* \,;$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant \mu_0 \mathbf{I}\,.$$

$L := \mathrm{L}^2(\Omega)^r$, $\mathcal{D} := \mathrm{C}_c^\infty(\Omega)^r$;
Define $T, \widetilde{T} : \mathcal{D} \to L$ by

$$T\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}\,, \qquad \widetilde{T}\mathsf{u} := -\sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \Big(\mathbf{C}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k\Big)\mathsf{u}$$

$(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators.

To solve $T\mathsf{u} = \mathsf{f}$ for $\mathsf{f} \in L$.

K. O. Friedrichs: *Symmetric positive linear differential equations*, Commun. Pure Appl. Math. **11** (1958) 333–418.

Unified treatment of linear hyperbolic systems like Maxwell's, Dirac's, or higher order equations (e.g. the wave equation).

– Contributions: C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...
– treating the equations of mixed type, such as the Tricomi equation:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;$$

– unified treatment of equations and systems of different type;
– more recently: better numerical properties.

## The classical theory in short

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Shortcommings:
– no satisfactory well-posedness result,
– no intrinsic (unique) way to pose boundary conditions.

# Back to the (new) abstract theory


A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems*, Comm. Partial Diff. Eq. **32** (2007) 317–341.

N. Antonić, K. Burazin: *Intrinsic boundary conditions for Friedrichs systems*, Comm. Partial Diff. Eq. **35** (2010) 1690–1715.


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It is more convenient to first extend $T$ and $\widetilde{T}$ and then seek for suitable restrictions. In [Ern at al., 2007] a construction of $(T_1, \widetilde{T}_1)$ such that

$$T \subseteq T_1 \ , \quad \widetilde{T} \subseteq \widetilde{T}_1 \ , \quad \operatorname{dom} T_1 = \operatorname{dom} \widetilde{T}_1 =: W \ ,$$

and $(W, \langle \cdot \mid \cdot \rangle_{T_1})$ is a Hilbert space ($\langle \cdot \mid \cdot \rangle_{T_1} := \langle \cdot \mid \cdot \rangle + \langle T_1 \cdot \mid T_1 \cdot \rangle$).

New goal: To find $V, \widetilde{V} \subseteq W$ such that $W_0 \subseteq V, \widetilde{V}$ and restrictions $T_1|_V : V \to L$, $\widetilde{T}_1|_{\widetilde{V}} : \widetilde{V} \to L$ are bijections (here $W_0 := \overline{(\mathcal{D}, \langle \cdot \mid \cdot \rangle_{T_1})}$).

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Questions:

1) Sufficient conditions on $V$
2) Existence of such $V$
3) Infinity of such $V$
4) Classification of such $V$

Boundary operator: $D : (W, \langle \cdot \mid \cdot \rangle_{T_1}) \to (W, \langle \cdot \mid \cdot \rangle_{T_1})'$,

$$_{W'}\langle\, Du, v \,\rangle_W := \langle\, T_1 u \mid v \,\rangle - \langle\, u \mid \widetilde{T}_1 v \,\rangle\,, \qquad u, v \in W\,.$$

Properties: $\ker D = W_0$ and $D$ symmetric, i.e.

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(V1)
$$ \begin{aligned} &(\forall\, u \in V) && {}_{W'}\langle\, Du, u \,\rangle_W \geqslant 0 \\ &(\forall\, v \in \widetilde{V}) && {}_{W'}\langle\, Dv, v \,\rangle_W \leqslant 0 \end{aligned} $$

(V2)
$$ V = D(\widetilde{V})^0 \,, \qquad \widetilde{V} = D(V)^0 \,. $$

**Theorem (Ern, Guermond, Caplain, 2007)**

*Let $(T, \widetilde{T})$ be a joint pair of Friedrichs systems and let $(V, \widetilde{V})$ satisfies (V1)–(V2). Then $T_1|_V : V \to L$ and $\widetilde{T}_1|_{\widetilde{V}} : \widetilde{V} \to L$ are closed bijective realisations of $T$ and $\widetilde{T}$, respectively.*

# Example 2 (Scalar elliptic PDE)

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$$\iff T\mathsf{v} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{v}) + \mathbf{C}\mathsf{v} = \mathsf{g}\,,$$

where $\mathsf{v} := [\mathsf{p}\ u]^\top$, $\mathsf{g} := [0\ f]^\top$, $(\mathbf{A}_k)_{ij} := \delta_{i,k}\delta_{j,d+1} + \delta_{i,d+1}\delta_{j,k}$, $\mathbf{C} := \operatorname{diag}\{1, \dots, 1, \mu\}$.
Assumtions (F1) and (F2) are satisfied.

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Assumtions (F1) and (F2) are satisfied.

$L = \mathrm{L}^2(\Omega)^{d+1}$, $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$

- $V = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)$ … Dirichelt boundary condition ($u = 0$ on $\Gamma$)
- $V = \mathrm{L}^2_{\mathrm{div},0}(\Omega) \times \mathrm{H}^1(\Omega)$ … Neumann boundary condition ($\mathsf{p} \cdot \nu = \nabla u \cdot \nu = 0$ on $\Gamma$)

**Theorem (Ern, Guermond, Caplain, 2007)**

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Can we say something more about extensions $T_1$, $\widetilde{T}_1$, and (V1)–(V2) conditions?

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Can we say something more about extensions $T_1$, $\widetilde{T}_1$, and (V1)–(V2) conditions?

**Theorem**

*Let $T, \widetilde{T} : \mathcal{D} \to L$. The pair $(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators iff*

(i) $T \subseteq \widetilde{T}^*$ *and* $\widetilde{T} \subseteq T^*$;

(ii) $\overline{T + \widetilde{T}}$ *is a bounded self-adjoint operator in $L$ with strictly positive bottom;*

(iii) $\operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} = W_0$ *and* $\operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* = W$.

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(i) $\operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} = W_0$ *and* $\operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* = W$;

(ii) $T_1 = \widetilde{T}^*$ *and* $\widetilde{T}_1 = T^*$.

As

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For $\operatorname{dom} \overline{T} = W_0 \subseteq V, \widetilde{V} \subseteq W = \operatorname{dom} T^*$

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## Theorem

*Let $(T, \widetilde{T})$ be a pair of operators on the Hilbert space $L$ satisfying conditions* (T1)-(T2), *and let $(V, \widetilde{V})$ be a pair of subspaces of $L$. Then*

$$\textit{condition } (\mathrm{V2}) \quad \iff \quad \begin{cases} W_0 \subseteq V \subseteq W, \;\; W_0 \subseteq \widetilde{V} \subseteq W \\ V \;\; \textit{and} \;\; \widetilde{V} \;\; \textit{closed in} \;\; W \\ (\widetilde{T}^*|_V)^* \; = \; T^*|_{\widetilde{V}} \\ (T^*|_{\widetilde{V}})^* \; = \; \widetilde{T}^*|_V \,. \end{cases}$$

We are seeking for bijective closed operators $S \equiv \widetilde{T}^*|_V$ such that

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and thus also $S^*$ is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$.

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In the rest we work with closed $T$ and $\widetilde{T}$.

## Definition

Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators on the Hilbert space $L$. For a closed $T \subseteq S \subseteq \widetilde{T}^*$ such that $(\mathrm{dom}\, S, \mathrm{dom}\, S^*)$ satisfies $(V1)$ we call $(S, S^*)$ an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.

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3) Infinity of such $V$
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# Existence and infinity of $V$'s

## Theorem (Antonić, E., Michelangeli, 2017)

*Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators on the Hilbert space $L$.*

(i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$. Moreover, there is an adjoint pair $(T_{\mathrm{r}}, T_{\mathrm{r}}^*)$ of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$ such that*

$$W_0 + \ker T^* \subseteq \operatorname{dom} T_{\mathrm{r}} \qquad \textit{and} \qquad W_0 + \ker \widetilde{T}^* \subseteq \operatorname{dom} T_{\mathrm{r}}^* .$$

(ii) *If both $\ker \widetilde{T}^* \neq \{0\}$ and $\ker T^* \neq \{0\}$, then the pair $(T, \widetilde{T})$ admits uncountably many adjoint pairs of bijective realisations with signed boundary map. On the other hand, if either $\ker \widetilde{T}^* = \{0\}$ or $\ker T^* = \{0\}$, then there is exactly one adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$. Such a pair is precisely $(\widetilde{T}^*, \widetilde{T})$ when $\ker \widetilde{T}^* = \{0\}$, and $(T, T^*)$ when $\ker T^* = \{0\}$.*

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The idea of the proof: $(\widehat{W}, [\,\cdot \mid \cdot\,])$ is a Kreĭn space, where $\widehat{W} := W/W_0$ and for $u, v \in W$

$$[\, u + W_0 \mid v + W_0 \,] := {}_{W'}\langle\, Du, v \,\rangle_W = \langle\, \widetilde{T}^* u \mid v \,\rangle - \langle\, u \mid T^* v \,\rangle \,.$$

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$$[\, u + W_0 \mid v + W_0 \,] := {}_{W'}\langle\, Du, v \,\rangle_W = \langle\, \widetilde{T}^* u \mid v \,\rangle - \langle\, u \mid T^* v \,\rangle \,.$$

Hence, there exist $X_+, X_- \subseteq \widetilde{W}$ such that $\widetilde{W} = X_+ [+] X_-$, and $(X_+, [\,\cdot \mid \cdot\,])$ and $(X_-, -[\,\cdot \mid \cdot\,])$ are Hilbert spaces.

Each choice of $X_+, X_-$ determines $V, \widetilde{V}$ satisfying (V1)–(V2).

Questions:

1) Sufficient conditions on $V$ ✓
2) Existence of $V \subseteq W$ such that $(\widetilde{T}^*|_V, (\widetilde{T}^*|_V)^*)$ is an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$ ✓
3) Infinity of such $V$ ✓
4) Classification of such $V$

# Grubb's universal classification 1/2

$$A_0 \subseteq (A_0')^* =: A_1 \quad \text{and} \quad A_0' \subseteq (A_0)^* =: A_1'$$

$(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$ are closed, satisfy $A_0 \subseteq A_{\mathrm{r}} \subseteq A_1$, equivalently $A_0' \subseteq A_{\mathrm{r}}^* \subseteq A_1'$, and are invertible with everywhere defined bounded inverses $A_{\mathrm{r}}^{-1}$ and $(A_{\mathrm{r}}^*)^{-1}$

$$\begin{aligned} \operatorname{dom} A_1 &= \operatorname{dom} A_{\mathrm{r}} \dotplus \ker A_1 \quad \text{and} \quad \operatorname{dom} A_1' = \operatorname{dom} A_{\mathrm{r}}^* \dotplus \ker A_1' \\ p_{\mathrm{r}} &= A_{\mathrm{r}}^{-1} A_1 , \qquad p_{\mathrm{r}'} = (A_{\mathrm{r}}^*)^{-1} A_1' , \\ p_{\mathrm{k}} &= \mathbb{1} - p_{\mathrm{r}} , \qquad p_{\mathrm{k}'} = \mathbb{1} - p_{\mathrm{r}'} , \end{aligned}$$

$$\left.\begin{array}{r} (A, A^*) \\ A_0 \subseteq A \subseteq A_1 \\ A_0' \subseteq A^* \subseteq A_1' \end{array}\right\} \longleftrightarrow \begin{cases} (B, B^*) \\ \mathcal{V} \subseteq \ker A_1 \ \text{closed} \\ \mathcal{W} \subseteq \ker A_1' \ \text{closed} \\ B : \mathcal{V} \to \mathcal{W} \ \text{densely defined} \end{cases}$$

$$B \mapsto A_B : \quad \operatorname{dom} A_B = \Big\{ u \in \operatorname{dom} A_1 \, : \, p_{\mathrm{k}} u \in \operatorname{dom} B , \ P_{\mathcal{W}}(A_1 u) = B(p_{\mathrm{k}} u) \Big\} ,$$

$$A \mapsto B_A : \quad \operatorname{dom} B_A = p_{\mathrm{k}} \operatorname{dom} A , \quad \mathcal{V} = \overline{\operatorname{dom} B_A} , \quad B_A(p_{\mathrm{k}} u) = P_{\mathcal{W}}(A_1 u) ,$$

where $P_{\mathcal{W}}$ is the *orthogonal* projections from $L$ onto $\mathcal{W}$.

Important: $A$ is injective, resp. surjective, resp. bijective, if and only if so is $B$.

When $A_B$ corresponds to $B$ as above, then

$$\operatorname{dom} A_B = \left\{ w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu \;\middle|\; \begin{array}{c} w_0 \in \operatorname{dom} A_0 \\ \nu \in \operatorname{dom} B \\ \nu' \in \ker A_1' \ominus \mathcal{W} \end{array} \right\},$$

$$A_B(w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu) = A_0 w_0 + B\nu + \nu'$$

G. Grubb: *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa **22** (1968) 425–513.

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We shall apply this theory on a joint pair of closed abstract Friedrichs systems.

For simplicity here we use the notation of Grubb's universal classification.

$(A_0, A_0')$ a joint pair of closed abstract Friedrichs operators, $A_1 := (A_0')^*$, $A_1' := A_0^*$, and let $(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$ be an adjoint pair of bijective realisations with signed boundary map relative to $(A_0, A_0')$.

$(A_B, A_B^*)$ a generic pair of closed extensions $A_0 \subseteq A_B \subseteq A_1$.

# Classification of bijective realisations with signed boundary map 2/2

(1)
$$(\forall\, \nu \in \operatorname{dom} B) \quad (\forall\, \nu' \in \ker A_1' \ominus \mathcal{W}) \qquad \begin{cases} \langle\, \nu \mid A_1'\nu \,\rangle - 2\,\mathfrak{Re}\langle\, p_{\mathrm{k}'}\nu \mid B\nu \,\rangle \leqslant 0 \\ \langle\, p_{\mathrm{k}'}\nu \mid \nu' \,\rangle = 0 \end{cases}$$

(2)
$$(\forall\, \mu' \in \operatorname{dom} B^*) \quad (\forall\, \mu \in \ker A_1 \ominus \mathcal{V}) \qquad \begin{cases} \langle\, A_1\mu' \mid \mu' \,\rangle - 2\,\mathfrak{Re}\langle\, B^*\mu' \mid p_{\mathrm{k}}\mu' \,\rangle \leqslant 0 \\ \langle\, \mu \mid p_{\mathrm{k}}\mu' \,\rangle = 0\,, \end{cases}$$

**Theorem (Antonić, E., Michelangeli, 2017)**

*Any of the following three facts,*

(a) *conditions* (1) *and* (2) *hold true, or*

(b) *condition* (1) *holds true and* $B : \operatorname{dom} B \to \mathcal{W}$ *is a bijection, or*

(c) *condition* (2) *holds true and* $B^* : \operatorname{dom} B^* \to \mathcal{V}$ *is a bijection,*

*is sufficient for* $(A_B, A_B^*)$ *to be another adjoint pair of bijective realisations with signed boundary map relative to* $(A_0, A_0')$.
*Assume further that* $\operatorname{dom} A_{\mathrm{r}} = \operatorname{dom} A_{\mathrm{r}}^*$. *Then the following properties are equivalent:*

(a) $(A_B, A_B^*)$ *is another adjoint pair of bijective realisations with signed boundary map relative to* $(A_0, A_0')$*;*

(b) *the mirror conditions* (1) *and* (2) *are satisfied.*

$L := \mathrm{L}^2(0,1)$, $\mathcal{D} := \mathrm{C}_c^\infty(0,1)$
$T, \widetilde{T} : \mathcal{D} \to L$,

$$T\phi := \frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi \qquad \text{and} \qquad \widetilde{T}\phi := -\frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi \,.$$

We have

$$\operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} = \mathrm{H}_0^1(0,1) =: W_0$$
$$\operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* = \mathrm{H}^1(0,1) =: W \,,$$

Define

$$A_0 := \overline{T} \,, \quad , A_0' := \overline{\widetilde{T}} \,, \quad A_1 := \widetilde{T}^* \,, A_1' := T^* \,.$$

As ${}_{W'}\langle \, Du, v \, \rangle_W = u(1)\overline{v(1)} - u(0)\overline{v(0)}$, for

$$V := \widetilde{V} := \{u \in \mathrm{H}^1(0,1) : u(0) = u(1)\}$$

we have that $A_\mathrm{r} := A_1|_V$, $A_\mathrm{r}^* = A_1'|_V$ for an adjoint pair of bijective realisations with signed boundary map.
$\ker A_1 = \operatorname{span}\{e^{-x}\}$ and $\ker A_1' = \operatorname{span}\{e^x\}$, so

$$p_\mathrm{k} u = -\frac{u(1) - u(0)}{1 - e^{-1}} e^{-x} \,, \quad p_{\mathrm{k}'} u = \frac{u(1) - u(0)}{e - 1} e^x \,.$$

$\mathcal{V} = \ker A_1$, $\mathcal{W} = \ker A_1'$, $B_{\alpha,\beta} : \mathcal{V} \to \mathcal{W}$,

$$B_{\alpha,\beta} e^{-x} = (\alpha + \mathrm{i}\beta) e^{x}$$

where $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$.
(1) simplifies to check

$$\begin{aligned} &\langle\, e^{-x} \mid A_1' e^{-x} \,\rangle - 2\Re\langle\, p_{\mathrm{k}' e^{-x}} \mid B_{\alpha,\beta} e^{-x} \,\rangle \leqslant 0 \\ \iff &\alpha \leqslant -e^{-1} \end{aligned}$$

$$\{(A_{\alpha,\beta}, A_{\alpha,\beta}^*) : \alpha \leqslant -e^{-1},\ \beta \in \mathbb{R}\} \cup \{(A_{\mathrm{r}}, A_{\mathrm{r}}^*)\}$$

$$\begin{aligned} \mathrm{dom}\, A_{\alpha,\beta}^{(*)} = \Big\{ u \in \mathrm{H}^1(0,1) : \Big(2e^{-1} - (+)\alpha(1+e) - \mathrm{i}\beta(1+e)\Big) u(1) \\ = \Big(2 + \alpha(1+e) - (+)\mathrm{i}\beta(1+e)\Big) u(0) \Big\} \end{aligned}$$

# ...thank you for your attention :)

N. Antonić, M.E., A. Michelangeli: *Friedrichs systems in a Hilbert space framework: solvability and multiplicity*, J. Differential Equations 263 (2017) 8264-8294.