Friedrichs operators as dual pairs and contact interactions

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Joint work with N. Antonić and A. Michelangeli

Classical Friedrichs operators

Assumptions:
$d, r \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary;
$\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega)^{r\times r}$, $k \in \{1, \dots, d\}$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega)^{r\times r}$ satisfying (a.e. on Ω):

(F1) $$\mathbf{A}_k = \mathbf{A}_k^* \,;$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant \mu_0 \mathbf{I}\,.$$

Define $\mathcal{L}, \widetilde{\mathcal{L}} : \mathrm{L}^2(\Omega)^r \to \mathcal{D}'(\Omega)^r$ by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{C}\mathsf{u}\,, \qquad \widetilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) + \Big(\mathbf{C}^* + \sum_{k=1}^d \partial_k\mathbf{A}_k\Big)\mathsf{u}\,.$$

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Aim: impose boundary conditions such that for any $\mathsf{f} \in \mathrm{L}^2(\Omega)^r$ we have a unique solution of $\mathcal{L}\mathsf{u} = \mathsf{f}$.
Gain: many important (semi)linear equations of mathematical physics can be written in the form of classical Friedrichs operators.

The classical theory in short

K. O. Friedrichs: *Symmetric positive linear differential equations*, Commun. Pure Appl. Math. **11** (1958) 333–418.

Unified treatment of linear hyperbolic systems like Maxwell's, Dirac's, or higher order equations (e.g. the wave equation).

– Contributions: C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...
– treating the equations of mixed type, such as the Tricomi equation:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;$$

– unified treatment of equations and systems of different type;
– more recently: better numerical properties.

Shortcommings:
– no satisfactory well-posedness result,
– no intrinsic (unique) way to pose boundary conditions.

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$\rightsquigarrow$ development of the abstract theory

$(L, \langle \cdot \mid \cdot \rangle)$ complex Hilbert space ($L' \equiv L$), $\| \cdot \| := \sqrt{\langle \cdot \mid \cdot \rangle}$
$\mathcal{D} \subseteq L$ dense subspace

Definition

Let $T, \widetilde{T} : \mathcal{D} \to L$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

(T1) $$(\forall\, \phi, \psi \in \mathcal{D}) \qquad \langle\, T\phi \mid \psi \,\rangle = \langle\, \phi \mid \widetilde{T}\psi \,\rangle\,;$$

(T2) $$(\exists\, c > 0)(\forall\, \phi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\phi\| \leqslant c\|\phi\|\,;$$

(T3) $$(\exists\, \mu_0 > 0)(\forall\, \phi \in \mathcal{D}) \qquad \langle\, (T + \widetilde{T})\phi \mid \phi \,\rangle \geqslant \mu_0 \|\phi\|^2\,.$$

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems*, Comm. Partial Diff. Eq. **32** (2007) 317–341.

N. Antonić, K. Burazin: *Intrinsic boundary conditions for Friedrichs systems*, Comm. Partial Diff. Eq. **35** (2010) 1690–1715.

Classical is *abstract*

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(T1) $\langle\, T\mathsf{u} \mid \mathsf{v}\,\rangle_{\mathrm{L}^2} = \langle\, \mathsf{u} \mid -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k^* \mathsf{v}) + \big(\mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\big)\mathsf{v}\,\rangle_{\mathrm{L}^2} \overset{\text{(F1)}}{=} \langle\, \mathsf{u} \mid \widetilde{T}\mathsf{v}\,\rangle_{\mathrm{L}^2}\,.$

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Since $(T+\widetilde{T})\mathsf{u} = \Big(\mathbf{C} + \mathbf{C}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k\Big)\mathsf{u}$,

(T2) $\|(T+\widetilde{T})\mathsf{u}\|_{\mathrm{L}^2} \leqslant \Big(2\|\mathbf{C}\|_{\mathrm{L}^\infty} + \sum_{k=1}^d \|\mathbf{A}_k\|_{\mathrm{W}^{1,\infty}}\Big)\|\mathsf{u}\|_{\mathrm{L}^2}\,,$

(T3) $\langle\, (T+\widetilde{T})\mathsf{u} \mid \mathsf{u}\,\rangle_{\mathrm{L}^2} \overset{\text{(F2)}}{\geqslant} \mu_0\|\mathsf{u}\|_{\mathrm{L}^2}^2\,.$

Well-posedness result

Goal: For $(T, \widetilde{T})$ satisfying (T1)–(T3) find $V \supseteq \mathcal{D}$ ($\widetilde{V} \supseteq \mathcal{D}$) such that T ($\widetilde{T}$) extended to V ($\widetilde{V}$) is a linear bijection.

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$\exists$ *maximal operators* :
$$\begin{aligned} T_1 : W \subseteq L \to L\,, \quad T \subseteq T_1\,, \\ T_1 : W \subseteq L \to L\,, \quad T \subseteq T_1\,. \end{aligned} \qquad (\operatorname{dom} T_1 = \operatorname{dom} \widetilde{T}_1 =: W)$$

Boundary map (form):
$$\begin{aligned} & D : W \times W \to \mathbb{C}\,, \\ & D[u,v] := \langle\, T_1 u \mid v \,\rangle - \langle\, u \mid \widetilde{T}_1 v \,\rangle\,. \end{aligned} \qquad (D[u,v] = \overline{D[v,u]})$$

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For $V, \widetilde{V} \subseteq W$ we introduce two conditions:

(V1)
$$(\forall\, u \in V) \qquad D[u,u] \geqslant 0$$
$$(\forall\, v \in \widetilde{V}) \qquad D[v,v] \leqslant 0$$

(V2)
$$V = \{u \in W : (\forall\, v \in \widetilde{V}) \quad D[v,u] = 0\}$$
$$\widetilde{V} = \{v \in W : (\forall\, u \in V) \quad D[u,v] = 0\} \qquad (\implies \mathcal{D} \subseteq V \cap \widetilde{V})$$

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Theorem (Ern, Guermond, Caplain, 2007)

(T1)–(T3) + (V1)–(V2) $\implies T_1|_V, \widetilde{T}_1|_{\widetilde{V}}$ *bijective realisations*

Theorem

$$(T1)-(T3) \iff \begin{cases} T \subseteq \widetilde{T}^* \quad \& \quad \widetilde{T} \subseteq T^* ; \\ \overline{T+\widetilde{T}} \text{ bounded self-adjoint in } L \text{ with strictly positive bottom}; \\ \operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} \quad \& \quad \operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* . \end{cases}$$

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$T_1 = \widetilde{T}^*$ *and* $\widetilde{T}_1 = T^*$.

Hilbert space framework

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Theorem

If $(T, \widetilde{T})$ *satisfies (T1)–(T2), then*

$$(V2) \iff \begin{cases} \mathcal{D} \subseteq V, \widetilde{V} \subseteq W \\ (\widetilde{T}^*|_V)^* = T^*|_{\widetilde{V}} \\ (T^*|_{\widetilde{V}})^* = \widetilde{T}^*|_V . \end{cases}$$

We are seeking for bijective closed operators $S \equiv \widetilde{T}^*|_V$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^* \,,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$. If $(\operatorname{dom} S, \operatorname{dom} S^*)$ satisfies $(V1)$ we call (S, S^*) an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.

Bijective realisations with signed boundary map

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Theorem

Let $(T, \widetilde{T})$ satisfies (T1)–(T3).

(i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.*

(ii)

$\ker \widetilde{T}^* \neq \{0\}$ & $\ker T^* \neq \{0\} \implies$ *uncountably many adjoint pairs of bijective realisations with signed boundary map*

$\ker \widetilde{T}^* = \{0\}$ *or* $\ker T^* = \{0\} \implies$ *only one adjoint pair of bijective realisations with signed boundary map*

Classification

For $(T, \widetilde{T})$ satisfying (T1)–(T3) we have

$$\overline{T} \subseteq \widetilde{T}^* \qquad \text{and} \qquad \overline{\widetilde{T}} \subseteq T^* ,$$

while by the previous theorem there exists closed T_{r} such that

- $\overline{T} \subseteq T_{\mathrm{r}} \subseteq \widetilde{T}^*$ ($\iff \overline{\widetilde{T}} \subseteq T_{\mathrm{r}}^* \subseteq T^*$),
- $T_{\mathrm{r}} : \operatorname{dom} T_{\mathrm{r}} \to L$ bijection,
- $(T_{\mathrm{r}})^{-1} : L \to \operatorname{dom} T_{\mathrm{r}}$ bounded.

Thus, we can apply a universal classification (classification of dual (adjoint) pairs).

We used Grubb's universal classification

G. Grubb: *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa **22** (1968) 425–513.

Result: complete classification of all adjoint pairs of bijective realisations with signed boundary map.
To do: apply this result to general classical Friedrichs operators form the beginning (*nice class of non-self-adjoint differential operators of interest*)

‘δ-extensions’ realised as Friedrichs systems 1/4

On $\mathrm{L}^2(\mathbb{R})$ we consider

$$\mathring{H} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \qquad \operatorname{dom}\mathring{H} := \mathrm{C}_c^\infty(\mathbb{R}\setminus\{0\})\,.$$

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Order reduction

$$f = \mathring{H}u \iff \begin{pmatrix} \dot{u} \\ f \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}x} \\ \frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix}}_{=:S} \begin{pmatrix} -\dot{u} \\ u \end{pmatrix}$$

$(S, -S)$ satisfies (T1) and (T2), but not coercivity condition (T3). Thus, on $L := \mathrm{L}^2(\mathbb{R}) \oplus \mathrm{L}^2(\mathbb{R})$ we define

$$\begin{aligned} T &:= S + \mathbb{1} \\ \widetilde{T} &:= -S + \mathbb{1} \end{aligned}\,, \qquad \operatorname{dom} T := \operatorname{dom} \widetilde{T} := \mathrm{C}_c^\infty(\mathbb{R}\backslash\{0\}) \oplus \mathrm{C}_c^\infty(\mathbb{R}\backslash\{0\})\,.$$

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How to return to the second order differential operator?

Definition

$$
\begin{aligned}
\Phi \ &: \ \mathfrak{L}(\mathrm{L}^2(\mathbb{R}) \oplus \mathrm{L}^2(\mathbb{R})) \longrightarrow \mathfrak{L}(\mathrm{L}^2(\mathbb{R}))\,, \\
\operatorname{dom} \Phi(A) \ &:= \ \left\{ u \in \mathrm{L}^2(\mathbb{R}) : (\exists\,!\, v_u \in \mathrm{L}^2(\mathbb{R})) \ \begin{pmatrix} v_u \\ u \end{pmatrix} \in \operatorname{dom} A \cap \ker P_1 A \right\}, \\
\Phi(A)\, u \ &:= \ P_2 A \begin{pmatrix} v_u \\ u \end{pmatrix}\,,
\end{aligned}
$$

where $\mathfrak{L}(X)$ is the space of *linear* (not necessarily bounded) maps on the vector space X and $P_j : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $j \in \{1, 2\}$, is the orthogonal projection onto the j-th component of L.

$$
\begin{aligned}
&\begin{pmatrix} v \\ u \end{pmatrix} \in \ker P_1 T \iff \dot{u} + v = 0 \iff -\dot{u} = v =: v_u \\
&\quad \implies \Phi(T)u = \dot{v_u} + u = -\ddot{u} + u
\end{aligned}
$$

Lemma

(i) $(T, \widetilde{T})$ *satisfies (T1)–(T3)*

(ii) $\operatorname{dom} \Phi(T) = \mathrm{C}_c^\infty(\mathbb{R}\setminus\{0\})$ *and* $\Phi(T_\lambda) \ = \ \mathring{H} + \mathbb{1}$.

$$\begin{array}{rcl} T^* & := & -S + \mathbb{1} \\ \widetilde{T}^* & := & S + \mathbb{1} \end{array}, \qquad \operatorname{dom} T^* \ := \ \operatorname{dom} \widetilde{T}^* \ := \ \mathrm{H}^1(\mathbb{R}\backslash\{0\}) \oplus \mathrm{H}^1(\mathbb{R}\backslash\{0\}) \,.$$

$\dim T^* = \dim \widetilde{T}^* = 2 \implies$ 4 parameter family of extensions
We focus on a specific one-parameter subfamily of extensions ($z \in \mathbb{C}$):

$T_z := \widetilde{T}^*|_{\operatorname{dom} T_z}$, where

$$\operatorname{dom} T_z \ = \ \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\mathbb{R}\backslash\{0\}) \oplus H^1(\mathbb{R}) \ : \ u_1(0^+) - u_1(0^-) = \frac{2}{z+1} u_2(0) \right\} .$$

$T_z^* = T^*|_{\operatorname{dom} T_z^*}$, where

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\operatorname{dom} T_z^* \;&=\; \left\{\begin{pmatrix}u_1\\u_2\end{pmatrix} \in H^1(\mathbb{R}\backslash\{0\}) \oplus H^1(\mathbb{R}) \;:\; u_1(0^+) - u_1(0^-) = \frac{-2}{\overline{z}+1}u_2(0)\right\}
\end{aligned}$$

Applying Φ we get ($u_2 \to u$, $u_1 \to -\dot{u}$)

$$\begin{aligned}
\operatorname{dom} \Phi(T_z) \;&=\; \left\{u \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R}) : \dot{u}(0^+) - \dot{u}(0^-) = \frac{-2}{z+1}u(0)\right\}\\
\Phi(T_z)\, u \;&=\; -\ddot{u} + u\,,
\end{aligned}$$

and analogously for T_z^* ($u_2 \to u$, $u_1 \to \dot{u}$)

$$\begin{aligned}
\operatorname{dom} \Phi(T_z^*) \;&=\; \left\{u \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R}) : \dot{u}(0^+) - \dot{u}(0^-) = \frac{-2}{\overline{z}+1}u(0)\right\}\\
\Phi(T_z^*)\, u \;&=\; -\ddot{u} + u\,;
\end{aligned}$$

It can be shown that in our case Φ preserves self-adjointness, i.e.

$$\Phi(T_z)^* \;=\; \Phi(T_z^*) \;\Longrightarrow\; \Big(\Phi(T_z) = \Phi(T_z)^* \iff z \in \mathbb{R}\Big)$$

...thank you for your attention :)

N. Antonić, M.E., A. Michelangeli: *Friedrichs systems in a Hilbert space framework: solvability and multiplicity*, J. Differential Equations 263 (2017) 8264-8294.

M.E., A. Michelangeli: *On contact interactions realised as Friedrichs systems*, SISSA;48/2017/MATE