Hilbert space approach to PDEs of Friedrichs type

Marko Erceg

Scuola Internazionale Superiore di Studi Avanzati (SISSA)
and
Department of Mathematics, Faculty of Science, University of Zagreb

Junior Trieste Quantum Days
Trieste, 19^{th} May, 2017

Joint work with N. Antonić, K. Burazin, I. Crnjac and A. Michelangeli

$(L, \langle \cdot \mid \cdot \rangle_L)$ complex Hilbert space ($L' \equiv L$), $\| \cdot \|_L := \sqrt{\langle \cdot \mid \cdot \rangle_L}$
$\mathcal{D} \subseteq L$ dense subspace

Definition

Let $T, \widetilde{T} : \mathcal{D} \to L$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

(T1) $\qquad (\forall\, \phi, \psi \in \mathcal{D}) \qquad \langle\, T\phi \mid \psi \,\rangle_L = \langle\, \phi \mid \widetilde{T}\psi \,\rangle_L\,;$

(T2) $\qquad (\exists\, c > 0)(\forall\, \phi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\phi\|_L \leqslant c\|\phi\|_L\,;$

(T3) $\qquad (\exists\, \mu_0 > 0)(\forall\, \phi \in \mathcal{D}) \qquad \langle\, (T + \widetilde{T})\phi \mid \phi \,\rangle_L \geqslant \mu_0\|\phi\|_L^2\,.$

Example 1 (Classical Friedrichs operators)

$\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary
$L := \mathrm{L}^2(\Omega)^r$, $\mathcal{D} := \mathrm{C}_c^\infty(\Omega)^r$
Let $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_{r\times r})$, $k \in \{1, \ldots, d\}$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_{r\times r})$ satisfy (a.e. on Ω):

(F1) $$\mathbf{A}_k = \mathbf{A}_k^* \,;$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant \mu_0 \mathbf{I}\,.$$

Define $T, \widetilde{T} : \mathcal{D} \to L$ by

$$T\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

$$\widetilde{T}\mathsf{u} := -\sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \Big(\mathbf{C}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k\Big)\mathsf{u}$$

$(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators.

Motivation

Goal: To find $V \supseteq \mathcal{D}$ $(\widetilde{V} \supseteq \mathcal{D})$ such that T $(\widetilde{T})$ extended to V $(\widetilde{V})$ is a linear bijection.

Motivation

Goal: To find $V \supseteq \mathcal{D}$ ($\widetilde{V} \supseteq \mathcal{D}$) such that T ($\widetilde{T}$) extended to V ($\widetilde{V}$) is a linear bijection.

It is more convenient to first extend T and $\widetilde{T}$ and then seek for suitable restrictions. In [Ern at al., 2007] a construction of $(T_1, \widetilde{T}_1)$ such that

$$T \subseteq T_1 \ , \quad \widetilde{T} \subseteq \widetilde{T}_1 \ , \quad \operatorname{dom} T_1 = \operatorname{dom} \widetilde{T}_1 =: W \ ,$$

and $(W, \langle \cdot \mid \cdot \rangle_{T_1})$ is a Hilbert space.

New goal: To find $V, \widetilde{V} \subseteq W$ such that $W_0 \subseteq V, \widetilde{V}$ and restrictions $T_1|_V : V \to L$, $\widetilde{T}_1|_{\widetilde{V}} : \widetilde{V} \to L$ are bijections (here $W_0 := \overline{(\mathcal{D}, \langle \cdot \mid \cdot \rangle_{T_1})}$).

Motivation

Goal: To find $V \supseteq \mathcal{D}$ ($\widetilde{V} \supseteq \mathcal{D}$) such that T ($\widetilde{T}$) extended to V ($\widetilde{V}$) is a linear bijection.

It is more convenient to first extend T and $\widetilde{T}$ and then seek for suitable restrictions. In [Ern at al., 2007] a construction of $(T_1, \widetilde{T}_1)$ such that

$$T \subseteq T_1 \ , \quad \widetilde{T} \subseteq \widetilde{T}_1 \ , \quad \operatorname{dom} T_1 = \operatorname{dom} \widetilde{T}_1 =: W \ ,$$

and $(W, \langle \cdot \mid \cdot \rangle_{T_1})$ is a Hilbert space.

New goal: To find $V, \widetilde{V} \subseteq W$ such that $W_0 \subseteq V, \widetilde{V}$ and restrictions $T_1|_V : V \to L$, $\widetilde{T}_1|_{\widetilde{V}} : \widetilde{V} \to L$ are bijections (here $W_0 := \overline{(\mathcal{D}, \langle \cdot \mid \cdot \rangle_{T_1})}$).

Questions:

1) Sufficient conditions on V
2) Existence of such V
3) Infinity of such V
4) Classification of such V

Well-posedness result

Boundary operator: $D : (W, \langle \cdot \mid \cdot \rangle_{T_1}) \to (W, \langle \cdot \mid \cdot \rangle_{T_1})'$,

$$_{W'}\langle Du, v \rangle_W := \langle T_1 u \mid v \rangle_L - \langle u \mid \widetilde{T}_1 v \rangle_L \,, \qquad u, v \in W \,.$$

Properties: $\ker D = W_0$ and D symmetric, i.e.

$$(\forall\, u, v \in W) \quad {}_{W'}\langle Du, v \rangle_W = {}_{W'}\langle Dv, u \rangle_W \,.$$

Well-posedness result

Boundary operator: $D : (W, \langle \cdot \mid \cdot \rangle_{T_1}) \to (W, \langle \cdot \mid \cdot \rangle_{T_1})'$,

$$ {}_{W'}\langle\, Du, v \,\rangle_W := \langle\, T_1 u \mid v \,\rangle_L - \langle\, u \mid \widetilde{T}_1 v \,\rangle_L \,, \qquad u, v \in W \,. $$

Properties: $\ker D = W_0$ and D symmetric, i.e.

$$ (\forall\, u, v \in W) \quad {}_{W'}\langle\, Du, v \,\rangle_W = {}_{W'}\langle\, Dv, u \,\rangle_W \,. $$

(V1)
$$ (\forall\, u \in V) \qquad {}_{W'}\langle\, Du, u \,\rangle_W \geqslant 0 $$
$$ (\forall\, v \in \widetilde{V}) \qquad {}_{W'}\langle\, Dv, v \,\rangle_W \leqslant 0 $$

(V2)
$$ V = D(\widetilde{V})^0 \,, \qquad \widetilde{V} = D(V)^0 \,. $$

Theorem (Ern, Guermond, Caplain, 2007)

Let $(T, \widetilde{T})$ be a joint pair of Friedrichs systems and let $(V, \widetilde{V})$ satisfies (V1)–(V2). Then $T_1|_V : V \to L$ and $\widetilde{T}_1|_{\widetilde{V}} : \widetilde{V} \to L$ are closed bijective realisations of T and $\widetilde{T}$, respectively.

Example 2 (Scalar elliptic PDE)

$\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary Γ, $\mu \in \mathrm{L}^\infty(\Omega)$ such that $\mu(x) \geqslant \mu_0 > 0$ (a.e. $x \in \Omega$).
For $f \in \mathrm{L}^2(\Omega)$ we consider

$$-\triangle u + \mu u = f$$

Example 2 (Scalar elliptic PDE)

$\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary Γ, $\mu \in \mathrm{L}^\infty(\Omega)$ such that $\mu(x) \geqslant \mu_0 > 0$ (a.e. $x \in \Omega$).
For $f \in \mathrm{L}^2(\Omega)$ we consider

$$-\triangle u + \mu u = f \iff -\operatorname{div} \nabla u + \mu u = f$$

$\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary Γ, $\mu \in \mathrm{L}^\infty(\Omega)$ such that $\mu(x) \geqslant \mu_0 > 0$ (a.e. $x \in \Omega$).
For $f \in \mathrm{L}^2(\Omega)$ we consider

$$-\triangle u + \mu u = f \iff -\operatorname{div} \nabla u + \mu u = f \iff \begin{cases} \nabla u + \mathsf{p} &= 0 \\ \operatorname{div} \mathsf{p} + \mu u &= f \end{cases}$$

Example 2 (Scalar elliptic PDE)

$\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary Γ, $\mu \in \mathrm{L}^\infty(\Omega)$ such that $\mu(x) \geqslant \mu_0 > 0$ (a.e. $x \in \Omega$).
For $f \in \mathrm{L}^2(\Omega)$ we consider

$$-\triangle u + \mu u = f \iff -\operatorname{div} \nabla u + \mu u = f \iff \begin{cases} \nabla u + \mathsf{p} = 0 \\ \operatorname{div} \mathsf{p} + \mu u = f \end{cases}$$

$$\iff T\mathsf{v} := \sum_{k=1}^{d} \partial_k (\mathbf{A}_k \mathsf{v}) + \mathbf{C}\mathsf{v} = \mathsf{g}\,,$$

where $\mathsf{v} := [\mathsf{p}\ u]^\top$, $\mathsf{g} := [0\ f]^\top$, $(\mathbf{A}_k)_{ij} := \delta_{i,k}\delta_{j,d+1} + \delta_{i,d+1}\delta_{j,k}$,
$\mathbf{C} := \operatorname{diag}\{1, \ldots, 1, \mu\}$.
Assumtions (F1) and (F2) are satisfied.

$\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary Γ, $\mu \in \mathrm{L}^\infty(\Omega)$ such that $\mu(x) \geqslant \mu_0 > 0$ (a.e. $x \in \Omega$).
For $f \in \mathrm{L}^2(\Omega)$ we consider

$$-\triangle u + \mu u = f \iff -\operatorname{div} \nabla u + \mu u = f \iff \begin{cases} \nabla u + \mathsf{p} = 0 \\ \operatorname{div} \mathsf{p} + \mu u = f \end{cases}$$

$$\iff T\mathsf{v} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{v}) + \mathbf{C}\mathsf{v} = \mathsf{g}\,,$$

where $\mathsf{v} := [\mathsf{p}\ u]^\top$, $\mathsf{g} := [0\ f]^\top$, $(\mathbf{A}_k)_{ij} := \delta_{i,k}\delta_{j,d+1} + \delta_{i,d+1}\delta_{j,k}$,
$\mathbf{C} := \mathrm{diag}\{1, \dots, 1, \mu\}$.
Assumtions (F1) and (F2) are satisfied.

$L = \mathrm{L}^2(\Omega)^{d+1}$, $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$

- $V = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)$ ... Dirichelt boundary condition ($u = 0$ on Γ)
- $V = \mathrm{L}^2_{\mathrm{div},0}(\Omega) \times \mathrm{H}^1(\Omega)$ ... Neumann boundary condition ($\mathsf{p} \cdot \nu = \nabla u \cdot \nu = 0$ on Γ)

$$\text{(P)} \qquad \begin{cases} \mathsf{u}'(t) + T_1\mathsf{u}(t) = \mathsf{f} \\ \mathsf{u}(0) = \mathsf{u}_0 \end{cases},$$

where $\mathsf{u} : [0, \tau] \to L$, for $\tau > 0$, is the unknown function, while the right-hand side $\mathsf{f} : \langle 0, \tau\rangle \to L$ (or $\mathsf{f} : \langle 0, \tau\rangle \times L \to L$ in the semi-linear case), the initial data $\mathsf{u}_0 \in L$ and the abstract Friedrichs operator T_1 (an extension of T as before), not depending on the time variable t, are given.

Theorem

Let $(T, \widetilde{T})$ be a joint pair of Friedrichs operators, and $(V, \widetilde{V})$ a pair of subspaces satisfying (V) conditions. Then $-T_1|_V$ is an infinitesimal generator of a contraction C_0-semigroup on L.

Theorem

Let $(T, \widetilde{T})$ be a joint pair of Friedrichs operators, and $(V, \widetilde{V})$ a pair of subspaces satisfying (V) conditions.

a) *If* $\mathsf{f} \in \mathrm{L}^1(\langle 0, \tau \rangle; L)$, *then for every* $\mathsf{u}_0 \in L$ *the problem* (P) *has the unique mild solution* $\mathsf{u} \in \mathrm{C}([0, \tau]; L)$ *given by*

$$\mathsf{u}(t) = S(t)\mathsf{u}_0 + \int_0^t S(t-s)\mathsf{f}(s)ds\,, \qquad t \in [0, \tau]\,,$$

where $(S(t))_{t \geqslant 0}$ *is a contraction* C_0*-semigroup generated by* $-T_1|_V$.

b) *If additionally* $\mathsf{u}_0 \in V$ *and* $\mathsf{f} \in \mathrm{W}^{1,1}(\langle 0, \tau \rangle; L) \cup \Big(\mathrm{C}([0, \tau]; L) \cap \mathrm{L}(\langle 0, \tau \rangle; V)\Big)$, *with* V *equipped by the graph norm, then the above weak solution is the classical solution of* (P) *on* $[0, \tau]$.

c) *If* $\mathsf{f} : [0, \tau] \times L \to L$ *is continuous and locally Lipschitz in the last variable, with Lipschitz constant not depending on the first variable, then for every* $\mathsf{u}_0 \in L$ *there exists* τ_{max}, *such that the semi-linear problem* (P) *has unique mild solution* $\mathsf{u} \in \mathrm{C}([0, \tau_{max}]; L)$.

Example 3 (Dirac system)

$$a\gamma^0\partial_t\psi + \gamma^1\partial_1\psi + \gamma^2\partial_2\psi + \gamma^3\partial_3\psi + B\psi = f\,,$$

where $\psi : [0,\tau]\times\mathbb{R}^3 \to \mathbb{C}^4$ is the unknown function, while $f : \langle 0,\tau\rangle \to \mathbb{C}^4$ (or $f : \langle 0,\tau\rangle \times \mathbb{C}^4 \to \mathbb{C}^4$ in the semi-linear case), $a > 0$ and $B = \begin{bmatrix} b_1 I & 0 \\ 0 & b_2 I \end{bmatrix}$, with $b_1, b_2 : \mathbb{R}^3 \to \mathbb{C}$ and I denotes 2×2 unit matrix, are given, and

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\,, \quad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}\,,$$

where σ^k are Pauli matrices.

Example 3 (Dirac system)

$$a\gamma^0\partial_t\psi + \gamma^1\partial_1\psi + \gamma^2\partial_2\psi + \gamma^3\partial_3\psi + B\psi = f\,,$$

where $\psi : [0,\tau] \times \mathbb{R}^3 \to \mathbb{C}^4$ is the unknown function, while $f : \langle 0,\tau\rangle \to \mathbb{C}^4$ (or $f : \langle 0,\tau\rangle \times \mathbb{C}^4 \to \mathbb{C}^4$ in the semi-linear case), $a > 0$ and $B = \begin{bmatrix} b_1 I & 0 \\ 0 & b_2 I \end{bmatrix}$, with $b_1, b_2 : \mathbb{R}^3 \to \mathbb{C}$ and I denotes 2×2 unit matrix, are given, and

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\,, \quad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}\,,$$

where σ^k are Pauli matrices.

$$\partial_t\psi + T\psi = F\,,$$

where $F = \frac{1}{a}\gamma^0 f$, while $T\psi = \sum_{k=1}^{3} A_k\partial_k\psi + C\psi$ with

$$A_k := \frac{1}{a}\begin{bmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{bmatrix} \quad \text{and} \quad C = \frac{1}{a}\gamma^0 B\,.$$

Example 3 (Dirac system)

$$a\gamma^0\partial_t\psi + \gamma^1\partial_1\psi + \gamma^2\partial_2\psi + \gamma^3\partial_3\psi + B\psi = f\,,$$

where $\psi : [0,\tau]\times\mathbb{R}^3 \to \mathbb{C}^4$ is the unknown function, while $f : \langle 0,\tau\rangle \to \mathbb{C}^4$ (or $f : \langle 0,\tau\rangle \times \mathbb{C}^4 \to \mathbb{C}^4$ in the semi-linear case), $a > 0$ and $B = \begin{bmatrix} b_1 I & 0 \\ 0 & b_2 I \end{bmatrix}$, with $b_1, b_2 : \mathbb{R}^3 \to \mathbb{C}$ and I denotes 2×2 unit matrix, are given, and

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\,, \quad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}\,,$$

where σ^k are Pauli matrices.

$$\partial_t\psi + T\psi = F\,,$$

where $F = \frac{1}{a}\gamma^0 f$, while $T\psi = \sum_{k=1}^{3} A_k\partial_k\psi + C\psi$ with

$$A_k := \frac{1}{a}\begin{bmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{bmatrix} \quad \text{and} \quad C = \frac{1}{a}\gamma^0 B\,.$$

T fits in Example 1, i.e. it is a Friedrichs operator.

Hilbert space framework

Theorem (Ern, Guermond, Caplain, 2007)

Let $(T, \widetilde{T})$ be a joint pair of Friedrichs systems and let $(V, \widetilde{V})$ satisfies (V1)–(V2). Then $T_1|_V : V \to L$ and $\widetilde{T}_1|_{\widetilde{V}} : \widetilde{V} \to L$ are closed bijective realisations of T and $\widetilde{T}$, respectively.

Can we say something more about extensions T_1, $\widetilde{T}_1$, and (V1)–(V2) conditions?

Theorem (Ern, Guermond, Caplain, 2007)

Let $(T, \widetilde{T})$ be a joint pair of Friedrichs systems and let $(V, \widetilde{V})$ satisfies (V1)–(V2). Then $T_1|_V : V \to L$ and $\widetilde{T}_1|_{\widetilde{V}} : \widetilde{V} \to L$ are closed bijective realisations of T and $\widetilde{T}$, respectively.

Can we say something more about extensions T_1, $\widetilde{T}_1$, and (V1)–(V2) conditions?

Theorem

Let $T, \widetilde{T} : \mathcal{D} \to L$. The pair $(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators iff

(i) $T \subseteq \widetilde{T}^*$ *and* $\widetilde{T} \subseteq T^*$;

(ii) $\overline{T + \widetilde{T}}$ *is a bounded self-adjoint operator in* L *with strictly positive bottom;*

(iii) $\operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} = W_0$ *and* $\operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* = W$.

Theorem (Ern, Guermond, Caplain, 2007)

Let $(T, \widetilde{T})$ be a joint pair of Friedrichs systems and let $(V, \widetilde{V})$ satisfies (V1)–(V2). Then $T_1|_V : V \to L$ and $\widetilde{T}_1|_{\widetilde{V}} : \widetilde{V} \to L$ are closed bijective realisations of T and $\widetilde{T}$, respectively.

Can we say something more about extensions T_1, $\widetilde{T}_1$, and (V1)–(V2) conditions?

Theorem

Let $T, \widetilde{T} : \mathcal{D} \to L$. The pair $(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators iff

(i) $T \subseteq \widetilde{T}^*$ *and* $\widetilde{T} \subseteq T^*$;

(ii) $\overline{T + \widetilde{T}}$ *is a bounded self-adjoint operator in* L *with strictly positive bottom;*

(iii) $\operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} = W_0$ *and* $\operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* = W$.

Theorem

(i) $\operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} = W_0$ *and* $\operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* = W$;

(ii) $T_1 = \widetilde{T}^*$ *and* $\widetilde{T}_1 = T^*$.

Theorem

Let $(T, \widetilde{T})$ *be a pair of operators on the Hilbert space* L *satisfying conditions* (T1)-(T2), *and let* $(V, \widetilde{V})$ *be a pair of subspaces of* L. *Then*

$$\text{condition (V2)} \quad \Leftrightarrow \quad \begin{cases} W_0 \subseteq V \subseteq W, \;\; W_0 \subseteq \widetilde{V} \subseteq W \\ V \;\; \text{and} \; \widetilde{V} \;\; \text{closed in} \; W \\ (\widetilde{T}^*|_V)^* \; = \; T^*|_{\widetilde{V}} \\ (T^*|_{\widetilde{V}})^* \; = \; \widetilde{T}^*|_V \, . \end{cases}$$

Theorem

Let $(T, \widetilde{T})$ *be a pair of operators on the Hilbert space* L *satisfying conditions* (T1)-(T2), *and let* $(V, \widetilde{V})$ *be a pair of subspaces of* L. *Then*

$$\textit{condition}\ \mathrm{(V2)} \quad \Leftrightarrow \quad \begin{cases} W_0 \subseteq V \subseteq W, \quad W_0 \subseteq \widetilde{V} \subseteq W \\ V \ \textit{and}\ \widetilde{V}\ \textit{closed in}\ W \\ (\widetilde{T}^*|_V)^* \ = \ T^*|_{\widetilde{V}} \\ (T^*|_{\widetilde{V}})^* \ = \ \widetilde{T}^*|_V\,. \end{cases}$$

We are seeking for bijective closed operators $S \equiv \widetilde{T}^*|_V$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^*\,,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$.

Theorem

Let $(T,\widetilde{T})$ be a pair of operators on the Hilbert space L satisfying conditions (T1)-(T2)*, and let $(V,\widetilde{V})$ be a pair of subspaces of L. Then*

$$\textit{condition (V2)} \quad \Leftrightarrow \quad \begin{cases} W_0 \subseteq V \subseteq W, \quad W_0 \subseteq \widetilde{V} \subseteq W \\ V \ \textit{and}\ \widetilde{V} \ \textit{closed in}\ W \\ (\widetilde{T}^*|_V)^* \ = \ T^*|_{\widetilde{V}} \\ (T^*|_{\widetilde{V}})^* \ = \ \widetilde{T}^*|_V \,. \end{cases}$$

We are seeking for bijective closed operators $S \equiv \widetilde{T}^*|_V$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^* \,,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$.

In the rest we work with closed T and $\widetilde{T}$.

Definition

Let $(T,\widetilde{T})$ be a joint pair of closed abstract Friedrichs operators on the Hilbert space L. For a closed $T \subseteq S \subseteq \widetilde{T}^*$ such that $(\mathrm{dom}\, S, \mathrm{dom}\, S^*)$ satisfies $(V1)$ we call (S,S^*) an adjoint pair of bijective realisations with signed boundary map relative to $(T,\widetilde{T})$.

Questions:

1) Sufficient conditions on V ✓

Questions:

1) Sufficient conditions on V ✓
2) Existence of $V \subseteq W$ such that $(\widetilde{T}^*|_V, (\widetilde{T}^*|_V)^*)$ is an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$
3) Infinity of such V
4) Classification of such V

Theorem

Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators on the Hilbert space L.

(i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$. Moreover, there is an adjoint pair $(T_{\mathrm{r}}, T_{\mathrm{r}}^*)$ of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$ such that*

$$W_0 + \ker T^* \subseteq \operatorname{dom} T_{\mathrm{r}} \qquad \text{and} \qquad W_0 + \ker \widetilde{T}^* \subseteq \operatorname{dom} T_{\mathrm{r}}^* .$$

(ii) *If both $\ker \widetilde{T}^* \neq \{0\}$ and $\ker T^* \neq \{0\}$, then the pair $(T, \widetilde{T})$ admits uncountably many adjoint pairs of bijective realisations with signed boundary map. On the other hand, if either $\ker \widetilde{T}^* = \{0\}$ or $\ker T^* = \{0\}$, then there is exactly one adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$. Such a pair is precisely $(\widetilde{T}^*, \widetilde{T})$ when $\ker \widetilde{T}^* = \{0\}$, and (T, T^*) when $\ker T^* = \{0\}$.*

Grubb's universal classification 1/2

$$A_0 \subseteq (A_0')^* =: A_1 \quad \text{and} \quad A_0' \subseteq (A_0)^* =: A_1'$$

$(A_\mathrm{r}, A_\mathrm{r}^*)$ are closed, satisfy $A_0 \subseteq A_\mathrm{r} \subseteq A_1$, equivalently $A_0' \subseteq A_\mathrm{r}^* \subseteq A_1'$, and are invertible with everywhere defined bounded inverses A_r^{-1} and $(A_\mathrm{r}^*)^{-1}$

$$\operatorname{dom} A_1 = \operatorname{dom} A_\mathrm{r} \dotplus \ker A_1 \quad \text{and} \quad \operatorname{dom} A_1' = \operatorname{dom} A_\mathrm{r}^* \dotplus \ker A_1'$$

$$p_\mathrm{r} = A_\mathrm{r}^{-1} A_1 , \qquad p_{\mathrm{r}'} = (A_\mathrm{r}^*)^{-1} A_1' ,$$

$$p_\mathrm{k} = \mathbf{1} - p_\mathrm{r} , \qquad p_{\mathrm{k}'} = \mathbf{1} - p_{\mathrm{r}'} ,$$

$$\left.\begin{array}{r} (A, A^*) \\ A_0 \subseteq A \subseteq A_1 \\ A_0' \subseteq A^* \subseteq A_1' \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} (B, B^*) \\ \mathcal{V} \subseteq \ker A_1 \ \text{closed} \\ \mathcal{W} \subseteq \ker A_1' \ \text{closed} \\ B : \mathcal{V} \to \mathcal{W} \ \text{densely defined} \end{array}\right.$$

$B \mapsto A_B$: $\quad \operatorname{dom} A_B = \Big\{ u \in \operatorname{dom} A_1 \,:\, p_\mathrm{k} u \in \operatorname{dom} B ,\ P_\mathcal{W}(A_1 u) = B(p_\mathrm{k} u) \Big\} ,$

$A \mapsto B_A$: $\quad \operatorname{dom} B_A = p_\mathrm{k} \operatorname{dom} A , \quad \mathcal{V} = \overline{\operatorname{dom} B_A} , \quad B_A(p_\mathrm{k} u) = P_\mathcal{W}(A_1 u) ,$

where $P_\mathcal{W}$ is the *orthogonal* projections from L onto $\mathcal{W}$.

Important: A is injective, resp. surjective, resp. bijective, if and only if so is B.

When A_B corresponds to B as above, then

$$\operatorname{dom} A_B = \left\{ w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu \;\middle|\; \begin{array}{c} w_0 \in \operatorname{dom} A_0 \\ \nu \in \operatorname{dom} B \\ \nu' \in \ker A_1' \ominus \mathcal{W} \end{array} \right\},$$

$$A_B(w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu) = A_0 w_0 + B\nu + \nu'$$

When A_B corresponds to B as above, then

$$\operatorname{dom} A_B = \left\{ w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu \;\middle|\; \begin{array}{c} w_0 \in \operatorname{dom} A_0 \\ \nu \in \operatorname{dom} B \\ \nu' \in \ker A_1' \ominus \mathcal{W} \end{array} \right\},$$

$$A_B(w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu) = A_0 w_0 + B\nu + \nu'$$

We shall apply this theory on a joint pair of closed abstract Friedrichs systems.

For simplicity here we use the notation of Grubb's universal classification.

(A_0, A_0') a joint pair of closed abstract Friedrichs operators, $A_1 := (A_0')^*$, $A_1' := A_0^*$, and let $(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$ be an adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0').
(A_B, A_B^*) a generic pair of closed extensions $A_0 \subseteq A_B \subseteq A_1$.

Classification of bijective realisations with signed boundary map 2/2

(1) $\quad \begin{array}{l}(\forall\, \nu \in \operatorname{dom} B)\\ (\forall\, \nu' \in \ker A_1' \ominus \mathcal{W})\end{array} \qquad \begin{cases} \langle\, \nu \mid A_1'\nu \,\rangle_L - 2\,\mathfrak{Re}\langle\, p_{\mathrm{k}'}\nu \mid B\nu \,\rangle_L \leqslant 0 \\ \langle\, p_{\mathrm{k}'}\nu \mid \nu' \,\rangle_L = 0 \end{cases}$

(2) $\quad \begin{array}{l}(\forall\, \mu' \in \operatorname{dom} B^*)\\ (\forall\, \mu \in \ker A_1 \ominus \mathcal{V})\end{array} \qquad \begin{cases} \langle\, A_1\mu' \mid \mu' \,\rangle_L - 2\,\mathfrak{Re}\langle\, B^*\mu' \mid p_{\mathrm{k}}\mu' \,\rangle_L \leqslant 0 \\ \langle\, \mu \mid p_{\mathrm{k}}\mu' \,\rangle_L = 0\,, \end{cases}$

Theorem

Any of the following three facts,

(a) *conditions* (1) *and* (2) *hold true, or*

(b) *condition* (1) *holds true and* $B : \operatorname{dom} B \to \mathcal{W}$ *is a bijection, or*

(c) *condition* (2) *holds true and* $B^* : \operatorname{dom} B^* \to \mathcal{V}$ *is a bijection,*

is sufficient for (A_B, A_B^*) *to be another adjoint pair of bijective realisations with signed boundary map relative to* (A_0, A_0').
Assume further that $\operatorname{dom} A_{\mathrm{r}} = \operatorname{dom} A_{\mathrm{r}}^*$. *Then the following properties are equivalent:*

(a) (A_B, A_B^*) *is another adjoint pair of bijective realisations with signed boundary map relative to* (A_0, A_0');

(b) *the mirror conditions* (1) *and* (2) *are satisfied.*

$L := \mathrm{L}^2(0,1)$, $\mathcal{D} := \mathrm{C}_c^\infty(0,1)$
$T, \widetilde{T} : \mathcal{D} \to L$,

$$T\phi := \frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi \qquad \text{and} \widetilde{T}\phi := -\frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi\,.$$

We have

$$\operatorname{dom}\overline{T} = \operatorname{dom}\overline{\widetilde{T}} = \mathrm{H}_0^1(0,1) =: W_0$$
$$\operatorname{dom}T^* = \operatorname{dom}\widetilde{T}^* = \mathrm{H}^1(0,1) =: W\,,$$

Define

$$A_0 := \overline{T}\ , \quad , A_0' := \overline{\widetilde{T}}\ , \quad A_1 := \widetilde{T}^*\ , A_1' := T^*\ .$$

As ${}_{W'}\langle\, Du, v\,\rangle_W = u(1)\overline{v(1)} - u(0)\overline{v(0)}$, for

$$V := \widetilde{V} := \{u \in \mathrm{H}^1(0,1) : u(0) = u(1)\}$$

we have that $A_\mathrm{r} := A_1|_V$, $A_\mathrm{r}^* = A_1'|_V$ for an adjoint pair of bijective realisations with signed boundary map.
$\ker A_1 = \operatorname{span}\{e^{-x}\}$ and $\ker A_1' = \operatorname{span}\{e^x\}$, so

$$p_\mathrm{k}u = -\frac{u(1)-u(0)}{1-e^{-1}}e^{-x}\ , \quad p_{\mathrm{k}'}u = \frac{u(1)-u(0)}{e-1}e^x\ .$$

$\mathcal{V} = \ker A_1$, $\mathcal{W} = \ker A_1'$, $B_{\alpha,\beta} : \mathcal{V} \to \mathcal{W}$,

$$B_{\alpha,\beta} e^{-x} = (\alpha + \mathrm{i}\beta) e^{x}$$

where $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$.
(1) simplifies to check

$$\begin{aligned} &\langle\, e^{-x} \mid A_1' e^{-x} \,\rangle_L - 2\Re\langle\, p_{\mathrm{k}' e^{-x}} \mid B_{\alpha,\beta} e^{-x} \,\rangle_L \leqslant 0 \\ \iff &\alpha \leqslant -e^{-1} \end{aligned}$$

$$\{(A_{\alpha,\beta}, A^*_{\alpha,\beta}) : \alpha \leqslant -e^{-1},\ \beta \in \mathbb{R}\} \cup \{(A_{\mathrm{r}}, A^*_{\mathrm{r}})\}$$

$$\begin{aligned} \operatorname{dom} A^{(*)}_{\alpha,\beta} = \Big\{ u \in \mathrm{H}^1(0,1) : \Big(2e^{-1} - (+)\alpha(1+e) - \mathrm{i}\beta(1+e)\Big) u(1) \\ = \Big(2 + \alpha(1+e) - (+)\mathrm{i}\beta(1+e)\Big) u(0) \Big\} \end{aligned}$$