Exploring Limit Behaviour of Non-quadratic Terms via H-measures. Application to Small Amplitude Homogenisation.

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September 5, 2016
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WeConMApp

Outline

H-measures

- introduced around 1990. by L. Tartar and P. Gérard
- Radon measures associated to bounded $\mathrm{L}^2(\mathbf{R}^d)$ sequences

$$\mu \sim (u_n)$$

- express limit of $\int u_n^2$
- a microlocal defect tool
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(u_n) - bounded in $\mathrm{L}^2(\mathbf{R}^d)$, $\quad u_n \rightharpoonup 0$.

$$\mu = 0 \iff u_n \longrightarrow 0 \text{ in } \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$$

H-measures

Theorem 1. (Existence) [a]

Let $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$. There exists a subsequence $(\mathsf{u}_{n'})$ and a non-negative Radon measure μ_H on $\mathbf{R}^d \times S^{d-1}$ such that for all $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$, $\psi \in \mathrm{C}(S^{d-1})$:

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_n)(\mathbf{x}) \overline{(\varphi_2 u_n)(\mathbf{x})} \, d\mathbf{x} = \langle \mu_H, \varphi \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times S^{d-1}} \varphi \, \psi(\boldsymbol{\xi}) \, d\mu(\mathbf{x}, \boldsymbol{\xi}) \, .$$

where: $\mathcal{A}_\psi$ is the (Fourier) multiplier operator $\mathcal{F}(\mathcal{A}_\psi u)(\boldsymbol{\xi}) = \psi(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})\hat{u}(\boldsymbol{\xi})$,

$\varphi = \varphi_1 \bar{\varphi}_2$.

Measure μ_H we call H-measure associated to the (sub)sequence (u_n).

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$$+ \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u)(\mathbf{x})\overline{(\varphi_2 u)(\mathbf{x})}\, d\mathbf{x}\,.$$

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The theorem is also valid:

- for u_n of class $\mathrm{L}^2_{\mathrm{loc}}$, but the associated H-measure does not need to be finite, test functions $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$,
- for vector functions $\mathsf{u}_n \in \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)$, the H-measure is a positive semi-definite matrix Radon measure.

Applications:

- Compensated compactness [1]
  – if $u_n, v_n \rightharpoonup 0$, does $u_n v_n \rightharpoonup 0$?

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- Velocity averaging [3]
  – under which conditions $\int_{\mathbf{R}_y} u_n(x,y)\rho(y)dy \longrightarrow 0$ in L^2?

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  – under which conditions $\int_{\mathbf{R}_y} u_n(x,y)\rho(y)dy \longrightarrow 0$ in L^2?
- (Averaged) control theory [4]
  – under which conditions can we control the averaged quantity $\int_{\mathbf{R}_y} u_n(x,y)\rho(y)dy$?

. . .

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[3] P. Gérard 1991; E. Yu. Panov 2009, 2010; M. L. & D. Mitrović 2011, 2012

[4] N. Burq & P. Gérard 1997; B. Dehman, M. Léautaud & J. Le Rousseau 2014; M. L. & E. Zuazua 2014

H-measures – restricted to quadratic terms of L^2 sequences.

H-distributions: [5]

- – a generalisation of the concept to the $\mathrm{L}^p, p \geq 1$ framework.
- – explore products of a form

$$\int u_n v_n \quad , u_n \in \mathrm{L}^p, v_n \in \mathrm{L}^{p'}.$$

The aim of the paper: [6]

– to deal with higher order terms

$$\int u_n^p, \quad u_n \in \mathrm{L}^p.$$

[5] N. Antonić, D. Mitrovic, H-distributions – an extension of the H-measures in $L^p - L^q$ setting, *Abstr. Appl. Anal.* **2011** (2011), 12 pp.

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More precisely

$$\lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\psi_1}(\varphi_1 u_n)(\mathbf{x}) \mathcal{A}_{\psi_2}(\varphi_2 u_n)(\mathbf{x}) \dots \mathcal{A}_{\psi_p}(\varphi_p u_n)(\mathbf{x}) d\mathbf{x} = ?$$

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Split the integrand into two parts

$$\int_{\mathbf{R}^d} \underbrace{\mathcal{A}_{\psi_1}(\varphi_1 u_n) \dots \mathcal{A}_{\psi_{p/2}}(\varphi_{p/2} u_n)}_{v_n} \underbrace{\mathcal{A}_{\psi_{p/2+1}}(\varphi_{p/2+1} u_n) \dots \mathcal{A}_{\psi_p}(\varphi_p u_n)}_{w_n} \, d\mathbf{x}$$

Theorem 2.

Let $u_n \longrightarrow 0$ *in* $\mathrm{L}^{p+\varepsilon}(\mathbf{R}^d), p \in \mathbf{N}, \varepsilon > 0$.

Then for any choice of test functions $\varphi_i \in \mathrm{C}_0(\mathbf{R}^d), \psi_i \in \mathrm{C}^d(\mathrm{S}^{d-1}), i = 1..p$ *it holds*

$$\lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\psi_1}(\varphi_1 u_n)(\mathbf{x}) \cdot \ldots \cdot \mathcal{A}_{\psi_p}(\varphi_p u_n)(\mathbf{x}) d\mathbf{x} = \langle \mu_{vw}, \varphi \boxtimes 1 \rangle + \int_{\mathbf{R}^d} (\varphi v)(\mathbf{x}) \overline{w}(\mathbf{x}) d\mathbf{x},$$

where:

- $\varphi = \prod_{i=1}^p \varphi_i$,
- μ_{vw} *– off-diagonal component of the matrix H-measure associated to* $(v_n - v, w_n - w)$.

Proof

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- the (First) commutation lemma

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- the (First) commutation lemma
  Let:
  $u_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d) \cap \mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 2, \infty]$.
  $C = \mathcal{A}_\psi \varphi - \varphi \mathcal{A}_\psi$ the commutator determined by $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$,
  $\psi \in \mathrm{C}^d(\mathrm{S}^{d-1})$.

  Then:
  $C u_n \longrightarrow 0$ in $\mathrm{L}^q(\mathbf{R}^d), q \in [2, p\rangle$.

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  Then:
  $Cu_n \longrightarrow 0$ in $\mathrm{L}^q(\mathbf{R}^d), q \in [2, p\rangle$.
- $u_n \in \mathrm{L}^p \Longrightarrow v_n, w_n \in \mathrm{L}^2$

❏

The periodic setting

Let (u_n) be a sequence of periodic functions

$$u_n(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^d} \hat{u}_{\mathbf{k}}\, e^{2\pi i n \mathbf{k}\cdot\mathbf{x}} \rightharpoonup 0.$$

The associated H-measure :

$$\mu(\mathbf{x},\boldsymbol{\xi}) = \sum_{\mathbf{k}} |\hat{u}_{\mathbf{k}}|^2 \delta_{\frac{\mathbf{k}}{|\mathbf{k}|}}(\boldsymbol{\xi})\, \lambda(\mathbf{x})\,.$$

Can we express $\lim \int \mathcal{A}_{\psi_1}(\varphi_1 u_n)\ldots\mathcal{A}_{\psi_p}(\varphi_p u_n)$ explicitly?

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Remark

$\mathcal{A}_\psi u_n$ is a periodic function:

$$\mathcal{A}_\psi u_n(\mathbf{x}) = \sum_{\mathbf{k}} \hat{u}_{\mathbf{k}}\, \psi(\mathbf{k}) e^{2\pi i n \mathbf{k}\cdot\mathbf{x}}.$$

p=4

Specially,

$$v_n(\mathbf{x}) = \big(\mathcal{A}_{\psi_1} u_n\big)\big(\mathcal{A}_{\psi_2} u_n\big)(\mathbf{x}) = \sum_{\mathbf{j},\mathbf{k}} \hat{u}_{\mathbf{j}}\, \hat{u}_{\mathbf{k}}\, \psi_1(\mathbf{j})\psi_2(\mathbf{k})\, e^{2\pi i n(\mathbf{j}+\mathbf{k})\cdot\mathbf{x}}$$

$$\longrightarrow \sum_{\mathbf{k}} \hat{u}_{\mathbf{k}}\, \hat{u}_{-\mathbf{k}}\, \psi_1(\mathbf{k})\psi_2(-\mathbf{k})\,.$$

Similarly for

$$\overline{w_n}(\mathbf{x}) = \big(\mathcal{A}_{\psi_3} u_n\big)\big(\mathcal{A}_{\psi_4} u_n\big)(\mathbf{x})\,.$$

The measure μ_{vw} determined by the sequences $(v_n - v)$ and $(w_n - w)$ reads

$$\mu_{vw} = \sum_{\substack{\mathbf{j},\mathbf{k} \\ \mathbf{j}+\mathbf{k}\neq\{0\}}} \Bigg(\sum_{\substack{\mathbf{l},\mathbf{m} \\ \mathbf{l}+\mathbf{m}=-(\mathbf{j}+\mathbf{k})}} \hat{u}_{\mathbf{j}}\, \hat{u}_{\mathbf{k}}\, \hat{u}_{\mathbf{l}}\, \hat{u}_{\mathbf{m}}\, \psi_1(\mathbf{j})\psi_2(\mathbf{k})\psi_3(\mathbf{l})\psi_4(\mathbf{m})\Bigg)\, \delta_{\frac{\mathbf{j}+\mathbf{k}}{|\mathbf{j}+\mathbf{k}|}}(\boldsymbol{\xi})\, \lambda(\mathbf{x})\,.$$

Taking into account the form of the limits v and w:

$$\lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\psi_1}(\varphi_1 u_n)(\mathbf{x}) \cdot \ldots \cdot \mathcal{A}_{\psi_4}(\varphi_4 u_n)(\mathbf{x}) d\mathbf{x}$$

$$= \sum_{\substack{\mathbf{j},\mathbf{k},\mathbf{l},\mathbf{m} \\ \mathbf{l}+\mathbf{m}=-(\mathbf{j}+\mathbf{k})}} \hat{u}_{\mathbf{j}}\, \hat{u}_{\mathbf{k}}\, \hat{u}_{\mathbf{l}}\, \hat{u}_{\mathbf{m}}\, \psi_1(\mathbf{j})\psi_2(\mathbf{k})\psi_3(\mathbf{l})\psi_4(\mathbf{m}) \int_{\mathbf{R}^d} \varphi(\mathbf{x}) d\mathbf{x},$$

where $\varphi = \prod_i \varphi_i$.

Explicit formula for general p

Theorem 3.

Let (u_n) *be a bounded sequence of periodic functions in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbf{R}^d)$, $u_n \longrightarrow 0$. *For any* $p \in \mathbf{N}$, $\varphi_i \in \mathrm{C}_c(\mathbf{R}^d), \psi_i \in \mathrm{C}^d(\mathrm{S}^{d-1}), i = 1..p$ *it holds*

$$\lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\psi_1}(\varphi_1 u_n)(\mathbf{x}) \cdot \ldots \cdot \mathcal{A}_{\psi_p}(\varphi_p u_n)(\mathbf{x}) d\mathbf{x} = \sum_{\substack{\mathbf{k}_i \in \mathbf{Z}^d, \\ \sum_i \mathbf{k}_i = 0}} \Big(\prod_{i=1}^p \hat{u}_{\mathbf{k}_i} \psi_i(\mathbf{k}_i) \Big) \int_{\mathbf{R}^d} \varphi(\mathbf{x}) d\mathbf{x},$$

where $\varphi = \prod_{i=1}^p \varphi_i$.

The theorem:

– easily generalises to a case when each factor in the integrand above is associated to a different sequence $(u_n^i)_n, i = 1..p$, just by adjusting the Fourier coefficients on the right hand side;

– incorporates the expression for an H-measure associated to a sequence of periodic functions (case $p = 2$).

Application to a stationary diffusion problem

A sequence of elliptic problems:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}^n \nabla u^n) = f \in \mathrm{H}^{-1}(\Omega) \\ u^n \in \mathrm{H}^1_0(\Omega)\,, \end{cases} \tag{1}$$

where $\Omega \subseteq \mathbf{R}^d$ is an open, bounded domain.
The coefficients $\mathbf{A}^n$ are taken from the set (with $0 < \alpha < \beta$)

$$\mathcal{M}(\alpha,\beta;\Omega) := \{\mathbf{A} \in \mathrm{L}^\infty(\Omega;\mathrm{M}_d(\mathbf{R}^d)) : \mathbf{A}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq \alpha|\boldsymbol{\xi}|^2\,,\ \mathbf{A}^{-1}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq \frac{1}{\beta}|\boldsymbol{\xi}|^2\}\,,$$

Then

$$\mathbf{A}^n \xrightarrow{H} \mathbf{A}^\infty \in \mathcal{M}(\alpha,\beta;\Omega)\,,$$

i.e. for any choice of $f \in \mathrm{H}^{-1}(\Omega)$, solutions to (1) satisfy:

$$\begin{aligned} u^n &\rightharpoonup u^\infty && \text{in } \mathrm{H}^1_0(\Omega) \\ \mathbf{A}^n\nabla u^n &\rightharpoonup \mathbf{A}^\infty\nabla u^\infty && \text{in } \mathrm{L}^2(\Omega)\,, \end{aligned}$$

where u^∞ is the solution of (1) with ∞ instead of n.

Small amplitude homogenisation

The coefficients $\mathbf{A}^n$ are perturbations of a constant:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma\mathbf{A}^n_1(t,\mathbf{x}) + \gamma^2\mathbf{A}^n_2(t,\mathbf{x}) + \gamma^3\mathbf{A}^n_3(t,\mathbf{x}) + o(\gamma^3) ,$$

where $\mathbf{A}^n_i \xrightarrow{\ *\ } \mathbf{0}$ in $\mathrm{L}^\infty(\Omega)$ for any $i \geq 1$.

Assuming $\mathbf{A}_0 \in \mathcal{M}(\alpha,\beta;\Omega)$, we have (for small values of γ)

$$\mathbf{A}^n_\gamma \xrightarrow{\ H\ } \mathbf{A}^\infty_\gamma = \mathbf{A}_0 + \gamma\mathbf{A}^\infty_1(t,\mathbf{x}) + \gamma^2\mathbf{A}^\infty_2(t,\mathbf{x}) + \gamma^3\mathbf{A}^\infty_3(t,\mathbf{x}) + o(\gamma^3) ,$$

where the limit $\mathbf{A}^\infty_\gamma$ is measurable in $\mathbf{x}$ and analytic in γ.

Existing results:

- $\mathbf{A}^\infty_1 = \mathbf{0}$
- $\mathbf{A}^\infty_2$ – the limit of a quadratic term in $\mathbf{A}^n_1$,
  – expressed via H-measure $\boldsymbol{\mu} \sim \mathbf{A}^n_1$.

Missing:

- Higher order correction terms, $\mathbf{A}^\infty_3$, etc.
  – the limit of expressions involving higher order powers,
  – beyond the scope of H-measures.

$\mathbf{A}_2^\infty$

Applying expansion in powers of γ.

For an arbitrary $u \in \mathrm{H}_0^1(\Omega)$

$$\mathbf{A}_2^\infty \nabla u = -\lim_n \mathbf{A}_1^n \mathcal{A}_\Psi \left(\mathbf{A}_1^n \nabla u\right),$$

where $\mathcal{A}_\psi$ is the multiplier operator with the symbol $\Psi(\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}}$.

It yields the (existing) expression for $\mathbf{A}_2^\infty$:

$$\int_\Omega \left(\mathbf{A}_2^\infty\right)_{ij}(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = -\sum_{k,l}\left\langle \mu_{11}^{iklj}, \phi\frac{\xi_k\xi_l}{\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}}\right\rangle,$$

with $\boldsymbol{\mu}_{11}$ standing for an H-measure (with four indices) associated to $\mathbf{A}_1^n$.

Higher order correction terms

Similarly

$$\mathbf{A}_3^\infty \nabla u = \lim_n \Big(-\mathbf{A}_1^n \mathcal{A}_\Psi \mathbf{A}_2^n \nabla u - \mathbf{A}_2^n \mathcal{A}_\Psi \mathbf{A}_1^n \nabla u + \mathbf{A}_1^n \mathcal{A}_\Psi \left(\mathbf{A}_1^n \mathcal{A}_\Psi \mathbf{A}_1^n \nabla u \right) \Big),$$

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providing:

$$\int_\Omega (\mathbf{A}_3^\infty)^{ij} \varphi \, d\mathbf{x} = -\langle 2\mathrm{Re}\, \boldsymbol{\mu}_{12}^{ij}, \varphi \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{\mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} \rangle + \langle \mathrm{tr} \boldsymbol{\mu}_{VW}^{ij}, \varphi \boxtimes 1 \rangle,$$

where $\boldsymbol{\mu}_{12} \sim (\mathbf{A}_1, \mathbf{A}_2)$
$\boldsymbol{\mu}_{VW} \sim (\mathbf{V}_n, \mathbf{W}^n)$.

And so on: $\mathbf{A}_4^\infty, \ldots$

Periodic setting

Periodic coefficients

$$\mathbf{A}_i^n(n\mathbf{x}) = \mathbf{A}_i(n\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^d} \hat{\mathbf{A}}_{i,\mathbf{k}} e^{2\pi i n\mathbf{k}\cdot\mathbf{x}}, \quad i \in \mathbf{N}$$

We have explicit expressions for H-measures associated to (an arbitrary power of $\mathbf{A}_i^n$).
Specially:

$$\mathbf{A}_2^\infty = -\sum_{\mathbf{k}\in\mathbf{Z}^d} \frac{1}{\mathbf{A}_0\mathbf{k}\cdot\mathbf{k}} (\hat{\mathbf{A}}_{1,\mathbf{k}}\mathbf{k}) \otimes \hat{\mathbf{A}}_{1,-\mathbf{k}}\mathbf{k}\,.$$

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and

$$\mathbf{A}_3^\infty = \sum_{\mathbf{k}\in\mathbf{Z}^d} \frac{1}{\mathbf{A}_0\mathbf{k}\cdot\mathbf{k}} (\hat{\mathbf{A}}_{1,\mathbf{k}}\mathbf{k}) \otimes \Big(-2\hat{\mathbf{A}}_{2,-\mathbf{k}}\mathbf{k} + \sum_{\substack{\mathbf{l},\mathbf{m}\in\mathbf{Z}^d \\ \mathbf{k}+\mathbf{l}+\mathbf{m}=0}} \frac{\hat{\mathbf{A}}_{1,\mathbf{l}}\mathbf{k}\cdot\mathbf{m}}{\mathbf{A}_0\mathbf{m}\cdot\mathbf{m}} \hat{\mathbf{A}}_{1,\mathbf{m}}\mathbf{m} \Big).$$

Periodic setting

Similarly:

$$
\begin{aligned}
\mathbf{A}_4^\infty = \sum_{\mathbf{k}\in\mathbf{Z}^d} \frac{1}{\mathbf{A}_0\mathbf{k}\cdot\mathbf{k}} \Bigg(& -2(\hat{\mathbf{A}}_{1,\mathbf{k}}\mathbf{k}) \otimes \hat{\mathbf{A}}_{3,-\mathbf{k}}\mathbf{k} - (\hat{\mathbf{A}}_{2,\mathbf{k}}\mathbf{k}) \otimes (\hat{\mathbf{A}}_{2},-\mathbf{k}) \\
& + \sum_{\substack{\mathbf{l},\mathbf{m}\in\mathbf{Z}^d \\ \mathbf{k}+\mathbf{l}+\mathbf{m}=0}} \frac{1}{\mathbf{A}_0\mathbf{m}\cdot\mathbf{m}} \Big(\hat{\mathbf{A}}_{1,\mathbf{l}}\mathbf{k}\cdot\mathbf{m}\,(\hat{\mathbf{A}}_{1,\mathbf{k}}\mathbf{k}) \otimes (\hat{\mathbf{A}}_{2,\mathbf{m}}\mathbf{m}) \\
& \qquad + \hat{\mathbf{A}}_{2,\mathbf{l}}\mathbf{k}\cdot\mathbf{m}\,(\hat{\mathbf{A}}_{1,\mathbf{k}}\mathbf{k}) \otimes (\hat{\mathbf{A}}_{1,\mathbf{m}}\mathbf{m}) + \hat{\mathbf{A}}_{1,\mathbf{l}}\mathbf{k}\cdot\mathbf{m}\,(\hat{\mathbf{A}}_{2,\mathbf{k}}\mathbf{k}) \otimes (\hat{\mathbf{A}}_{1,\mathbf{m}}\mathbf{m}) \\
& - \sum_{\substack{\mathbf{j}\in\mathbf{Z}^d \\ \mathbf{l}+\mathbf{m}=-(\mathbf{j}+\mathbf{k})}} \frac{1}{\mathbf{A}_0(\mathbf{j}+\mathbf{k})\cdot(\mathbf{j}+\mathbf{k})} (\hat{\mathbf{A}}_{1,\mathbf{j}}\mathbf{k} + \hat{\mathbf{A}}_{1,\mathbf{l}}\mathbf{m})\cdot(\mathbf{j}+\mathbf{k})(\hat{\mathbf{A}}_{1,\mathbf{k}}\mathbf{k}) \otimes (\hat{\mathbf{A}}_{1,\mathbf{m}}\mathbf{m}) \Big) \Bigg)
\end{aligned}
$$

Conclusion

Presented:

- a method for expressing limits of non-quadratic terms by means of original H-measures,
- application to the small amplitude homogenisation problem for a stationary diffusion equation.

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Perspectives:

- non-stationary diffusion problems [7] by means of parabolic H-measures, [8]
- coefficients $\mathbf{A}_i^n$ oscillating on different scales – multiscale H-measures, [9]
- optimal design problems conducting so far up to the second expansion term. [10]

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Thanks for your attention!

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