H-distributions and compactness by compensation

Marin Mišur

University of Zagreb

doctoral thesis defence

02/06/2017

What are H-measures?

Mathematical objects introduced (1989/90) by:

- Luc Tartar, who was motivated by possible applications in homogenisation, and independently by
- Patrick Gérard, whose motivation were problems in kinetic theory.

Theorem 1. *If* $u_n \rightharpoonup 0$ *and* $v_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)$*, then there exist their subsequences and a complex valued Radon measure* μ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$*, such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} (\psi \circ \pi) d\boldsymbol{\xi} = \langle \mu, \varphi_1 \overline{\varphi_2} \boxtimes \psi \rangle,$$

where $\pi : \mathbf{R}^d \setminus \{0\} \longrightarrow \mathrm{S}^{d-1}$ *is the projection along rays.* ∎

Question: How to replace L^2 with L^p?

Notice: if we denote by $\mathcal{A}_\psi$ the Fourier multiplier operator with symbol $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$:

$$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee,$$

we can rewrite the equality from the theorem as

$$\begin{aligned}\langle \mu, \varphi_1 \overline{\varphi_2} \boxtimes \psi \rangle &= \lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} (\psi \circ \pi) d\boldsymbol{\xi} \\ &= \lim_{n'} \int_{\mathbf{R}^d} \varphi_1 u_{n'}(\mathbf{x}) \overline{\mathcal{A}_{\overline{\psi \circ \pi}}(\varphi_2 u_{n'})(\mathbf{x})} d\mathbf{x} \ .\end{aligned}$$

Hörmander-Mihlin Theorem

Theorem 2. *Let* $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ *have partial derivatives of order less than or equal to* $\kappa = [d/2] + 1$. *If for some* $k > 0$

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \quad |\boldsymbol{\alpha}| \leq \kappa \Longrightarrow \int_{r/2 \leq |\boldsymbol{\xi}| \leq r} |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leq k^2 r^{d-2|\boldsymbol{\alpha}|},$$

then for any $p \in \langle 1, \infty \rangle$ *and the associated multiplier operator* $\mathcal{A}_\psi$ *there exists a constant* C_d *such that*

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leq C_d \max\{p, 1/(p-1)\}(k + \|\psi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}).$$

∎

For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d \setminus \{0\}$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}$.

Y. Heo, F. Nazarov, A. Seeger, *Radial Fourier multipliers in high dimensions*, Acta Mathematica **206** (2011) 55-92.

What says the First commutation lemma?

- $\mathcal{A}_\psi u := (\psi \hat{u})^\vee$
- $M_b u := bu$

$$[\mathcal{A}_\psi, M_b] := \mathcal{A}_\psi M_b - M_b \mathcal{A}_\psi$$

Question: Why do we need such a result?

$$\begin{aligned}\langle \mu, \varphi_1 \overline{\varphi_2} \boxtimes \psi \rangle &= \lim_{n'} \int_{\mathbf{R}^d} \varphi_1 u_{n'}(\mathbf{x}) \overline{\mathcal{A}_{\overline{\psi\circ\pi}}(\varphi_2 u_{n'})(\mathbf{x})} d\mathbf{x} \\ &= \lim_{n'} \int_{\mathbf{R}^d} M_{\varphi_1} u_{n'}(\mathbf{x}) \, \overline{\mathcal{A}_{\overline{\psi\circ\pi}}(M_{\varphi_2} u_{n'})(\mathbf{x})} d\mathbf{x} \; .\end{aligned}$$

Compactness on L^2 - Cordes' result[1]

Theorem

If bounded continuous functions b and ψ satisfy

$$\lim_{|\boldsymbol{\xi}|\to\infty} \sup_{|\mathbf{h}|\leq 1} \{|\psi(\boldsymbol{\xi}+\mathbf{h})-\psi(\boldsymbol{\xi})|\} = 0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty} \sup_{|\mathbf{h}|\leq 1} \{|b(\mathbf{x}+\mathbf{h})-b(\mathbf{x})|\} = 0 \, ,$$

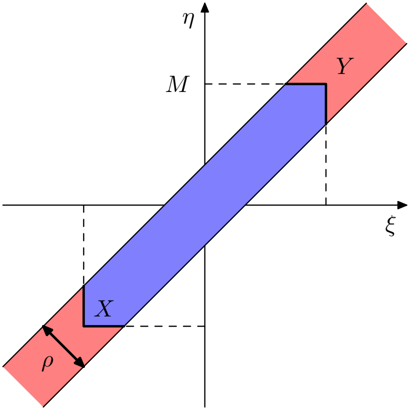
then the commutator $[\mathcal{A}_\psi, M_b]$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$.

[1] H. O. Cordes, *On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators*, J. Funct. Anal. **18** (1975) 115–131.

Compactness on L^2 - Tartar's version

For given $M, \varrho \in \mathbf{R}^+$ we denote the set

$$Y(M,\varrho) = \{(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geq M \& |\boldsymbol{\xi}-\boldsymbol{\eta}| \leq \varrho\} \ .$$



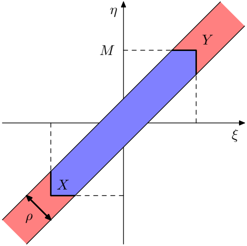
Compactness on L^2 - Tartar's version[2]

Lemma (general form of the First commutation lemma)

If $b \in \mathrm{C}_0(\mathbf{R}^d)$, *while* $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ *satisfies the condition*

$$(\forall\, \varrho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |\psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\eta})| \leqslant \varepsilon \ \ (\mathrm{s.s.}\ (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \varrho))\ , \quad (1)$$

then $[\mathcal{A}_\psi, M_b]$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$.



Lemma

Let $\pi : {\mathbf{R}^d}_* \to \Sigma$ *be a smooth projection to a smooth compact hypersurface* Σ, *such that* $\|\nabla\pi(\boldsymbol{\xi})\| \to 0$ *for* $|\boldsymbol{\xi}| \to \infty$, *and let* $\psi \in \mathrm{C}(\Sigma)$. *Then* $\psi \circ \pi$ *(*ψ *extended by homogeneity of order 0) satisfies* (1).

---

[2] L. Tartar, *The general theory of homogenization: A personalized introduction*, Springer, 2009.

Where is it used?

- L. Tartar, *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations,* Proc. Roy. Soc. Edinburgh **115A** (1990) 193–230.[3]
- E. Ju. Panov, *Ultra-parabolic H-measures and compensated compactness*, Ann. Inst. H. Poincaré Anal. Non Linéaire C **28** (2011) 47–62.
- N. Antonić, M. Lazar, *Parabolic H-measures*, J. Funct. Anal. **265** (2013) 1190–1239.
- Z. Lin, *Instability of nonlinear dispersive solitary waves*, J. Funct. Anal. **255** (2008) 1191–1224.
- Z. Lin, *On Linear Instability of 2D Solitary Water Waves*, International Mathematics Research Notices **2009** (2009) 1247–1303.
- S. Richard, R. T. de Aldecoa, *New Formulae for the Wave Operators for a Rank One Interaction*, Integr. Equ. Oper. Theory **66** (2010) 283–292.

[3] P. Gérard, *Microlocal defect measures,* Comm. Partial Diff. Eq. **16** (1991) 1761–1794.

What about the L^p variant of the First commutation lemma?

One variant can be found in the article by Cordes - complicated proof and higher regularity assumptions. Namely, the symbol is requird to satisfy:

- $\psi \in \mathrm{C}^{2\kappa}(\mathbf{R}^d)$,
- for every $\boldsymbol{\alpha} \in \mathbf{N}_0^d$, $|\boldsymbol{\alpha}| \leq 2\kappa$:

$$(1+|\boldsymbol{\xi}|)^{|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi}) \qquad \text{is bounded.}$$

A different variant was given by Antonić and Mitrović[4]:

Lemma

Assume $\psi \in \mathrm{C}^{\kappa}(\mathrm{S}^{d-1})$ and $b \in \mathrm{C}_0(\mathbf{R}^d)$. Let (v_n) be a bounded sequence, both in $\mathrm{L}^2(\mathbf{R}^d)$ and in $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 2, \infty]$, and such that $v_n \rightharpoonup 0$ in the sense of distributions.

Then $[\mathcal{A}_\psi, M_b]v_n \longrightarrow 0$ strongly in $\mathrm{L}^q(\mathbf{R}^d)$, for any $q \in [2, r\rangle$.

The proof was based on a simple interpolation inequality of L^p spaces: $\|f\|_{\mathrm{L}^q} \leqslant \|f\|_{\mathrm{L}^2}^{\theta} \|f\|_{\mathrm{L}^r}^{1-\theta}$, where $1/q = \theta/2 + (1-\theta)/r$.

---

[4] N. Antonić, D. Mitrović, *H-distributions: an extension of H-measures to an $\mathrm{L}^p - \mathrm{L}^q$ setting*, Abs. Appl. Analysis **2011** Article ID 901084 (2011) 12 pp.

A variant of Krasnoselskij's type of result[5]

Lemma

Assume that linear operator A is compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty\rangle \setminus \{2\}$. Then A is also compact on $\mathrm{L}^p(\mathbf{R}^d)$, for any p between 2 and r (i.e. such that $1/p = \theta/2 + (1-\theta)/r$, for some $\theta \in \langle 0, 1\rangle$).

Corollary

If $b \in \mathrm{C}_0(\mathbf{R}^d)$, while $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d)$ satisfies the conditions of the Hörmander-Mihlin theorem, then the commutator $[\mathcal{A}_\psi, M_b]$ is a compact operator on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty\rangle$.

---

[5] M. A. Krasnoselskij, *On a theorem of M. Riesz*, Dokl. Akad. Nauk SSSR **131** (1960) 246–248 (in russian); translated as Soviet Math. Dokl. **1** (1960) 229–231.

Theorem

Let $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d \setminus \{0\})$ *be bounded and satisfy Hörmander's condition, while* $b \in \mathrm{C}_c(\mathbf{R}^d)$. *Then for any* $u_n \xrightarrow{*} 0$ *in* $\mathrm{L}^\infty(\mathbf{R}^d)$ *and* $p \in \langle 1, \infty\rangle$ *one has:*

$$(\forall \varphi, \phi \in \mathrm{C}_c^\infty(\mathbf{R}^d)) \qquad \phi C(\varphi u_n) \longrightarrow 0 \quad \textit{in} \quad \mathrm{L}^p(\mathbf{R}^d)\ .$$

Corollary

Let (u_n) *be a bounded, uniformly compactly supported sequence in* $\mathrm{L}^\infty(\mathbf{R}^d)$*, converging to 0 in the sense of distributions. Assume that* $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d \setminus \{0\})$ *satisfies Hörmander's condition and condition from the general form of the First commutation lemma.*
Then for any $b \in \mathrm{L}^s(\mathbf{R}^d)$, $s > 1$ *arbitrary, it holds*

$$\lim_{n\to\infty} \|b\mathcal{A}_\psi(u_n) - \mathcal{A}_\psi(bu_n)\|_{\mathrm{L}^r(\mathbf{R}^d)} = 0, \quad r \in \langle 1, s\rangle.$$

H-distributions

H-distributions were introduced by N. Antonić and D. Mitrović as an extension of H-measures to the $\mathrm{L}^p - \mathrm{L}^q$ context.

Existing applications are related to the velocity averaging[6] and $\mathrm{L}^p - \mathrm{L}^q$ compactness by compensation[7].

[6] M. Lazar, D. Mitrović, *On an extension of a bilinear functional on* $\mathrm{L}^p(\mathbf{R}^d) \times \mathrm{E}$ *to Bochner spaces with an application to velocity averaging*, C. R. Math. Acad. Sci. paris **351** (2013) 261–264.

[7] M. Mišur, D. Mitrović, *On a generalization of compensated compactness in the* $\mathrm{L}^p - \mathrm{L}^q$ *setting*, Journal of Functional Analysis **268** (2015) 1904–1927.

Existence of H-distributions

Theorem 3. *If* $u_n \longrightarrow 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ *and* $v_n \xrightarrow{*} v$ *in* $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ *for some* $p \in \langle 1, \infty\rangle$ *and* $q \geq p'$*, then there exist subsequences* $(u_{n'})$, $(v_{n'})$ *and a complex valued distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$*, such that, for every* $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$*, for* $\kappa = [d/2] + 1$*, one has:*

$$\begin{aligned}\lim_{n' \to \infty} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} &= \lim_{n' \to \infty} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} \\ &= \langle \mu, \varphi_1 \overline{\varphi}_2 \boxtimes \psi \rangle,\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ *is the Fourier multiplier operator with symbol* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$*.* ∎

Distributions of anisotropic order

Let X and Y be open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$ (or C^∞ manifolds of dimenions d and r) and $\Omega \subseteq X \times Y$ an open set. By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ and $\boldsymbol{\beta} \in \mathbf{N}_0^r$, if $|\boldsymbol{\alpha}| \leq l$ and $|\boldsymbol{\beta}| \leq m$, $\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega)$.

$\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$p_{K_n}^{l,m}(f) := \max_{|\boldsymbol{\alpha}| \leq l, |\boldsymbol{\beta}| \leq m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} \ ,$$

where $K_n \subseteq \Omega$ are compacts, such that $\Omega = \cup_{n\in\mathbf{N}} K_n$ and $K_n \subseteq \mathrm{Int} K_{n+1}$, Consider the space

$$\mathrm{C}_c^{l,m}(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}_{K_n}^{l,m}(\Omega) \ ,$$

and equip it by the topology of *strict inductive limit*.

Conjecture

Definition. A *distribution of order* l *in* $\mathbf{x}$ *and order* m *in* $\mathbf{y}$ is any linear functional on $\mathrm{C}_c^{l,m}(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

Conjecture. *Let* X, Y *be* C^∞ *manifolds and let* u *be a linear functional on* $\mathrm{C}_c^{l,m}(X \times Y)$. *If* $u \in \mathcal{D}'(X \times Y)$ *and satisfies*
$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^\infty(X))(\forall \psi \in \mathrm{C}_L^\infty(Y))$

$$|\langle u, \varphi \boxtimes \psi \rangle| \leq C p_K^l(\varphi) p_L^m(\psi),$$

then u *can be uniquely extended to a continuous functional on* $\mathrm{C}_c^{l,m}(X \times Y)$ *(i.e. it can be considered as an element of* $\mathcal{D}'_{l,m}(X \times Y)$*).* ■

From the proof of the existence theorem, we already have $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ and the following bound with $\varphi := \varphi_1\overline{\varphi_2}$:

$$|\langle \mu, \varphi \boxtimes \psi \rangle| \leq C\|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}\|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} \;,$$

where C does not depend on φ and ψ.

If the conjecture were true, then the H-distribution μ from the preceeding theorem belongs to the space $\mathcal{D}'_{0,\kappa}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, i.e. it is a distribution of order 0 in $\mathbf{x}$ and of order not more than κ in $\boldsymbol{\xi}$.

But the conjecture is not true. Indeed, take a distribution $u = \frac{-1}{\pi}\partial_y \ln|x-y|$ on $\mathbf{R}^2$. It is an element of $\mathcal{D}'_{0,1}(\mathbf{R} \times \mathbf{R})$. It holds:

$$\langle u, \varphi(x)\psi(y)\rangle = \frac{1}{\pi}\int_{\mathbf{R}} \varphi(x)\int_{\mathbf{R}} \ln|x-y|\psi'(y)\,dydx = \int_{\mathbf{R}} \varphi(x)H\psi(x)\,dx\,,$$

$$|\langle u, \varphi(x)\psi(y)\rangle| \leq C_{\operatorname{supp}\varphi,\operatorname{supp}\psi}\|\varphi\|_{\mathrm{L}^\infty}\|\psi\|_{\mathrm{L}^\infty}\;.$$

If u were locally finite measure on $\mathbf{R}^2$, in case $\operatorname{supp} g$ does not intersect the diagonal we would get $\langle u, g\rangle = \frac{1}{\pi}\int_{\mathbf{R}^2} \frac{g(x,y)}{x-y}\,dxdy$.

Schwartz kernel theorem[8]

Let X and Y be two C^∞ manifolds. Then the following statements hold:

a) Let $K \in \mathcal{D}'(X \times Y)$. Then for every $\varphi \in \mathcal{D}(X)$, the linear form K_φ defined as $\psi \mapsto \langle K, \varphi \boxtimes \psi\rangle$ is a distribution on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathcal{D}(X)$ to $\mathcal{D}'(Y)$ is linear and continuous.

b) Let $A : \mathcal{D}(X) \to \mathcal{D}'(Y)$ be a continous linear operator. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$

$$\langle K, \varphi \boxtimes \psi\rangle = \langle K_\varphi, \psi\rangle = \langle A\varphi, \psi\rangle.$$

[8] Theorem 23.9.2 of J. Dieudonné, *Éléments d'Analyse, Tome VII, Éditions Jacques Gabay, 2007.*

Schwartz kernel theorem for anisotropic distributions

Let X and Y be two C^∞ manifolds of dimensions d and r, respectively. Then the following statements hold:

a) Let $K \in \mathcal{D}'_{l,m}(X \times Y)$. Then for every $\varphi \in \mathrm{C}^l_c(X)$, the linear form K_φ defined as $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$ is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.

b) Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continous linear operator. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$

Furthermore, $K \in \mathcal{D}'_{l,r(m+2)}(X \times Y)$.

How to prove it?

Use the structure theorem of distributions (Dieudonné).

Two steps:

Step I: assume the range of A is $\mathrm{C}(Y)$
Step II: use structure theorem and go back to Step I

Consequence: H-distributions are of order 0 in $\mathbf{x}$ and of finite order not greater than $d(\kappa+2)$ with respect to $\boldsymbol{\xi}$.

Div-rot lemma in L^2

Theorem 4. *Assume that Ω is open and bounded subset of $\mathbf{R}^3$, and that it holds:*

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } \mathrm{L}^2(\Omega; \mathbf{R}^3),$$

$$\mathbf{v}_n \rightharpoonup \mathbf{v} \text{ in } \mathrm{L}^2(\Omega; \mathbf{R}^3),$$

$$\operatorname{rot} \mathbf{u}_n \text{ bounded in } \mathrm{L}^2(\Omega; \mathbf{R}^3), \ \operatorname{div} \mathbf{v}_n \text{ bounded in } \mathrm{L}^2(\Omega).$$

Then

$$\mathbf{u}_n \cdot \mathbf{v}_n \rightharpoonup \mathbf{u} \cdot \mathbf{v}$$

in the sense of distributions. ∎

Quadratic theorem

Theorem 5. *(**Quadratic theorem**) Assume that $\Omega \subseteq \mathbf{R}^d$ is open and that $\Lambda \subseteq \mathbf{R}^r$ is defined by*

$$\Lambda := \left\{ \boldsymbol{\lambda} \in \mathbf{R}^r \,:\, (\exists\, \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}) \quad \sum_{k=1}^{d} \xi_k \mathbf{A}^k \boldsymbol{\lambda} = 0 \right\},$$

where Q is a real quadratic form on $\mathbf{R}^r$, which is nonnegative on Λ, i.e.

$$(\forall\, \boldsymbol{\lambda} \in \Lambda) \quad Q(\boldsymbol{\lambda}) \geqslant 0\,.$$

Furthermore, assume that the sequence of functions $(\mathbf{u}_n)$ satisfies

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{weakly in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{R}^r)\,,$$

$$\Big(\sum_k \mathbf{A}^k \partial_k \mathbf{u}_n \Big) \quad \text{relatively compact in} \quad \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{R}^q)\,.$$

Then every subsequence of $(Q \circ \mathbf{u}_n)$ which converges in distributions to it's limit L, satisfies

$$L \geqslant Q \circ \mathbf{u}$$

in the sense of distributions. ∎

The most general version of the classical L^2 results has recently been proved by E. Yu. Panov[9]:

Assume that the sequence $(\mathbf{u}_n)$ is bounded in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)$, $2 \le p < \infty$, and converges weakly in $\mathcal{D}'(\mathbf{R}^d)$ to a vector function $\mathbf{u}$.
Let $q = p'$ if $p < \infty$, and $q > 1$ if $p = \infty$. Assume that the sequence

$$\sum_{k=1}^{\nu} \partial_k(\mathbf{A}^k \mathbf{u}_n) + \sum_{k,l=\nu+1}^{d} \partial_{kl}(\mathbf{B}^{kl}\mathbf{u}_n)$$

is precompact in the anisotropic Sobolev space $\mathrm{W}^{-1,-2;q}_{loc}(\mathbf{R}^d;\mathbf{R}^m)$, where $m \times r$ matrices $\mathbf{A}^k$ and $\mathbf{B}^{kl}$ have variable coefficients belonging to $\mathrm{L}^{2\bar{q}}(\mathbf{R}^d)$, $\bar{q} = \frac{p}{p-2}$ if $p > 2$, and to the space $\mathrm{C}(\mathbf{R}^d)$ if $p = 2$.

[9]E. Yu. Panov, *Ultraparabolic H-measures and compensated compactness*, Annales Inst. H.Poincaré **28** (2011) 47–62.

We introduce the set $\Lambda(\mathbf{x})$

$$\Lambda(\mathbf{x}) = \Bigg\{ \boldsymbol{\lambda} \in \mathbf{C}^r \Big| \, (\exists \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}) : \qquad (2)$$

$$\left(i \sum_{k=1}^{\nu} \xi_k \mathbf{A}^k(\mathbf{x}) - 2\pi \sum_{k,l=\nu+1}^{d} \xi_k \xi_l \mathbf{B}^{kl}(\mathbf{x}) \right) \boldsymbol{\lambda} = \mathbf{0}_m \Bigg\},$$

and consider the bilinear form on $\mathbf{C}^r$

$$q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\eta}) = \mathbf{Q}(\mathbf{x}) \boldsymbol{\lambda} \cdot \boldsymbol{\eta}, \qquad (3)$$

where $\mathbf{Q} \in \mathrm{L}^{\bar{q}}_{loc}(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p > 2$ and $\mathbf{Q} \in \mathrm{C}(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p = 2$.
Finally, let $q(\mathbf{x}, \mathbf{u}_n, \mathbf{u}_n) \rightharpoonup \omega$ weakly in the space of distributions.

Result by Panov

The following theorem holds

Theorem 6. *Assume that* $(\forall \boldsymbol{\lambda} \in \Lambda(\mathbf{x}))\ q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\lambda}) \geq 0$ *(a.e.* $\mathbf{x} \in \mathbf{R}^d$*) and* $\mathbf{u}_n \rightharpoonup \mathbf{u}$*, then* $q(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})) \leq \omega$*.* ∎

The connection between q and Λ given in the previous theorem, we shall call *the consistency condition*.

Appropriate symbols

We need Fourier multiplier operators with symbols defined on a manifold P determined by d-tuple $\boldsymbol{\alpha} \in (\mathbf{R}^+)^d$:

$$\mathrm{P} = \Big\{ \boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{2\alpha_k} = 1 \Big\},$$

where $\alpha_k \in \mathbf{N}$ or $\alpha_k \geq d$. In order to associate an L^p Fourier multiplier to a function defined on P, we extend it to $\mathbf{R}^d \backslash \{0\}$ by means of the projection

$$(\pi_{\mathrm{P}}(\boldsymbol{\xi}))_j = \xi_j \left(|\xi_1|^{2\alpha_1} + \cdots + |\xi_d|^{2\alpha_d} \right)^{-1/2\alpha_j}, \quad j = 1, \ldots, d.$$

We need the following extension of the results given above.

Theorem 7. *Let (u_n) be a bounded sequence in $\mathrm{L}^p(\mathbf{R}^d)$, $p>1$, and let (v_n) be a bounded sequence of uniformly compactly supported functions in $\mathrm{L}^q(\mathbf{R}^d)$, $1/q+1/p<1$. Then, after passing to a subsequence (not relabelled), for any $\bar{s}\in(1,\frac{pq}{p+q})$ there exists a continuous bilinear functional B on $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d)\otimes\mathrm{C}^d(\mathrm{P})$ such that for every $\varphi\in\mathrm{L}^{\bar{s}'}(\mathbf{R}^d)$ and $\psi\in\mathrm{C}^d(\mathrm{P})$, it holds*

$$B(\varphi,\psi)=\lim_{n\to\infty}\int_{\mathbf{R}^d}\varphi(\mathbf{x})u_n(\mathbf{x})\big(\mathcal{A}_{\psi_\mathrm{P}}v_n\big)(\mathbf{x})d\mathbf{x}\,,$$

where $\mathcal{A}_{\psi_\mathrm{P}}$ is the Fourier multiplier operator on $\mathbf{R}^d$ associated to $\psi\circ\pi_\mathrm{P}$. The bilinear functional B can be continuously extended[10] *as a linear functional on $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d;\mathrm{C}^d(\mathrm{P}))$.* ∎

[10] M. Lazar, D. Mitrović, *On an extension of a bilinear functional on $\mathrm{L}^p(\mathbf{R}^d)\times\mathrm{E}$ to Bochner spaces with an application to velocity averaging*, C. R. Math. Acad. Sci. paris **351** (2013) 261–264.

For separable Banach space E, the dual of $\mathrm{L}^p(\mathbf{R}^d; E)$ consists of all weakly-$*$ measurable functions $B : \mathbf{R}^d \to E'$ such that

$$\int_{\mathbf{R}^d} \|B(\mathbf{x})\|_{E'}^{p'} d\mathbf{x}$$

is finite[11].

Sometimes the dual is denoted by $\mathrm{L}^{p'}_{w*}(\mathbf{R}^d; E')$.

[11] p. 606 of R.E. Edwards, *Functional Analysis*, Holt, Rinehart and Winston, 1965.

Localisation principle

Lemma

Assume that sequences $(\mathbf{u}_n)$ *and* $(\mathbf{v}_n)$ *are bounded in* $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)$ *and* $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^r)$, *respectively, and converge toward* $\mathbf{0}$ *and* $\mathbf{v}$ *in the sense of distributions.*
Furthermore, assume that sequence $(\mathbf{u}_n)$ *satisfies:*

$$\mathbf{G}_n := \sum_{k=1}^{d} \partial_k^{\alpha_k}(\mathbf{A}^k \mathbf{u}_n) \ \to \mathbf{0} \ \textit{in} \ \mathrm{W}^{-\alpha_1,\dots,-\alpha_d;p}(\Omega;\mathbf{R}^m), \tag{4}$$

where either $\alpha_k \in \mathbf{N}$, $k = 1,\dots,d$ *or* $\alpha_k > d$, $k = 1,\dots,d$, *and elements of matrices* $\mathbf{A}^k$ *belong to* $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d)$, $\bar{s} \in (1, \frac{pq}{p+q})$.
Finally, by $\boldsymbol{\mu}$ *denote a matrix* H*-distribution corresponding to subsequences of* $(\mathbf{u}_n)$ *and* $(\mathbf{v}_n - \mathbf{v})$. *Then the following relation holds*

$$\Big(\sum_{k=1}^{d} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k\Big)\boldsymbol{\mu} = \mathbf{0}.$$

Strong consistency condition

Introduce the set

$$\Lambda_{\mathcal{D}} = \Big\{ \boldsymbol{\mu} \in \mathrm{L}^{\bar{s}}(\mathbf{R}^d; (\mathrm{C}^d(\mathrm{P}))')^r : \Big(\sum_{k=1}^{n} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k \Big) \boldsymbol{\mu} = \mathbf{0}_m \Big\},$$

where the given equality is understood in the sense of $\mathrm{L}^{\bar{s}}(\mathbf{R}^d; (\mathrm{C}^d(\mathrm{P}))')^m$.

Let us assume that coefficients of the bilinear form q from (3) belong to space $\mathrm{L}^t_{loc}(\mathbf{R}^d)$, where $1/t + 1/p + 1/q < 1$.

Definition

We say that set $\Lambda_{\mathcal{D}}$, bilinear form q from (3) and matrix $\boldsymbol{\mu} = [\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r], \boldsymbol{\mu}_j \in \mathrm{L}^{\bar{s}}(\mathbf{R}^d; (\mathrm{C}^d(\mathrm{P}))')^r$ satisfy the strong consistency condition if $(\forall j \in \{1, \ldots, r\})\ \boldsymbol{\mu}_j \in \Lambda_{\mathcal{D}}$, and it holds

$$\langle \phi \mathbf{Q} \otimes 1, \boldsymbol{\mu} \rangle \geq \mathbf{0}, \quad \phi \in \mathrm{L}^{\bar{s}}(\mathbf{R}^d; \mathbf{R}^+_0).$$

Theorem 8. *Assume that sequences* $(\mathbf{u}_n)$ *and* $(\mathbf{v}_n)$ *are bounded in* $\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)$ *and* $\mathrm{L}^q(\mathbf{R}^d;\mathbf{R}^r)$*, respectively, and converge toward* $\mathbf{u}$ *and* $\mathbf{v}$ *in the sense of distributions.*
Assume that (4) *holds and that*

$$q(\mathbf{x};\mathbf{u}_n,\mathbf{v}_n) \rightharpoonup \omega \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

If the set $\Lambda_{\mathcal{D}}$*, the bilinear form* (3)*, and matrix* H*-distribution* $\boldsymbol{\mu}$*, corresponding to subsequences of* $(\mathbf{u}_n-\mathbf{u})$ *and* $(\mathbf{v}_n-\mathbf{v})$*, satisfy the strong consistency condition, then*

$$q(\mathbf{x};\mathbf{u},\mathbf{v}) \leq \omega \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

∎

Application to the parabolic type equation

Now, let us consider the non-linear parabolic type equation

$$L(u) = \partial_t u - \mathsf{div}\,\mathsf{div}\,(g(t,\mathbf{x},u)\mathbf{A}(t,\mathbf{x})),$$

on $(0,\infty)\times\Omega$, where Ω is an open subset of $\mathbf{R}^d$. We assume that

$$u \in \mathrm{L}^p((0,\infty)\times\Omega), \quad g(t,\mathbf{x},u(t,\mathbf{x})) \in \mathrm{L}^q((0,\infty)\times\Omega), \quad 1 < p, q,$$
$$\mathbf{A} \in \mathrm{L}^s_{loc}((0,\infty)\times\Omega)^{d\times d}, \quad \text{where} \quad 1/p + 1/q + 1/s < 1,$$

and that the matrix $\mathbf{A}$ is strictly positive definite, i.e.

$$\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi} > 0, \quad \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{\mathbf{0}\}, \quad (a.e.(t,\mathbf{x}) \in (0,\infty)\times\Omega).$$

Furthermore, assume that g is a Carathèodory function and non-decreasing with respect to the third variable.

Then we have the following theorem.

Theorem 9. *Assume that sequences* (u_r) *and* $g(\cdot, u_r)$ *are such that* $u_r, g(u_r) \in \mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d)$ *for every* $r \in \mathbf{N}$*; assume that they are bounded in* $\mathrm{L}^p(\mathbf{R}^+ \times \mathbf{R}^d)$, $p \in (1, 2]$, *and* $\mathrm{L}^q(\mathbf{R}^+ \times \mathbf{R}^d)$, $q > 2$, *respectively, where* $1/p + 1/q < 1$*; furthermore, assume* $u_r \rightharpoonup u$ *and, for some,* $f \in \mathrm{W}^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d)$, *the sequence*

$$L(u_r) = f_r \to f \quad \textit{strongly in } \mathrm{W}^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d).$$

Under the assumptions given above, it holds

$$L(u) = f \;\; \text{in} \;\; \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d).$$

∎

References

- N. Antonić, M. Mišur, D. Mitrović, *On the First commutation lemma*, submitted, 18 pages

- N. Antonić, M. Erceg, M. Mišur, *Distributions of anisotropic order and applications*, in preparation, 24 pages

- M. Mišur, D. Mitrović, *On a generalization of compensated compactness in the* $\mathrm{L}^p - \mathrm{L}^q$ *setting*, Journal of Functional Analysis **268** (2015) 1904–1927.