Distributions of anisotropic order and applications

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International Workshop on PDEs: analysis and modelling
Zagreb, $21^{\rm st}$ June 2016
(Celebrating $80^{\rm th}$ anniversary of professor Nedžad Limić)

Joint work with Marko Erceg and Marin Mišur

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Theorem. (u^n) *a sequence in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, $\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there is a subsequence* $(\mathsf{u}^{n'})$ *and* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that:*

$$\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{F}\Big(\varphi_1 \mathsf{u}^{n'}\Big) \otimes \mathcal{F}\Big(\varphi_2 \mathsf{u}^{n'}\Big) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi})\ .$$

∎

Notation:

$$\mathsf{v} \cdot \mathsf{u} := \sum v_i \bar{u}_i$$

$$(\mathsf{v} \otimes \mathsf{u})\mathsf{a} := (\mathsf{a} \cdot \mathsf{u})\mathsf{v}$$

L^p case: the Hörmander-Mihlin theorem

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

$$\bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d) \,, \qquad \text{for } \theta \in \mathcal{S}(\mathbf{R}^d),$$

and

$$\mathcal{S}(\mathbf{R}^d) \ni \theta \mapsto \bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)$$

can be extended to a continuous mapping $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^d)$.

Theorem. [Hörmander-Mihlin] *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa = [\frac{d}{2}] + 1$. If for some $k > 0$*

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \implies \int_{\frac{r}{2} \leqslant |\boldsymbol{\xi}| \leqslant r} |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leqslant k^2 r^{d-2|\boldsymbol{\alpha}|} \,,$$

then for any $p \in \langle 1, \infty \rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a C_d (depending only on the dimension d) such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_\infty) \,.$$

∎

For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d_*$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa}$.

Existence of H-distributions

Theorem. [N.A. & D. Mitrović (2011)] *If* $u_n \longrightarrow 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$ *and* $v_n \xrightarrow{*} v$ *in* $\mathrm{L}^q(\mathbf{R}^d)$ *for some* $q \geqslant \max\{p', 2\}$*, then there exist subsequences* $(u_{n'})$*,* $(v_{n'})$ *and a complex valued distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$*, such that for every* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ *we have:*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} = \lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x}$$
$$= \langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle \ .$$

∎

μ is the *H-distribution* corresponding to (a subsequence of) (u_n) and (v_n).

If (u_n), (v_n) are defined on $\Omega \subseteq \mathbf{R}^d$, extension by zero to $\mathbf{R}^d$ preserves the convergence, and we can apply the Theorem. μ is supported on $\mathrm{Cl}\,\Omega \times \mathrm{S}^{d-1}$.

We distinguish $u_n \in \mathrm{L}^p(\mathbf{R}^d)$ and $v_n \in \mathrm{L}^q(\mathbf{R}^d)$. For $p \geqslant 2$, $p' \leqslant 2$ and we can take $q \geqslant 2$; this covers the L^2 case (including $u_n = v_n$).
The assumptions imply $u_n, v_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, resulting in a distribution μ of order zero (an unbounded Radon measure, not a general distribution).
The novelty in Theorem is for $p < 2$.

For vector-valued $\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^k)$ and $\mathsf{v}_n \in \mathrm{L}^q(\mathbf{R}^d; \mathbf{C}^l)$, the result is a *matrix valued distribution* $\boldsymbol{\mu} = [\mu^{ij}]$, $i \in 1..k$ and $j \in 1..l$.

The H-distribution would correspond to a non-diagonal block for an H-measure.

A particular Nemyckiĭ operator

Canonical choice of $\mathrm{L}^{p'}$ sequence corresponding to an L^p, $p \in \langle 1, \infty \rangle$, sequence (u_n) is given by $v_n = \Phi_p(u_n)$, where Φ_p is an operator from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ defined by $\Phi_p(u) = |u|^{p-2}u$.

Φ_p is a nonlinear Nemytskiĭ operator, continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ and additionally we have the following bound

$$\|\Phi_p(u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} \leqslant \|u\|^{p/p'}_{\mathrm{L}^p(\mathbf{R}^d)}.$$

It maps bounded sets in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ topology to bounded sets in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ topology. Hence for a bounded sequence (u_n), we get that $(\Phi_p(u_n))$ is weakly precompact in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

It is continuous from $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

Example: concentration

$u \in \mathrm{L}^p_c(\mathbf{R}^d)$, and define $u_n(\mathbf{x}) = n^{\frac{d}{p}} u(n(\mathbf{x} - \mathbf{z}))$ for some $\mathbf{z} \in \mathbf{R}^d$.

Simple change of variables: $\|u_n\|_{\mathrm{L}^p(\mathbf{R}^d)} = \|u\|_{\mathrm{L}^p(\mathbf{R}^d)}$ and $u_n \rightharpoonup 0$ in $\mathrm{L}^p(\mathbf{R}^d)$.

Indeed, the sequence is bounded, while for $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$

$$\begin{aligned}
\int_{\mathbf{R}^d} u_n(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} &= \int_{\mathbf{R}^d} n^{d/p} u(n(\mathbf{x}-\mathbf{z}))\varphi(\mathbf{x})d\mathbf{x} \\
&= \int_{\mathbf{R}^d} n^{d/p-d} u(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \\
&= \frac{1}{n^{d/p'}} \int_{\mathbf{R}^d} u(\mathbf{y})\chi_{\operatorname{supp} u}(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \\
&\leqslant \left(\frac{\mathrm{vol}(\operatorname{supp} u)}{n^d}\right)^{1/p'} \|u\|_{\mathrm{L}^p(\mathbf{R}^d)} \max_{\mathbf{R}^d} |\varphi|.
\end{aligned}$$

Passing to the limit, we get our claim.

Actually, the H-distribution corresponding to sequences (u_n) and $(\Phi_p(u_n))$ is given by $\delta_{\mathbf{z}} \boxtimes \nu$, where ν is a distribution on $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ defined for $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by

$$\langle \nu, \psi \rangle = \int_{\mathbf{R}^d} u(\mathbf{x})\overline{\mathcal{A}_{\bar{\psi}}(|u|^{p-2}u)(\mathbf{x})}d\mathbf{x}.$$

H-distributions
- H-measures
- Existence
- An example

Distributions of anisotropic order
- Functions of anisotropic smoothness
- Definition and tensor products
- Conjectures
- Consequence for H-distributions

Functions of anisotropic smoothness

Let X and Y be open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$ (or C^∞ manifolds), $\Omega \subseteq X \times Y$.

By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ and $\boldsymbol{\beta} \in \mathbf{N}_0^r$, if $|\boldsymbol{\alpha}| \leqslant l$ and $|\boldsymbol{\beta}| \leqslant m$,

$$\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega) .$$

$\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$p_{K_n}^{l,m}(f) := \max_{|\boldsymbol{\alpha}| \leqslant l, |\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} ,$$

where $K_n \subseteq \Omega$ are compacts, such that $\Omega = \cup_{n \in \mathbf{N}} K_n$ and $K_n \subseteq \mathrm{Int} K_{n+1}$.

For a compact set $K \subseteq \Omega$ we define a subspace of $\mathrm{C}^{l,m}(\Omega)$

$$\mathrm{C}_K^{l,m}(\Omega) := \left\{ f \in \mathrm{C}^{l,m}(\Omega) : \ \mathrm{supp}\, f \subseteq K \right\} .$$

This subspace inherits the topology from $\mathrm{C}^{l,m}(\Omega)$, which is, when considered only on the subspace, a norm topology determined by

$$\|f\|_{l,m,K} := p_K^{l,m}(f) ,$$

and $\mathrm{C}_K^{l,m}(\Omega)$ is a Banach space (it can be identified with a proper subspace of $\mathrm{C}^{l,m}(K)$). However, if $m = \infty$ (or $l = \infty$), then we shall not get a Banach space, but a Fréchet space. As in the isotropic case, an increasing sequence of seminorms that makes $\mathrm{C}_{K_n}^{l,\infty}(\Omega)$ a Fréchet space is given by $(p_{K_n}^{l,k}), k \in \mathbf{N}_0$.

Functions of anisotropic smoothness (cont.)

We can also consider the space

$$\mathrm{C}_c^{l,m}(\Omega) := \bigcup_{n \in \mathbf{N}} \mathrm{C}_{K_n}^{l,m}(\Omega) ,$$

of all functions with compact support in $\mathrm{C}^{l,m}(\Omega)$, and equip it by a stronger topology than the one induced from $\mathrm{C}^{l,m}(\Omega)$: by the topology of *strict inductive limit*. More precisely, it can easily be checked that

$$\mathrm{C}_{K_n}^{l,m}(\Omega) \hookrightarrow \mathrm{C}_{K_{n+1}}^{l,m}(\Omega) ,$$

the inclusion being continuous. Also, the topology induced on $\mathrm{C}_{K_n}^{l,m}(\Omega)$ by that of $\mathrm{C}_{K_{n+1}}^{l,m}(\Omega)$ coincides with the original one, and $\mathrm{C}_{K_n}^{l,m}(\Omega)$ (as a Banach space in that topology) is a closed subspace of $\mathrm{C}_{K_{n+1}}^{l,m}(\Omega)$. Then we have that the inductive limit topology on $\mathrm{C}_c^{l,m}(\Omega)$ induces on each $\mathrm{C}_{K_n}^{l,m}(\Omega)$ the original topology, while a subset of $\mathrm{C}_c^{l,m}(\Omega)$ is bounded if and only if it is contained in one $\mathrm{C}_{K_n}^{l,m}(\Omega)$, and bounded there.

Of course, $\mathrm{C}_c^{\infty}(\Omega) \hookrightarrow \mathrm{C}_c^{l,m}(\Omega)$ is a continuous and dense imbedding.

Definition. A *distribution of order l in $\mathbf{x}$ and order m in $\mathbf{y}$* is any linear functional on $\mathrm{C}_c^{l,m}(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

Clearly, $\mathrm{C}_c^{\infty}(\Omega) \hookrightarrow \mathrm{C}_c^{l,m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$, with continuous and dense imbeddings, thus $\mathrm{C}_c^{l,m}(\Omega)$ is a normal space of distributions, hence its dual $\mathcal{D}'_{l,m}(\Omega)$ forms a subspace of $\mathcal{D}'(\Omega)$. If we equip it with a strong topology, it is even continuously imbedded in $\mathcal{D}'(\Omega)$.

In order to better understand the properties of elements of $\mathcal{D}'_{l,m}(\Omega)$, we shall relate them to tensor products.
The first step is to consider the algebraic tensor product $\mathrm{C}_c^{l}(X) \boxtimes \mathrm{C}_c^{m}(Y)$, the vector space of all (finite) linear combinations of functions of the form $(\phi \boxtimes \psi)(\mathbf{x}, \mathbf{y}) := \phi(\mathbf{x})\psi(\mathbf{y})$. This is a vector subspace of $\mathrm{C}_c^{l,m}(X \times Y)$.

Tensor product of distributions

Theorem. *Let X and Y be C^∞ manifolds, $u \in \mathcal{D}'_l(X)$ and $v \in \mathcal{D}'_m(Y)$. Then*

$$\Big(\exists! w \in \mathcal{D}'_{l,m}(X \times Y)\Big)\Big(\forall \varphi \in \mathrm{C}^l_c(X)\Big)\Big(\forall \psi \in \mathrm{C}^m_c(Y)\Big) \quad \langle w, \varphi \boxtimes \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle.$$

Furthermore, for any $\Phi \in \mathrm{C}^{l,m}_c(X \times Y)$, function $V : \mathbf{x} \mapsto \langle v, \Phi(\mathbf{x}, \cdot) \rangle$ is in $\mathrm{C}^l_c(X)$, while $U : \mathbf{y} \mapsto \langle u, \Phi(\cdot, \mathbf{y}) \rangle$ is in $\mathrm{C}^m_c(Y)$, and we have that

$$\langle w, \Phi \rangle = \langle u, V \rangle = \langle v, U \rangle.$$

∎

Lemma. *If $u \in \mathcal{D}'_{l,m}(X \times Y)$ then, for any $\psi \in \mathrm{C}^{l,m}(X \times Y)$, ψu is a well defined distribution of order at most (l, m).*

∎

Theorem. *Let $u \in \mathcal{D}'_{l,m}(X \times Y)$ and take $F \subseteq X \times Y$ relatively compact set such that $\operatorname{supp} u \subseteq F$. Then there exists unique linear functional $\tilde{u}$ on $\mathcal{Q} := \{\varphi \in \mathrm{C}^{l,m}(X \times Y) : F \cap \operatorname{supp} \varphi \Subset X \times Y\}$ such that*

a) $(\forall \varphi \in \mathrm{C}^{l,m}_c(X \times Y)) \quad \langle \tilde{u}, \varphi \rangle = \langle u, \varphi \rangle$,

b) $(\forall \varphi \in \mathrm{C}^{l,m}(X \times Y)) \quad F \cap \operatorname{supp} \varphi = \emptyset \quad \Longrightarrow \quad \langle \tilde{u}, \varphi \rangle = 0.$

The domain of $\tilde{u}$ is largest for $F = \operatorname{supp} u$.

∎

First conjecture

Let X, Y be C^∞ manifolds and u a linear functional on $\mathrm{C}_c^{l,m}(X \times Y)$.
If $u \in \mathcal{D}'(X \times Y)$ and satisfies
$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^\infty(X))(\forall \psi \in \mathrm{C}_L^\infty(Y))$

$$|\langle u, \varphi \boxtimes \psi \rangle| \leqslant C p_K^l(\varphi) p_L^m(\psi),$$

then u can be uniquely extended to $\mathcal{D}'_{l,m}(X \times Y)$.

It is not true!

We need a more complicated result.

We are aware of the abstract approach via nuclear spaces.

Second conjecture: the Schwartz kernel theorem

Theorem. *Let X and Y be two differentiable manifolds. Then the following statements hold:*

a) *Let $K \in \mathcal{D}'_{l,m}(X \times Y)$. Then for every $\varphi \in \mathrm{C}^l_c(X)$ the linear form K_φ defined as $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$ is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.*

b) *Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathrm{C}^\infty_c(X)$ and $\psi \in \mathrm{C}^\infty_c(Y)$*

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$

Furthermore, $K \in \mathcal{D}'_{l,d(m+2)}(X \times Y)$. ∎

We are reasonably confident that it is true!

Consequence for H-distributions

By the previous theorem the H-distribution μ mentioned at the beginning belongs to the space $\mathcal{D}'_{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, i.e. it is a distribution of order 0 in $\mathbf{x}$ and of order not more than $d(\kappa+2)$ in $\boldsymbol{\xi}$.

Indeed, we already have $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ and the following bound with $\varphi := \varphi_1 \overline{\varphi_2}$:

$$|\langle \mu, \varphi \boxtimes \psi \rangle| \leqslant C \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} \|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} ,$$

where C does not depend on φ and ψ.

Now we just need to apply the Schwartz kernel theorem given above to conclude that μ is a continuous linear functional on $\mathrm{C}_c^{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$.

Thank you for your attention.