Explicit solutions of multiple state optimal design problems

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Joint work with **Marko Vrdoljak**

WeConMApp

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Outline

Multiple state optimal design problem

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f_1, \ldots, f_m \in \mathrm{L}^2(\Omega)$ given; stationary diffusion equations with homogenous Dirichlet b. c.:

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{array} \right. , \quad i = 1, \ldots, m \tag{1}$$

where $\mathbf{A}$ is a mixture of two isotropic materials with conductivities $0 < \alpha < \beta$: $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where $\chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$, $\int_\Omega \chi\, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$.
For given Ω, α, β, q_α, f_i, and some given weights $\mu_i > 0$, we want to find such material $\mathbf{A}$ which minimizes the weighted sum of energies (total amounts of heat/electrical energy dissipated in Ω):

$$I(\chi) := \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min , \quad \chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$$

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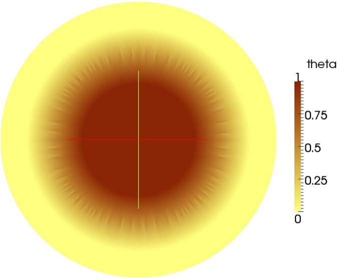
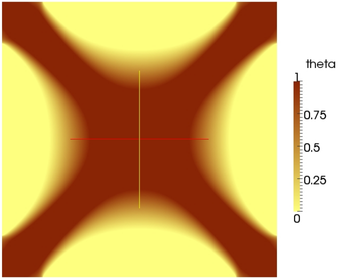
single state, $f \equiv 1$, Ω circle / square

Murat & Tartar



Lurie & Cherkaev



$\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$ $\cdots$ $\theta \in \mathrm{L}^\infty(\Omega; [0,1])$

$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$ $\quad$ $\mathbf{A} \in \mathcal{K}(\theta)$ a.e. on Ω

classical material $\quad$ composite mateiral - relaxation

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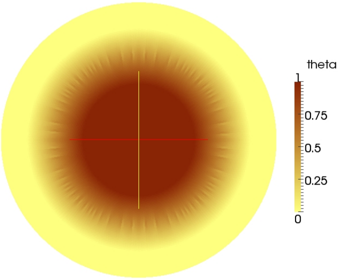

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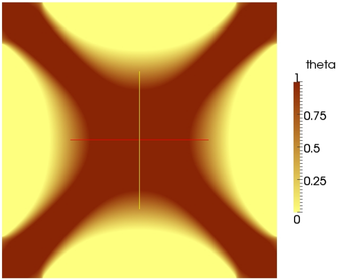

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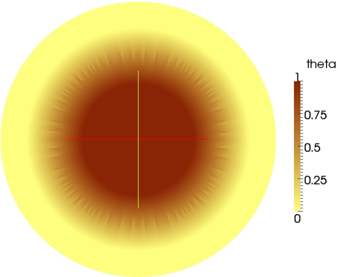

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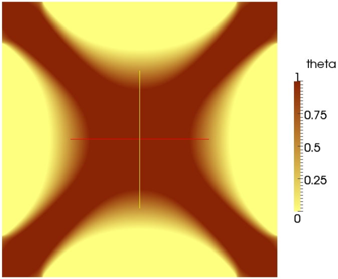

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Composite material

Definition

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega; \{0,1\})$ *and conductivities* $\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1-\chi_\varepsilon(x))\beta\mathbf{I}$ *satisfy* $\chi_\varepsilon \rightharpoonup \theta$ *weakly* $*$ *and* $\mathbf{A}^\varepsilon$ *H-converges to* $\mathbf{A}^*$*, then it is said that* $\mathbf{A}^*$ *is homogenised tensor of two-phase composite material with proportions* θ *of first material and microstructure defined by the sequence* (χ_ε)*.*

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A}^* = \mathrm{diag}(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \ldots, \lambda_\theta^+)\,,$$

where

$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\,.$$

Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,,\ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.}\}$$

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Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

$$
\begin{aligned}
\lambda_\theta^- \leq \quad \lambda_j \quad &\leq \lambda_\theta^+ \quad j = 1, \ldots, d \\
\sum_{j=1}^d \frac{1}{\lambda_j - \alpha} \quad &\leq \quad \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha} \\
\sum_{j=1}^d \frac{1}{\beta - \lambda_j} \quad &\leq \quad \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},
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2D:

3D:

$\min_{\mathcal{A}} J$ is a proper relaxation of $\min_{L^\infty(\Omega;\{0,1\})} I$

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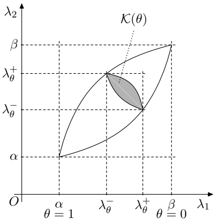

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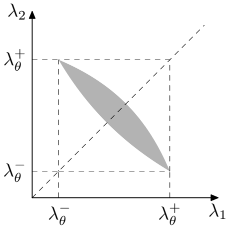

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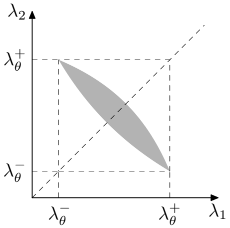

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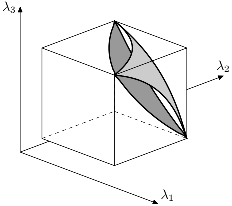

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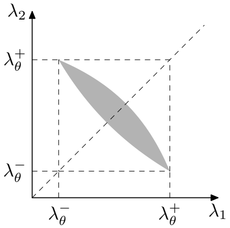

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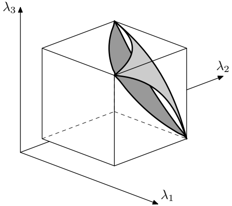

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \textstyle\int_\Omega \theta = q_\alpha\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

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B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

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Goal: find explicit solution for some simple domains (circle)

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If $m < d$ then $\min_{\mathcal{A}} J = \min_{\mathcal{T}} I$ and:

- *There is unique* $\mathsf{u}^* \in \mathrm{H}^1_0(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{A}} J$ *and* $\min_{\mathcal{T}} I$.
- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{A}} J$*, then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$.
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Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric, and let the right-hand sides $f_i = f_i(r), r \in \omega, i = 1, \ldots, m$ be radial functions. Then $\min_{\mathcal{A}} J = \min_{\mathcal{T}} I$ and there is unique (radial) u^* which is the state for any solution of $\min_{\mathcal{A}} J$ and $\min_{\mathcal{T}} I$. Moreover,

a) *For any minimizer θ of functional I over $\mathcal{T}$, let us define a radial function $\theta^* : \Omega \longrightarrow \mathbf{R}$ as the average value over spheres of θ: for $r \in \omega$ we take*

$$\theta^*(r) := \fint_{\partial B(\mathbf{0},r)} \theta \, dS \,,$$

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Spherical symmetry... cont.

Theorem

b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define $\mathbf{A}^* \in \mathcal{K}(\theta^*)$ as a simple laminate with the lamination direction orthogonal to the radial vector $\mathbf{e}_r$, almost everywhere on Ω. To be specific, we define*

$$\mathbf{A}^*(\mathbf{x}) := diag\left(\lambda^+_{\theta^*}(|\mathbf{x}|), \lambda^-_{\theta^*}(|\mathbf{x}|), \lambda^+_{\theta^*}(|\mathbf{x}|), \dots, \lambda^+_{\theta^*}(|\mathbf{x}|)\right) .$$

in spherical basis $(\mathbf{e}_r(\mathbf{x}), \mathbf{e}_{\phi_1}(\mathbf{x}), \mathbf{e}_{\phi_2}(\mathbf{x}), \dots, \mathbf{e}_{\phi_{d-1}}(\mathbf{x}))$. Then $(\theta^*, \mathbf{A}^*)$ is an optimal design for $\min_{\mathcal{A}} J$.

c) *If $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$ is a solution of the relaxed problem $\min_{\mathcal{A}} J$ then θ^* is optimal for $\min_{\mathcal{T}} I$, and $\mathbf{A}^*\nabla u_i^* = \lambda^+_{\theta^*}\nabla u_i^*$, almost everywhere, $i = 1, \dots, m$.*

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Uniqueness on a ball

Lemma

Let Ω be ball $B(\mathbf{0}, R)$, and let the right-hand sides f_i be radial functions, such that mappings $r \mapsto r^{\frac{d-1}{2}} f_i(r)$ belong to $\mathrm{L}^2(\langle 0, R\rangle)$, $i = 1, \dots, m$. Then there are unique radial fluxes

$$\sigma_i^*(r) = -\frac{1}{r^{d-1}} \int_0^r \rho^{d-1} f_i(\rho)\, d\rho\, \mathbf{e}_r$$

corresponding to each minimizer of $\min_{\mathcal{T}} I$, and this minimizer is radial and unique on the set where at least one σ_i^* does not vanish. If the Lagrange multiplier c is positive, this holds true on the whole $B(\mathbf{0}, R)$.

Optimality conditions for $\min_{\mathcal{T}} I$

Lemma

$\theta^* \in \mathcal{T}$ is a solution $\min_{\mathcal{T}} I$ if and only if there exists a Lagrange multiplier $c \geq 0$ such that

$$\begin{aligned} \theta^* \in \langle 0,1\rangle \quad &\Rightarrow \quad \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 = c\,, \\ \theta^* = 0 \quad &\Rightarrow \quad \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 \geq c\,, \\ \theta^* = 1 \quad &\Rightarrow \quad \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 \leq c\,, \end{aligned}$$

or equivalently

$$\begin{aligned} \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 > c \quad &\Rightarrow \quad \theta^* = 0\,, \\ \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 < c \quad &\Rightarrow \quad \theta^* = 1\,. \end{aligned}$$

Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation, $f(r) = 1 - r$

State equation in polar coordinates $$-\frac{1}{r}\left(r\lambda^{+}_{\theta(r)}u'\right)' = 1 - r\,.$$

Integration gives $|u'(r)| = \frac{\psi(r)}{\alpha\theta(r)+\beta(1-\theta(r))}$, where $\psi(r) = \frac{|2r^2-3r|}{6}$.

Conditions of optimality: there exists a constant $\gamma := \sqrt{c} > 0$ such that for optimal θ^* we have:

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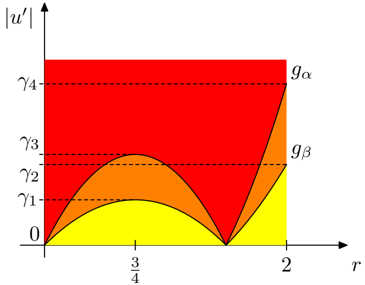

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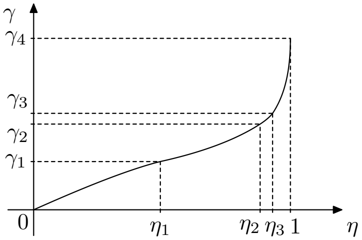

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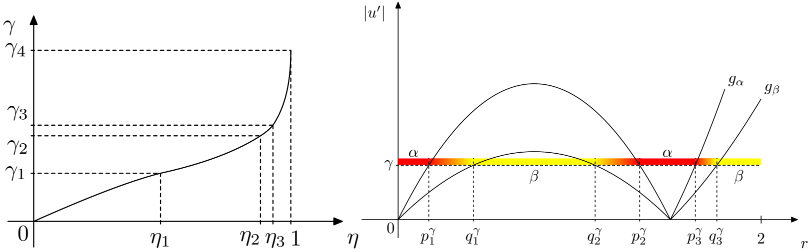

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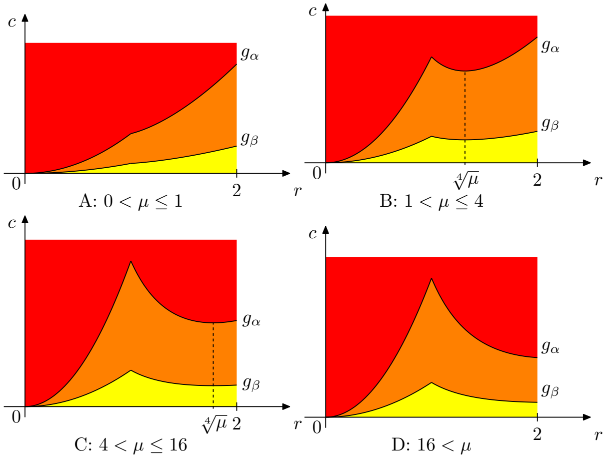
Geometric interpretation of optimality conditions



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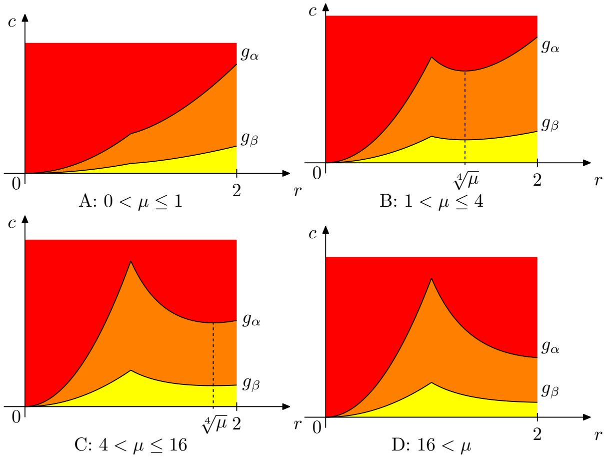

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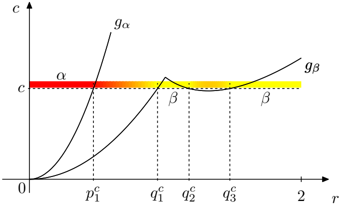
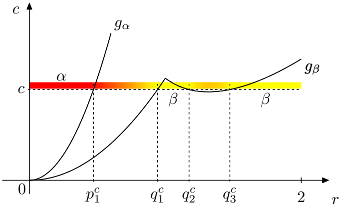


In orange region:

$$\theta^*(r) = \frac{1}{\beta - \alpha}\left(\beta - \sqrt{\frac{\psi(r)}{c}}\right)$$

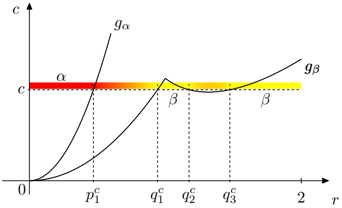
Optimal θ^* for case B



In orange region:

$$\theta^*(r) = \frac{1}{\beta - \alpha}\left(\beta - \sqrt{\frac{\psi(r)}{c}}\right)$$

Optimal θ^* for case B



In orange region:

$$\theta^*(r) = \frac{1}{\beta - \alpha}\left(\beta - \sqrt{\frac{\psi(r)}{c}}\right)$$