Explicit solutions of multiple state optimal design problems

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WeConMApp

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Multiple state optimal design problem

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f_1, \ldots, f_m \in \mathrm{L}^2(\Omega)$ given; stationary diffusion equations with homogenous Dirichlet b. c.:

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{array} \right. , \quad i = 1, \ldots, m \tag{1}$$

where $\mathbf{A}$ is a mixture of two isotropic materials with conductivities $0 < \alpha < \beta$: $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$, $\int_\Omega \chi\, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$.
For given Ω, α, β, q_α, f_i, and some given weights $\mu_i > 0$, we want to find such material $\mathbf{A}$ which minimizes the weighted sum of energies (total amounts of heat/electrical energy dissipated in Ω):

$$I(\chi) := \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min\,, \quad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$$

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single state, $f \equiv 1$, Ω circle / square

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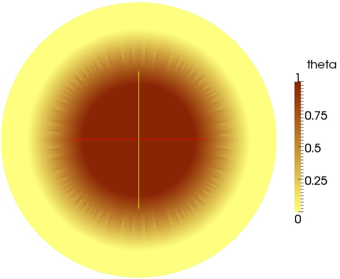
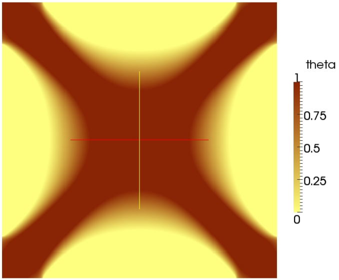


Lurie & Cherkaev



$\chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$ $\cdots$ $\theta \in \mathrm{L}^\infty(\Omega; [0, 1])$

$\mathbf{A} = \chi\alpha\mathbf{I} + (1 - \chi)\beta\mathbf{I}$ $\mathbf{A} \in \mathcal{K}(\theta)$ a.e. on Ω

classical material composite mateiral - relaxation

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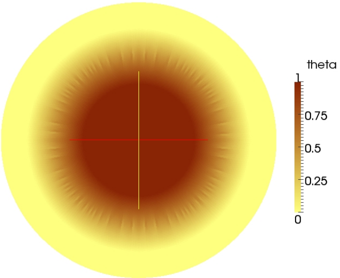

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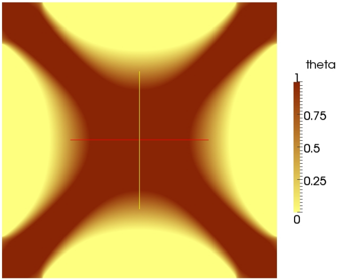

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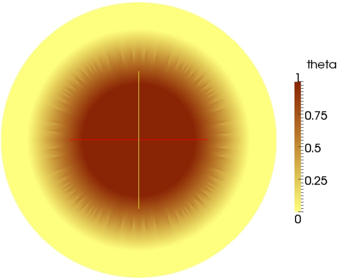

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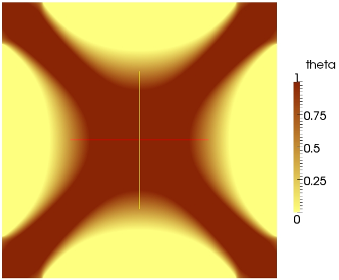

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Composite material

Definition

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega;\{0,1\})$ and conductivities $\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1-\chi_\varepsilon(x))\beta\mathbf{I}$ satisfy $\chi_\varepsilon \rightharpoonup \theta$ weakly $*$ and $\mathbf{A}^\varepsilon$ H-converges to $\mathbf{A}^*$, then it is said that $\mathbf{A}^*$ is homogenised tensor of two-phase composite material with proportions θ of first material and microstructure defined by the sequence (χ_ε).

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A}^* = \mathrm{diag}(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \dots, \lambda_\theta^+)\,,$$

where

$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\,.$$

Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,,\ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.}\}$$

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Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

2D:

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

$$\lambda_\theta^- \leq \quad \lambda_j \quad \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \quad \leq \quad \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

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$\min_{\mathcal{A}} J$ is a proper relaxation of $\min_{\mathrm{L}^\infty(\Omega;\{0,1\})} I$

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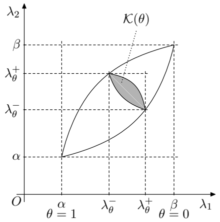

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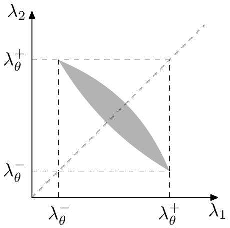

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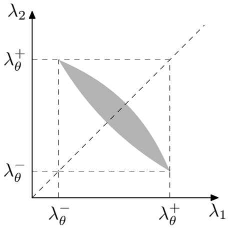

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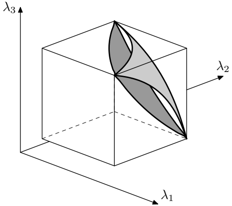

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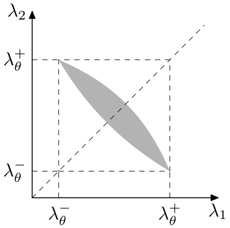

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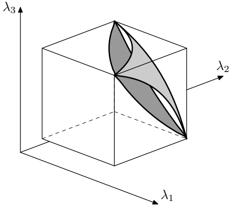

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

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$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

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B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

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How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

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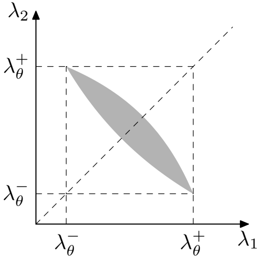
$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \, , \ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$

2D:



Further relaxation:

$\mathcal{B}$ ... $\int_\Omega \theta \, d\mathbf{x} = q_\alpha$

$$\lambda_\theta^- \le \lambda_{\min}(\mathbf{A}) \, , \ \lambda_{\max}(\mathbf{A}) \le \lambda_\theta^+$$

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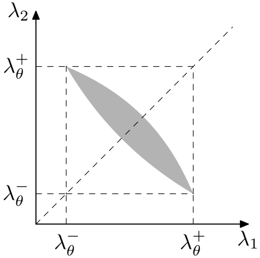

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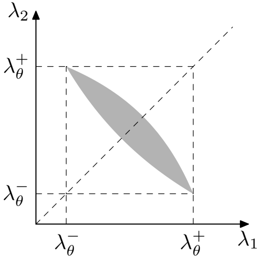

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$\min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I \iff \min_{\mathcal{A}} J$ **if** $m < d$

Theorem

- *There is unique* $\mathrm{u}^* \in \mathrm{H}_0^1(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$*.*
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Spherical symmetry: $\min_{\mathcal{A}} J \iff \min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$

Theorem

Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric, and let the right-hand sides $f_i = f_i(r), r \in \omega, i = 1, \dots, m$ be radial functions. Then $\min_{\mathcal{A}} J = \min_{\mathcal{B}} J = \min_{\mathcal{T}} I$, and there exists a minimizer $(\theta^*, \mathbf{A}^*)$ of the optimal design problem $\min_{\mathcal{A}} J$ which is a radial function. More precisely,

a) *For any minimizer θ of functional I over $\mathcal{T}$, let us define a radial function $\theta^* : \Omega \longrightarrow \mathbf{R}$ as the average value over spheres of θ: for $r \in \omega$ we take*

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Spherical symmetry... cont.

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b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define $\mathbf{A}^* \in \mathcal{K}(\theta^*)$ as a simple laminate with the lamination direction orthogonal to the radial vector $\mathbf{e}_r$, almost everywhere on Ω. To be specific, we define*

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Optimality conditions for $\min_{\mathcal{T}} I$

Lemma

$\theta^* \in \mathcal{T}$ is a solution $\min_{\mathcal{T}} I$ if and only if there exists a Lagrange multiplier $c \geq 0$ such that

$$\begin{aligned} \theta^* \in \langle 0,1\rangle \quad &\Rightarrow \quad \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 = c\,, \\ \theta^* = 0 \quad &\Rightarrow \quad \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 \geq c\,, \\ \theta^* = 1 \quad &\Rightarrow \quad \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 \leq c\,, \end{aligned}$$

or equivalently

$$\begin{aligned} \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 > c \quad &\Rightarrow \quad \theta^* = 0\,, \\ \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 < c \quad &\Rightarrow \quad \theta^* = 1\,. \end{aligned}$$

Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

optimality conditions: $\gamma := \sqrt{c} > 0$

γ is uniquely determined by

$$\int_\Omega \theta^* \, d\mathbf{x} = \eta := \frac{q_\alpha}{|\Omega|} \in [0, 1] \,,$$

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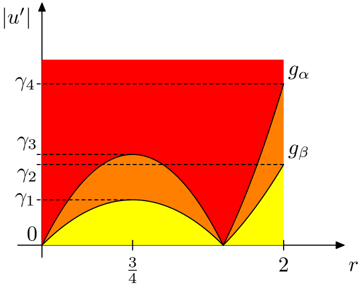

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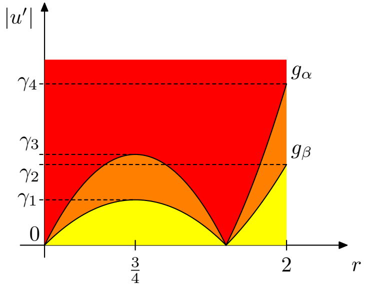

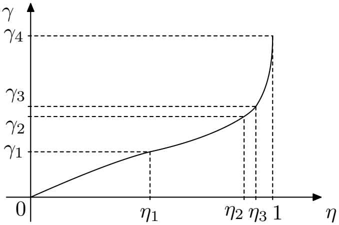

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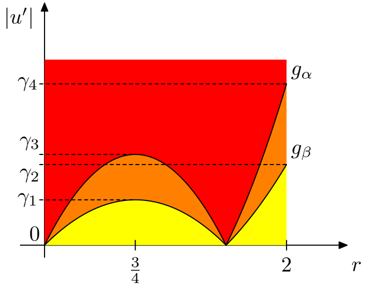

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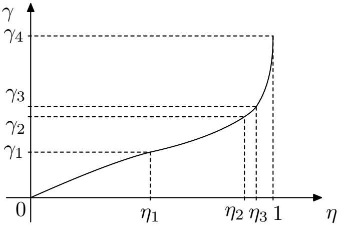

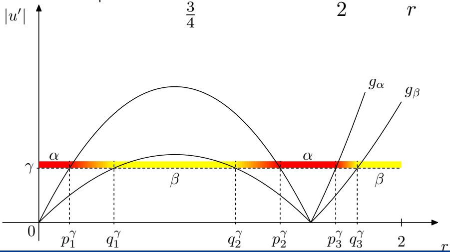

Two state equations on a ball $\Omega = B(\mathbf{0}, 2)$

- $f_1 = \chi_{B(\mathbf{0},1)}\,,\; f_2 \equiv 1\,,$
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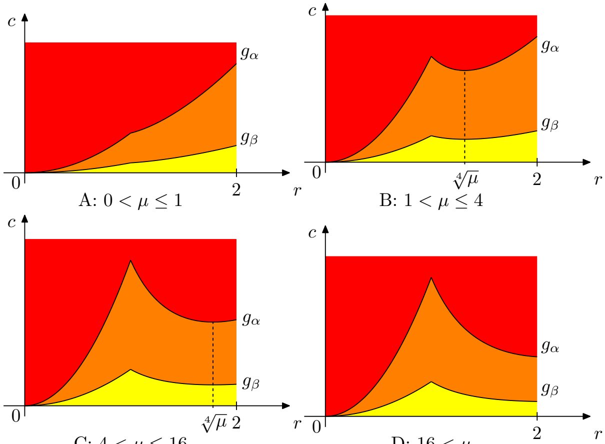

A: $0 < \mu \leq 1$

B: $1 < \mu \leq 4$

C: $4 < \mu \leq 16$

D: $16 < \mu$

Optimal θ^* for case B

As before, Lagrange multiplier can be numerically calculated from corresponding algebraic equation $\fint_\Omega \theta^* \, d\mathbf{x} = \eta$.

Thank you for your attention!

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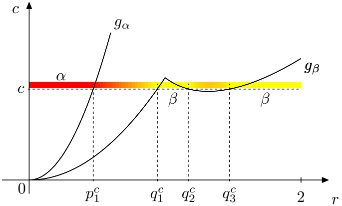

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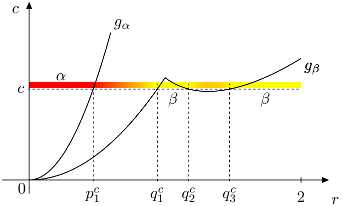
Optimal θ^* for case B

As before, Lagrange multiplier can be numerically calculated from corresponding algebraic equation $⨍_\Omega \theta^* \, d\mathbf{x} = \eta$.



Thank you for your attention!