Homogenization of Kirchhoff-Love plate equation and composite plates

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The physical idea of homogenization is to average a heterogeneous media in order to derive effective properties.

Sequence of similar problems

$$\begin{cases} A_n u_n = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition} . \end{cases}$$

Identify topologies (and limits) s.t.
$u_n \to u$, $A_n u_n \to Au$.
Then the limit (effective) problem is

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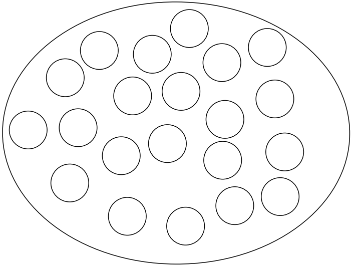

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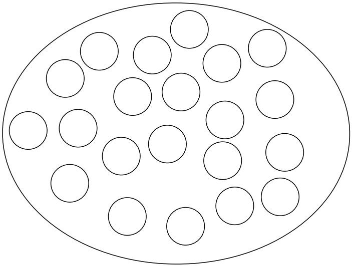

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Kirchhoff-Love plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega). \end{cases}$$

- $\Omega \subseteq \mathbb{R}^d$ bounded domain ($d = 2 \ldots$ plate)
- $f \in H^{-2}(\Omega)$ external load
- $u \in H_0^2(\Omega)$ vertical displacement of the plate
- $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{\mathbf{N} \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\, \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \alpha \mathbf{S} : \mathbf{S} \text{ and } \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \frac{1}{\beta}\mathbf{S} : \mathbf{S} \text{ a.e. } \mathbf{x}\}$ describes properties of material of the given plate

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H-convergence

Definition

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

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coverges weakly to a limit u in $H_0^2(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $L^2(\Omega; \mathrm{Sym})$.

Theorem (Compactness of H-topology)

Let $(\mathbf{M}^n)$ be a sequence in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. Then there is a subsequence $(\mathbf{M}^{n_k})$ and a tensor function $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ such that $(\mathbf{M}^{n_k})$ H-converges to $\mathbf{M}$.

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Compactness

Antonić, Balenović, 1999.

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Theorem (Compactness by compensation)

Let the following convergences be valid:

$$w^n \rightharpoonup w^\infty \quad \text{in} \quad \mathrm{H}^2_{\mathrm{loc}}(\Omega)\,,$$
$$\mathbf{D}^n \rightharpoonup \mathbf{D}^\infty \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})\,,$$

with an additional assumption that the sequence $(\operatorname{div}\operatorname{div}\mathbf{D}^n)$ *is contained in a precompact (for the strong topology) set of the space* $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$. *Then we have*

$$\nabla\nabla w^n : \mathbf{D}^n \overset{*}{\rightharpoonup} \nabla\nabla w^\infty : \mathbf{D}^\infty$$

in the space of Radon measures.

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Outline

- Properties of the H-convergence: locality, irrelevance of boundary conditions, energy convergence, ordering property, metrizability
- Corrector results
- Small-amplitude homogenization, smooth dependence of H-limit on a parameter, H-limit of periodic sequence
- Composite plates: G-closure problem, density of periodic mixtures, laminated materials, Hashin-Shtrikman bounds

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Locality and irrelevance of boundary conditions

Theorem (Locality of the H-convergence)

Let $(\mathbf{M}^n)$ *and* $(\mathbf{O}^n)$ *be two sequences of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, which H-converge to* $\mathbf{M}$ *and* $\mathbf{O}$*, respectively. Let* ω *be an open subset compactly embedded in* Ω*. If* $\mathbf{M}^n(\mathbf{x}) = \mathbf{O}^n(\mathbf{x})$ *in* ω*, then* $\mathbf{M}(\mathbf{x}) = \mathbf{O}(\mathbf{x})$ *in* ω*.*

Theorem (Irrelevance of boundary conditions)

Let $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converges to* $\mathbf{M}$*. For any sequence* (z_n) *such that*

$$\begin{aligned} z_n &\longrightarrow z && \text{in } \mathrm{H}^2_{\mathrm{loc}}(\Omega) \\ \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla z_n) = f_n &\longrightarrow f && \text{in } \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega), \end{aligned}$$

the weak convergence $\mathbf{M}^n \nabla\nabla z_n \rightharpoonup \mathbf{M}\nabla\nabla z$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$ *holds.*

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Energy convergence

Theorem (Energy convergence)

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to $\mathbf{M}$. *For any $f \in H^{-2}(\Omega)$, the sequence (u_n) of solutions of*

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satisfies $\mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \rightharpoonup \mathbf{M}\nabla\nabla u : \nabla\nabla u$ weakly-$*$ in the space of Radon measures and $\int_\Omega \mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \, d\mathbf{x} \to \int_\Omega \mathbf{M}\nabla\nabla u : \nabla\nabla u \, d\mathbf{x}$, where u is the solution of the homogenized equation

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M} \nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega)\,. \end{cases}$$

Ordering property

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Let $(\mathbf{M}^n)$ *and* $(\mathbf{O}^n)$ *be two sequences of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converge to the homogenized tensors* **M** *and* **O***, respectively. Assume that, for any* n*,*

$\mathbf{M}^n\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mathbf{O}^n\boldsymbol{\xi} : \boldsymbol{\xi}, \quad \forall \boldsymbol{\xi} \in \mathrm{Sym}.$

Then the homogenized limits are also ordered:

$\mathbf{M}\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mathbf{O}\boldsymbol{\xi} : \boldsymbol{\xi}, \quad \forall \boldsymbol{\xi} \in \mathrm{Sym}.$

Theorem

Let $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that either converges strongly to a limit tensor* **M** *in* $\mathrm{L}^1(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, or converges to* **M** *almost everywhere in* Ω*. Then,* $\mathbf{M}^n$ *also H-converges to* **M***.*

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Theorem (Metrizability of H-topology)

Let $F = \{f_n : n \in \mathbf{N}\}$ *be a dense countable family in* $H^{-2}(\Omega)$, **M** *and* **O** *tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$, *and* (u_n), (v_n) *sequences of solutions to*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u_n) = f_n \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

and

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{O}\nabla\nabla v_n) = f_n \\ v_n \in \mathrm{H}^2_0(\Omega) \end{cases},$$

respectively. Then,

$$d(\mathbf{M}, \mathbf{O}) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|u_n - v_n\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{M}\nabla\nabla u_n - \mathbf{O}\nabla\nabla v_n\|_{H^{-1}(\Omega;\mathrm{Sym})}}{\|f_n\|_{H^{-2}(\Omega)}}$$

is a metric function on $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *and H-convergence is equivalent to the convergence with respect to* d.

Definition of correctors

Definition

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit **M**. Let $(w_n^{ij})_{1\leq i,j\leq d}$ be a family of test functions satisfying

$$w_n^{ij} \rightharpoonup \frac{1}{2}x_i x_j \quad \text{in} \quad \mathrm{H}^2(\Omega)$$

$$\mathbf{M}^n \nabla\nabla w_n^{ij} \rightharpoonup \cdot \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$$

$$\operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla w_n^{ij}) \to \cdot \quad \text{in} \quad \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega).$$

The sequence of tensors $\mathbf{W}^n$ defined with $\mathbf{W}^n_{ijkm} = [\nabla\nabla w_n^{km}]_{ij}$ is called a sequence of correctors.

Uniqueness of correctors

Theorem

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor $\mathbf{M}$. A sequence of correctors $(\mathbf{W}^n)$ is unique in the sense that, if there exist two sequences of correctors $(\mathbf{W}^n)$ and $(\tilde{\mathbf{W}}^n)$, their difference $(\mathbf{W}^n - \tilde{\mathbf{W}}^n)$ converges strongly to zero in $L^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Corrector result

Theorem

Let $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *which H-converges to* $\mathbf{M}$. *For* $f \in H^{-2}(\Omega)$, *let* (u_n) *be the solution of*

$$\begin{cases} \mathrm{div}\,\mathrm{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \quad \mathrm{in} \quad \Omega \\ u_n \in H_0^2(\Omega)\,, \end{cases}$$

and let u *be the weak limit of* (u_n) *in* $H_0^2(\Omega)$, *i.e., the solution of the homogenized equation*

$$\begin{cases} \mathrm{div}\,\mathrm{div}\,(\mathbf{M} \nabla\nabla u) = f \quad \mathrm{in} \quad \Omega \\ u \in H_0^2(\Omega)\,. \end{cases}$$

Then, $r_n := \nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u \to 0$ *strongly in* $L^1_{\mathrm{loc}}(\Omega; Sym)$.

Small-amplitude homogenization

$$\mathbf{M}_p^n(\mathbf{x}) := \mathbf{A}_0 + p\mathbf{B}^n(\mathbf{x}),\ p \in \mathbf{R}$$

$$\mathbf{M}_p := \mathbf{A}_0 + p\mathbf{B}_0 + p^2\mathbf{C}_0 + o(p^2),\ p \in \mathbf{R}$$

If $p \mapsto \mathbf{M}_n^p$ is a C^k mapping (for any $n \in \mathbf{N}$) from some subset of $\mathbf{R}$ to $L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, what can we say about $p \mapsto \mathbf{M}_p$?

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Smoothness with respect to a parameter

Theorem

Let $\mathbf{M}^n : \Omega \times P \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ *be a sequence of tensors, such that* $\mathbf{M}^n(\cdot, p) \in \mathfrak{M}_2(\alpha, \beta; \Omega)$*, for* $p \in P$*, where* $P \subseteq \mathbf{R}$ *is an open set. Assume that (for some* $k \in \mathbf{N}_0$*) a mapping* $p \mapsto \mathbf{M}^n(\cdot, p)$ *is of class* C^k *from* P *to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, with derivatives (up to order* k*) being equicontinuous on every compact set* $K \subseteq P$*:*

$$(\forall K \in \mathcal{K}(P))\,(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})(\forall i \leq k)$$
$$|p - q| < \delta \Rightarrow \|(\mathbf{M}^n)^{(i)}(\cdot, p) - (\mathbf{M}^n)^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} < \varepsilon.$$

Then there is a subsequence $(\mathbf{M}^{n_k})$ *such that for every* $p \in P$

$$\mathbf{M}^{n_k}(\cdot, p) \xrightarrow{H} \mathbf{M}(\cdot, p) \quad \text{in} \quad \mathfrak{M}_2(\alpha, \beta; \Omega)$$

and $p \mapsto \mathbf{M}(\cdot, p)$ *is a* C^k *mapping from P to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*.*

Periodic case

- $Y = [0,1]^d$, $\mathbf{M} \in \mathrm{L}^\infty_\#(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) \cap \mathfrak{M}_2(\alpha, \beta; Y)$
- $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), \mathbf{x} \in \Omega \subseteq \mathbf{R}^d$ (open and bounded)
- $H^2_\#(Y) := \{f \in H^2_{\mathrm{loc}}(\mathbf{R}^d) \,\mathrm{such\ that\ f\ is\ Y-periodic}\}$ with the norm $\|\cdot\|_{H^2(Y)}$
- $H^2_\#(Y)/\mathbf{R}$ equipped with the norm $\|\nabla\nabla\cdot\|_{L^2(Y)}$
- $\mathbf{E}_{ij}, 1 \leq i,j \leq d$ are $M_{d\times d}$ matrices defined as

$$[\mathbf{E}_{ij}]_{kl} = \begin{cases} 1, & \text{if } i=j=k=l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise.} \end{cases}$$

H-limit of a periodic sequence

Theorem

Let $(\mathbf{M}^n)$ *be a sequence of tensors defined by* $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), x \in \Omega$. *Then* $(\mathbf{M}^n)$ *H-converges to a constant tensor* $\mathbf{M}^* \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ *defined as*

$$m^*_{klij} = \int_Y \mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x})) : (\mathbf{E}_{kl} + \nabla\nabla w_{kl}(\mathbf{x}))\, d\mathbf{x},$$

where $(w_{ij})_{1\leq i,j\leq d}$ *is the family of unique solutions in* $H^2_\#(Y)/\mathbf{R}$ *of boundary value problems*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x}))) = 0 \text{ in Y} \\ \mathbf{x} \to w_{ij}(\mathbf{x}) \quad \text{is } Y\text{-periodic.} \end{cases}$$

Small-amplitude assumptions

Theorem

Let $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym};\mathrm{Sym})$ *be a constant coercive tensor,* $P \subseteq \mathbf{R}$ *an open set,* $\mathbf{B}^n(\mathbf{x}) := \mathbf{B}(n\mathbf{x})$, $\mathbf{x} \in \Omega$, *where* $\Omega \subseteq \mathbf{R}^d$ *is a bounded, open set, and* $\mathbf{B}$ *is a Y-periodic,* L^∞ *tensor function, satisfying* $\int_Y \mathbf{B}(\mathbf{x})\,d\mathbf{x} = 0$.

Then

$$\mathbf{M}_p^n(\mathbf{x}) := \mathbf{A}_0 + p\mathbf{B}^n(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

H-converges (for any $p \in P$*) to a tensor*

$$\mathbf{M}_p := \mathbf{A}_0 + p\mathbf{B}_0 + p^2\mathbf{C}_0 + o(p^2)$$

with coefficients $\mathbf{B}_0 = 0$ *and*

Small-amplitude limit

$$
\begin{aligned}
\mathsf{C}_0\mathbf{E}_{mn} : \mathbf{E}_{rs} = {} & (2\pi i)^2 \sum_{\mathbf{k}\in J} a^{mn}_{-\mathbf{k}} \mathsf{B}_{\mathbf{k}}(\mathbf{k}\otimes\mathbf{k}) : \mathbf{E}_{rs} + \\
& + (2\pi i)^4 \sum_{\mathbf{k}\in J} a^{mn}_{\mathbf{k}} a^{rs}_{-\mathbf{k}} \mathsf{A}_0(\mathbf{k}\otimes\mathbf{k}) : \mathbf{k}\otimes\mathbf{k} + \\
& + (2\pi i)^2 \sum_{\mathbf{k}\in J} a^{rs}_{-\mathbf{k}} \mathsf{B}_{\mathbf{k}} \mathbf{E}_{mn} : \mathbf{k}\otimes\mathbf{k}\,.
\end{aligned}
$$

where $m, n, r, s \in \{1, 2, \cdots, d\}$, $J := \mathbf{Z}^d/\{0\}$, and

$$
a^{mn}_{\mathbf{k}} = -\frac{\mathsf{B}_{\mathbf{k}}\mathbf{E}_{mn}\mathbf{k}\cdot\mathbf{k}}{(2\pi i)^2\mathsf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}, \quad \mathbf{k}\in J, \quad m, n \in \{1, 2, \cdots, d\}
$$

and $\mathsf{B}_{\mathbf{k}}$, $\mathbf{k}\in J$, are Fourier coefficients of function B.

Two-phase composite

Let **A** and **B** be two constant tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$.
We are interested in a material that is their mixture on a fine scale, i.e. in the H-limit of a sequence

$$\mathbf{M}^n(\mathbf{x}) = \chi_n(\mathbf{x})\mathbf{A} + (1 - \chi_n(\mathbf{x}))\mathbf{B}.$$

Here, every χ_n is a characteristic functions of a subset of Ω that is filled with **A** material.

Definition

If a sequence of characteristic functions $\chi_n \in \mathrm{L}^\infty(\Omega; \{0, 1\})$ and above tensors $\mathbf{M}^n$ satisfy $\chi_n \rightharpoonup \theta$ weakly $*$ in $\mathrm{L}^\infty(\Omega; [0, 1])$ and $\mathbf{M}^n$ H-converges to $\mathbf{M}$, then it is said that $\mathbf{M}$ is homogenised tensor of two-phase composite material with proportions θ of the first material and microstructure defined by the sequence (χ_n).

Two-phase composite

Let **A** and **B** be two constant tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$.
We are interested in a material that is their mixture on a fine scale, i.e. in the H-limit of a sequence

$$\mathbf{M}^n(\mathbf{x}) = \chi_n(\mathbf{x})\mathbf{A} + (1 - \chi_n(\mathbf{x}))\mathbf{B}.$$

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More general composites

Fill $\Omega \subseteq \mathbf{R}^d$ with m constant materials $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_m \in \mathfrak{M}_2(\alpha, \beta; \Omega)$:

$$\mathbf{M}^n(\mathbf{x}) = \sum_{i=1}^{m} \chi_i^n(\mathbf{x}) R_n^T(\mathbf{x}) \mathbf{M}_i R_n(\mathbf{x}), \tag{4.1}$$

where $R_n \in \mathrm{L}^\infty(\Omega; SO(\mathbf{R}^d))$, $\boldsymbol{\chi}^n \in \mathrm{L}^\infty(\Omega; T)$ is a sequence of characteristic functions and $T = \{\boldsymbol{\vartheta} \in \{0,1\}^m : \sum_{i=1}^m \vartheta_i = 1\}$. If

$$\boldsymbol{\chi}^n \xrightarrow{*} \boldsymbol{\theta} \text{ in } \mathrm{L}^\infty(\Omega, \mathbf{R}^m),$$

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By $\mathcal{G}_{\boldsymbol{\theta}}$ we denote all $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ obtained in such way (for fixed $\boldsymbol{\theta} \in \mathrm{L}^\infty(\Omega; \overline{T})$)

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G-closure problem

For $\boldsymbol{\theta} \in \overline{T} := \mathrm{Cl}\,\mathrm{conv}T = \{\boldsymbol{\vartheta} \in [0,1]^m : \sum_{i=1}^m \vartheta_i = 1\}$, let

$$G_{\boldsymbol{\theta}} := \{\mathbf{M} \in \mathcal{L}(\mathrm{Sym}) : (\exists\,(\boldsymbol{\chi}^n)\,\&\,(R_n)) \quad \boldsymbol{\chi}^n \xrightarrow{*} \boldsymbol{\theta} \text{ in } \mathrm{L}^\infty(\Omega;\mathbf{R}^m)\,\&$$

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Theorem

For $\boldsymbol{\theta} \in \mathrm{L}^\infty(\Omega, \overline{T})$ it holds

$$\mathcal{G}_{\boldsymbol{\theta}} = \{\mathbf{M} \in \mathcal{L}(\mathrm{Sym}) : \mathbf{M}(x) \in G_{\boldsymbol{\theta}(x)},\ \textit{a.e. } \mathbf{x} \in \Omega\}. \qquad (4.2)$$

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Density of periodic mixtures

Let $\mathbf{M}(x) := \sum_{i=1}^{m} \chi_i(\mathbf{x}) R^T(\mathbf{x}) \mathbf{M}_i R(\mathbf{x})$, for $\boldsymbol{\chi} \in \mathrm{L}^\infty(Y; T)$, $R \in \mathrm{L}^\infty(Y; SO(\mathbf{R}^d))$, and let us extend these functions periodically to $\mathbf{R}^d$. Take $\boldsymbol{\chi}_n(\mathbf{x}) := \boldsymbol{\chi}(n\mathbf{x})$, $R_n(\mathbf{x}) := R(n\mathbf{x})$ and $\mathbf{M}_n(\mathbf{x}) = \mathbf{M}(n\mathbf{x})$, so that

$$\boldsymbol{\chi}_n \xrightarrow{*} \boldsymbol{\theta} := \int_Y \boldsymbol{\chi} \, d\mathbf{x}$$

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$$\mathbf{M}_n \xrightarrow{H} \mathbf{M}^*$$

For fixed $\boldsymbol{\theta} \in \overline{T}$, all H-limits $\mathbf{M}^*$ obtained in this way we denote by $P_{\boldsymbol{\theta}}$.

Theorem

For every $\boldsymbol{\theta} \in \overline{T}$ it holds $G_{\boldsymbol{\theta}} = \mathrm{Cl} P_{\boldsymbol{\theta}}$.

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Two-phase simple laminates

Lemma

Let **A** *and* **B** *be two constant tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *and* $\chi_n(x_1)$ *be a sequence of characteristic functions that converges to* $\theta(x_1)$ *in* $\mathrm{L}^\infty(\Omega; [0, 1])$ *weakly-*$*$*. Then, a sequence* $(\mathbf{M}^n)$ *of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, defined as*

$$\mathbf{M}^n(x_1) = \chi_n(x_1)\mathbf{A} + (1 - \chi_n(x_1))\mathbf{B}$$

H-converges to

$$\begin{aligned}\mathbf{M}^* &= \theta\mathbf{A} + (1-\theta)\mathbf{B} \\ &- \frac{\theta(1-\theta)(\mathbf{A}-\mathbf{B})(\mathsf{e}_1 \otimes \mathsf{e}_1) \otimes (\mathbf{A}-\mathbf{B})^T(\mathsf{e}_1 \otimes \mathsf{e}_1)}{(1-\theta)\mathbf{A}(\mathsf{e}_1 \otimes \mathsf{e}_1) : (\mathsf{e}_1 \otimes \mathsf{e}_1) + \theta\mathbf{B}(\mathsf{e}_1 \otimes \mathsf{e}_1) : (\mathsf{e}_1 \otimes \mathsf{e}_1)},\end{aligned}$$

which also depends only on x_1.

Two-phase simple laminates cont.

Corollary

If we take some other unit vector e $\in \mathbf{R}^d$ *for lamination direction, and let* $\theta(x \cdot \mathsf{e})$ *be the weak limit of the sequence* $\chi_n(x \cdot \mathsf{e})$*, then the corrsponding H-limit is*

$$\mathbf{M}^* = \theta\mathbf{A} + (1-\theta)\mathbf{B} - \frac{\theta(1-\theta)(\mathbf{A}-\mathbf{B})(\mathsf{e}\otimes\mathsf{e})\otimes(\mathbf{A}-\mathbf{B})^T(\mathsf{e}\otimes\mathsf{e})}{(1-\theta)\mathbf{A}(\mathsf{e}\otimes\mathsf{e}):(\mathsf{e}\otimes\mathsf{e})+\theta\mathbf{B}(\mathsf{e}\otimes\mathsf{e}):(\mathsf{e}\otimes\mathsf{e})}.$$

Corollary

If $(\mathbf{A}-\mathbf{B})$ *is invertible, then the above formula is equivalent to*

$$\theta(\mathbf{M}^*-\mathbf{B})^{-1} = (\mathbf{A}-\mathbf{B})^{-1} + \frac{1-\theta}{\mathbf{B}(\mathsf{e}\otimes\mathsf{e}):(\mathsf{e}\otimes\mathsf{e})}(\mathsf{e}\otimes\mathsf{e})\otimes(\mathsf{e}\otimes\mathsf{e}).$$

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Sequential laminates

rank-1 (sim.) laminate $\mathbf{A}_1^*$: mix **A**, **B**; proportions $\theta_1, 1-\theta_1$; direction e_1.
rank-2 sequential lam. $\mathbf{A}_2^*$: mix $\mathbf{A}_1^*$, **B**; proportions $\theta_2, 1-\theta_2$; direction e_2.
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rank-p seq. lam. $\mathbf{A}_p^*$: mix $\mathbf{A}_{p-1}^*$, **B**; proportions $\theta_p, 1-\theta_p$; direction e_p.
A - core phase, **B** - matrix phase

Theorem

Let $\theta_i \in [0,1]$ and let $\mathrm{e}_i \in \mathbf{R}^d$ be unit vectors, $1 \le i \le p$. Then

$$\Big(\prod_{j=1}^{p} \theta_j\Big)(\mathbf{A}_p^* - \mathbf{B})^{-1} = (\mathbf{A} - \mathbf{B})^{-1} + \sum_{i=1}^{p}\Big((1-\theta_i)\prod_{j=1}^{i-1}\theta_j\Big)\frac{(\mathrm{e}_i \otimes \mathrm{e}_i)\otimes(\mathrm{e}_i \otimes \mathrm{e}_i)}{\mathbf{B}(\mathrm{e}_i \otimes \mathrm{e}_i):(\mathrm{e}_i \otimes \mathrm{e}_i)}.$$

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Hashin-Shtrikman bounds

Theorem

For any $\boldsymbol{\xi} \in \mathrm{Sym}$, *the effective energy of a composite material* $\mathbf{A}^* \in G_\theta$ *satisfies the following bounds:*

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq \mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} + (1-\theta) \max_{\boldsymbol{\eta} \in \mathrm{Sym}} [2\boldsymbol{\xi} : \boldsymbol{\eta} - (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - \theta g(\boldsymbol{\eta})],$$

where $g(\boldsymbol{\eta})$ *is defined by*

$$g(\boldsymbol{\eta}) = \sup_{\mathbf{k} \in \mathbf{Z}^d, \mathbf{k} \neq 0} \frac{|(\mathbf{k} \otimes \mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{k} \otimes \mathbf{k}) : (\mathbf{k} \otimes \mathbf{k})}$$

and

Hashin-Shtrikman bounds cont.

Theorem

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mathbf{B}\boldsymbol{\xi} : \boldsymbol{\xi} + \theta \min_{\boldsymbol{\eta}\in\mathrm{Sym}} [2\boldsymbol{\xi} : \boldsymbol{\eta} + (\mathbf{B} - \mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta)h(\boldsymbol{\eta})],$$

where $h(\boldsymbol{\eta})$ *is defined by*

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Moreover, these bounde are optimal, and optimality is achieved by a finite-rank sequential laminate.

Now what?

- Small-amplitude homogenization - non-periodic case
- Explicit Hashin-Shtrikman bounds (2D, isotropic case, first correction in small-amplitude regime)
- G-closure problem
- Optimal design of plates

Thank you for your attention!

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