# Stability of Observations of Partial Differential Equations under Uncertain Perturbations

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# Outline

# Observability problem for the wave equation

We consider the wave equation:

$$\partial_{tt}u - \operatorname{div}(\mathbf{A}(t,\mathbf{x})\nabla u) = 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega$$
$$u(0,\cdot) = u^0 \in \mathrm{L}^2(\Omega)$$
$$\partial_t u(0,\cdot) = \tilde{u}^0 \in \mathrm{H}^{-1}\Omega$$
$$+ \text{ bounded conditions}\,.$$

Observability problem: Under which conditions can we recover the (initial) energy of the system by observing the solution on a suitable subdomain?

$$E(0) \leqslant C \int_0^T \int_\omega |u|^2 d\mathbf{x}dt?$$

$E(0) := \|u^0\|^2_{\mathrm{L}^2} + \|\tilde{u}^0\|_{\mathrm{H}^{-1}}$

**Answer**[1]: The observability region has to satisfy the Geometric Control Condition (GCC), stating that each characteristic ray has to enter the region in a finite time.

[1] C. Bardos, G. Lebeau, J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30(5)** (1992) 1024–1065.

# Controllability problem for the wave equation

observability $\Longleftrightarrow$ controllability

The mentioned observability problem is equivalent to the following control one for the adjoint system:

$$\begin{aligned} \partial_{tt} v - \mathsf{div}\,(\mathbf{A}(t,\mathbf{x})\nabla v) &= f\chi_{\langle 0,T\rangle\times\omega}, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\ v(0,\cdot) &= v_0 \in \mathrm{H}^1\Omega \\ \partial_t v(0,\cdot) &= \tilde{v}_0 \in \mathrm{L}^2(\Omega)\,. \end{aligned}$$

For any given initial data can we find the control $f$ such that the system is driven to an arbitrary (e.g. zero) state in a finite time:

$$v(T,\cdot) = \partial_t v(T,\cdot) = 0\,?$$

# Robust observability for the wave equation

We consider the system

$$
\begin{aligned}
P_1 u_1 = \partial_{tt} u_1 - \mathsf{div}\,(\mathbf{A}_1(t,\mathbf{x})\nabla u_1) &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\
P_2 u_2 &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\
u_1 &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \partial\Omega \\
u_1(0,\cdot) &= u_1^0 \in \mathrm{L}^2(\Omega) \\
\partial_t u_1(0,\cdot) &= \tilde{u}_1^0 \in \mathrm{H}^{-1}(\Omega)\,,
\end{aligned}
\tag{1}
$$

$\Omega$ – is an open, bounded set in $\mathbf{R}^d$
$\mathbf{A}_1$ – bounded, positive definite matrix field
$P_2$ – perturbation operator
$\mathcal{A}_1 = -\mathsf{div}\,(\mathbf{A}_1\nabla)$ – the elliptic part of $P_1$
Coefficients of both the operators are bounded and continuous.

# Robust observability for the wave equation

We assume for the $1^{\text{st}}$ component

$$E_1(0) := \|u_1^0\|_{\mathrm{L}^2}^2 + \|\tilde{u}_1^0\|_{\mathrm{H}^{-1}}^2 \leq \tilde{C} \int_0^T \int_\omega |u_1|^2 d\mathbf{x}dt \,.$$

## The key problem

Under which conditions the observability estimate remains stable under perturbation of the solution by $u_2$?

$$E_1(0) \leq C \int_0^T \int_\omega |u_1 + u_2|^2 d\mathbf{x}dt \,.$$

# The relaxed observability inequality

**Theorem 1.**

*Suppose that characteristic sets* $\{p_i(t, \mathbf{x}, \tau, \boldsymbol{\xi}) = 0\}, i = 1, 2$ *have no intersection for* $(t, \mathbf{x}) \in \langle 0, T\rangle \times \omega, (\tau, \boldsymbol{\xi}) \in \mathrm{S}^d$*.*
*Then there exists a constant* $\tilde{C}$ *such that the observability inequality*

$$E_1(0) \leq \tilde{C} \left( \int_0^T \int_\omega |u_1 + u_2|^2 d\mathbf{x}dt + \|u_1^0\|^2_{\mathrm{H}^{-1}} + \|\tilde{u}_1^0\|^2_{\mathrm{H}^{-2}} \right) \qquad (2)$$

*holds for any pair of solutions* $(u_1, u_2)$ *to* (1)*.*

- The theorem allows for quite a general class of perturbation operators.
- No assumption on initial/boundary data for the $2^{\mathrm{nd}}$ component.
- Non-hyperbolic operators directly satisfy the assumption.

# The relaxed observability inequality

- Non-hyperbolic operators directly satisfy the assumption.

**Example: $P_2$ – Scrödinger operator**

$$p_2 = \mathbf{A}_2 \boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0 \Longleftrightarrow \boldsymbol{\xi} = 0$$

has no intersection with

$$p_1 = \tau^2 - \mathbf{A}_1 \boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0.$$

- For the wave operator $P_2$ the assumption reads as $(\mathbf{A}_1 - \mathbf{A}_2)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \neq 0$ on the observability region.
- The obstacle – compact term on the right hand side.

The proof

- goes by contradiction:
- based on H-measures.

# H-measures

A microlocal defect tool: Suppose $u_n \rightharpoonup u$ in $\mathrm{L}^2_{\mathrm{loc}}$

$$\mu = 0 \ \text{ iff } \ u_n \longrightarrow u \text{ in } \mathrm{L}^2_{\mathrm{loc}} \text{ strongly}$$

## Localisation principle

P – (pseudo)differential operator

$(u_n)$ – ($\mathrm{L}^2$ bounded) sequence of solutions to the equation

$$P u_n = 0$$

For H-measure $\mu \sim (u^n)$

$$p\mu = 0,$$

p - the principal symbol of $P$.

Specially for the wave equation

$$\partial_{tt} u^n - \mathrm{div}\,(\mathbf{A}(t,\mathbf{x})\nabla u^n) = 0,$$

the measure $\mu \sim (\nabla u^n)$ satisfies

$$(\tau^2 - \mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi})\mu = 0.$$

# The strong observability inequality

We want to get rid of the compact term in (2).
Additional assumptions:

- $P_2$ – an evolution operator

$$P_2 = (\partial_t)^k + c_2(\mathbf{x})\mathcal{A}_1, \quad k \in \mathbf{N},$$

$\mathcal{A}_1$ – an elliptic part of the wave operator $P_1$,
$c_2$ – a bounded and continuous function, $\begin{cases} \neq 1, & k = 2 \\ \neq 0, & k \neq 2 \end{cases}$ on $\omega$.

- initial data - related by a linear operator such that

$$\Big( (u_1(0) + u_2(0))|_\omega = 0 \Big) \Longrightarrow \Big( u_1(0)|_\omega = u_2(0)|_{|\omega} = 0 \Big),$$

and similarly for initial derivatives.

Then there is $C \in \mathbf{R}^+$ such that the strong observability inequality holds:

$$E_1(0) \leq C \int_0^T \int_\omega |u_1 + u_2|^2 d\mathbf{x} dt \,. \tag{3}$$

# Relation to the control theory

If:

- $P_2$ is a wave operator
- initial data coincide on the whole domain $\Omega$

the strong observability inequality (3) is equivalent to the averaged controllability of the adjoint system under a single control

$$\begin{aligned} \partial_{tt} v_i - \mathrm{div}\,(c_i \nabla v_i) &= \chi_{\langle 0,T\rangle\times\omega} f, \qquad (t, \mathbf{x}) \in \mathbf{R}^+ \times \Omega \\ v_i(0,\cdot) &= v_i^0 \in \mathrm{H}^1(\Omega) \\ \partial_t v_i(0,\cdot) &= \tilde{v}_i^0 \in \mathrm{L}^2(\Omega), \quad i = 1, 2\,, \end{aligned} \tag{4}$$

with $f \in \mathrm{L}^2(\mathbf{R}^+ \times \Omega)$. More precisely, the following result holds.

**Theorem**

*For any choice of initial data of the system* (4) *and any final target* $(v^T, \tilde{v}^T) \in \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega)$ *there exists a control* $f$ *such that*

$$(v_1 + v_2)(T,\cdot) = v^T, \quad \partial_t (v_1 + v_2)(T,\cdot) = \tilde{v}^T.$$

# Relation to the existing results

The last, averaged control result already obtained in:

LZ14 M. L., E. Zuazua, *Averaged control and observation of parameter-depending wave equations*, C. R. Acad. Sci. Paris, Ser. I **352** (2014) 497–502

Presented work generalises the observability results of [LZ14] by allowing for a general evolution operator $P_2$ which does not have to be the wave one.

The proof of the relaxed observability inequality (2) does not rely on the propagation property of H-measures, which allows for system's coefficients to be merely continuous (instead of $\mathrm{C}^{1,1}$).

Such approach avoid technical issues related to the reflection of H-measures on the domain boundary.

# Observation of the Schrödinger equation under non-parabolic perturbations

We consider a system in which the first component, the one for which observation is made, satisfies the Schrödinger equation:

$$\begin{aligned} P_1 u_1 = i\partial_t u_1 + \mathsf{div}\,(\mathbf{A}_1(\mathbf{x})\nabla u_1) &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\ P_2 u_2 &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\ u_1 &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \partial\Omega \\ u_1(0,\cdot) = u_1^0 &\in \mathrm{L}^2(\Omega)\,. \end{aligned} \tag{5}$$

We suppose the observability inequality holds for the main component

$$E_1(0) := \|u_1^0\|_{\mathrm{L}^2}^2 \leq \tilde{C} \int_0^T \int_\omega |u_1|^2 d\mathbf{x}dt\,.$$

Does it remain stable under additive perturbations by $u_2$?

# Observation of the Schrödinger equation under non-parabolic perturbations

As for the wave equation we need to assume separation of characteristic sets $\{p_i(t, \mathbf{x}, \tau, \boldsymbol{\xi}) = 0\}, i = 1, 2$.
The problem - how to obtain the separation for two Schrödinger operators?

$$p_1 = \mathbf{A}_1 \boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0 \Longleftrightarrow \boldsymbol{\xi} = 0$$

and similarly

$$p_2 = \mathbf{A}_2 \boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0 \Longleftrightarrow \boldsymbol{\xi} = 0$$

No separation of coefficients $\mathbf{A}_i$ will imply separation of characteristic sets.
The solution: – a microlocal defect tool better adopted to a study of parabolic problems:

parabolic H-measures .

# Parabolic H-measures [2]

– similar to the original ones
– difference in scaling of the dual variable, take into account 1:2 ratio between time and space variables

**Localisation principle for parabolic H-measures**

$P$ – a (pseudo)differential operator whose *principal part* is of the type

$$\partial_t^m(a_0\,\cdot) + \sum_{|\boldsymbol{\alpha}|=2m} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(a_{\boldsymbol{\alpha}}\,\cdot)$$

$a_0, a_{\boldsymbol{\alpha}}$ – bounded and continuous coefficients
$(u_n)$ – ($\mathrm{L}^2(\mathbf{R}^{1+d})$ bounded) sequence of solutions to the equation

$$Pu_n = 0$$

For a parabolic H-measure $\mu \sim (u^n)$

$$\Big((2\pi i\tau)^m a_0 + \sum_{|\boldsymbol{\alpha}|=2m} (2\pi i\boldsymbol{\xi})^{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}\Big)\mu = 0.$$

[2] N. Antonić, M.L.: *Parabolic H-measures*, *J. Funct. Anal.* **265** (2013) 1190–1239.

## Example: – the Scrödinger operator

Let $(u_n)$ be a sequence of solutions to the Schrödinger equation

$$i\partial_t u_n + \operatorname{div}\left(\mathbf{A}(t,\mathbf{x})\nabla u_n\right) = 0.$$

The associated parabolic H-measure $\mu$ satisfies

$$\left(2\pi\tau + 4\pi^2\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi}\right)\mu = 0,$$

The measure $\boldsymbol{\mu}$ is supported within the *parabolic characteristic set*

$$2\pi\tau = -4\pi^2\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi}.$$

Consequence: two Scrödinger operators with separated coefficients

$$(\mathbf{A}_1 - \mathbf{A}_2)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \neq 0$$

have disjoint characteristic sets, as well as supports of corresponding parabolic H-measures.

# Robust observation of the Schrödinger equation

We reconsider the system (5)

$$
\begin{aligned}
P_1 u_1 = i\partial_t u_1 + \mathsf{div}\,(\mathbf{A}_1(\mathbf{x})\nabla u_1) &= 0, & (t,\mathbf{x}) &\in \mathbf{R}^+ \times \Omega \\
P_2 u_2 &= 0, & (t,\mathbf{x}) &\in \mathbf{R}^+ \times \Omega \\
u_1 &= 0, & (t,\mathbf{x}) &\in \mathbf{R}^+ \times \partial\Omega \\
u_1(0,\cdot) &= u_1^0 \in \mathrm{L}^2(\Omega)\,.
\end{aligned}
$$

We suppose the observability inequality holds for the main component

$$
E_1(0) := \|u_1^0\|_{\mathrm{L}^2}^2 \leq \tilde{C} \int_0^T \int_\omega |u_1|^2 d\mathbf{x}dt\,.
$$

# Robust observation of the Schrödinger equation

As for the wave equation we assume:

- $P_2$ – an evolution operator

$$P_2 = (\partial_t)^k + c_2(\mathbf{x})\mathcal{A}_1 , \quad k >= 1,$$

$\mathcal{A}_1$ – an elliptic part of the wave operator $P_1$,
$c_2$ – a bounded and continuous function, $\begin{cases} \neq 1, & k = 1 \\ \neq 0, & k > 1 \end{cases}$ on $\omega$.

- initial data - related by a linear operator such that

$$\Big( \big(u_1(0) + u_2(0)\big)|_\omega = 0 \Big) \Longrightarrow \Big( u_1(0)|_\omega = u_2(0)|_{|\omega} = 0 \Big),$$

Then there is $C \in \mathbf{R}^+$ such that the strong observability inequality holds:

$$E_1(0) \leq C \int_0^T \int_\omega |u_1 + u_2|^2 d\mathbf{x} dt \, .$$

# Open problems and perspectives

Ongoing work:

- to obtain the result for a more general perturbation operator $P_2$
- to remove constraints on initial data
- to consider larger systems:
  – with $N$ components;
  – or even infinite number of them (both discrete and continuous).

Such generalisations already obtained for a relaxed observability inequality.
Technical difficulties related to the passage to the strong inequality.

The solution - a better microlocal defect tool:

1-scale H-measures.

Happy anniversary!