Some new applications of microlocal defect functionals

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Motivation

In [5], a problem of type (9) was considered, but with flux and diffusion independent of t and $\mathbf{x}$. The homogeneity of the flux allowed the separation of coefficients from the unknown u_n (by applying the Fourier transform). In our work (in progress) we consider the situation of inhomogeneous rough flux.

In [2], a problem of the type $\partial_t u = \triangle_{\mathbf{x}'} u + \partial_y f(u)$, where $\mathbf{x} = (\mathbf{x}', y) \in \mathbf{R}^d$, was considered. A matter is convected in the y-direction, while it is at the same type diffused in all other orthogonal directions. Such problem arose while studying asymptotic behaviour of nonlinear diffusion-convection model $\partial_t u = \triangle_{\mathbf{x}} u + \partial_y f(u)$.

In some applications, when a flow occurs in the highly heterogeneous porous media (e.g. in the CO_2 sequestration problems [4]), we get rough coefficients and flux in the resulting model.

H-measures

Theorem. *If $(u_n)_{n\in\mathbf{N}}$ is a sequence in $\mathrm{L}^2_{loc}(\Omega;\mathbf{R}^r)$, $\Omega \subset \mathbf{R}^{d+1}$, such that $u_n \rightharpoonup 0$ in $\mathrm{L}^2_{loc}(\Omega)$, then there exists subsequence $(u_{n'})_{n'} \subset (u_n)_n$ and positive complex bounded measure $\mu = \{\mu^{jk}\}_{j,k=1,\dots,r}$ on $\mathbf{R}^{d+1}\times \mathrm{S}^d$ such that for all $\varphi_1, \varphi_2 \in \mathrm{C}_0(\Omega)$ and $\psi \in \mathrm{C}(\mathrm{S}^d)$,*

$$\begin{aligned}\lim_{n'\to\infty} \int_\Omega (\varphi_1 u^j_{n'})(\boldsymbol{\xi}) \overline{\mathcal{A}_{\psi(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})}(\varphi_2 u^k_{n'})(\boldsymbol{\xi})} d\mathbf{x} &= \langle \mu^{jk}, \varphi_1\bar{\varphi}_2\psi\rangle \\ &= \int_{\mathbf{R}^{d+1}\times \mathrm{S}^d} \varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})}\psi(\boldsymbol{\xi}) d\mu^{jk}(\mathbf{x},\boldsymbol{\xi})\end{aligned} \tag{1}$$

where $\mathcal{A}_{\psi(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})}$ is the multiplier operator with the symbol $\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)$. ∎

H-measure *see* only derivatives of the same highest order. For example, we can change the scaling and put $\boldsymbol{\xi}/(|(\xi_1,\dots,\xi_k)| + |(\xi_{k+1},\dots,\xi_d)|^2)$ instead of $\boldsymbol{\xi}/|\boldsymbol{\xi}|$, but such H-measure will be able to *see* the first order derivatives with respect to $(x_1,\dots,x_k)$ and second order derivatives with respect to $(x_{i+1},\dots,x_d)$.
In other words, no changing of the highest order of the equation is permitted. We overcome this situation by considering multiplier operators with symbols of the form

$$\psi\left(\frac{(\tau,\boldsymbol{\xi})}{|(\tau,\boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\right), \quad \psi \in \mathrm{C}(\mathbf{R}^d),$$

where the matrix a represents the diffusion matrix in the degenerate parabolic equation.
By $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^+\times\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^+\times\mathbf{R}^d)$ we shall denote the Fourier multiplier operator with respect to $(t,\mathbf{x})$-variables.

Preliminaries from matrix analysis

Let A be a non-negative definite symmetric matrix of order d. we can write $A = \sigma^T\sigma$, where

$$\sigma = \begin{bmatrix} [\sigma_{11}] & [\sigma_{12}] \\ O & O \end{bmatrix},$$

where we assume that $[\sigma_{11}]$ is regular matrix of order $k\times k$. We will need a change of variables $\boldsymbol{\eta} = M\boldsymbol{\xi}$, where

$$M = \begin{bmatrix} [\sigma_{11}] & [\sigma_{12}] \\ O & I \end{bmatrix}.$$

If $[\sigma_{11}]$ were not regular, then we would just define matrix M in a different way:

$$M = \begin{bmatrix} [\sigma_{11}] & [\sigma_{12}] \\ \tilde{I}_k & \tilde{I}_{d-k} \end{bmatrix},$$

where $\tilde{I}_k$ is a matrix with ones on the main diagonal on the places of columns of $[\sigma_{11}]$ which do not form a linearly independent set, and zeroes otherwise. Similarly for $\tilde{I}_{d-k}$.

Preliminaries from matrix analysis II

Clearly, it is a regular change of variables and it holds

$$\boldsymbol{\eta} = (\tilde{\boldsymbol{\eta}}, \mathring{\boldsymbol{\eta}}) = ([\sigma\boldsymbol{\xi}]_{1,\dots,k}, \xi_{k+1},\dots,\xi_d). \tag{2}$$

Its inverse is given by:

$$M^{-1} = \begin{bmatrix} [\sigma_{11}]^{-1} & -[\sigma_{12}][\sigma_{11}]^{-1} \\ O & I \end{bmatrix}.$$

Since A is only assumed to be non-negative definite, we can not obtain the bound of $\|M^{-1}\|_2$ only in terms of A. For matrix M one easily gets $\|M\|_2 \le \max\{1, \|A\|_2\} + \|A\|_2$.
In the case where $A(t)$ depends continuously only one one parameter, we get that the corresponding norms depend continuously on t as well.

Marczinkiewicz multiplier result

Corollary. *Suppose that $\psi \in \mathrm{C}^d(\mathbf{R}^d \setminus \cup_{j=1}^d \{\xi_j = 0\})$ is a bounded function such that for some constant $C > 0$ it holds*

$$|\boldsymbol{\xi}^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})| \le C, \quad \boldsymbol{\xi} \in \mathbf{R}^d\setminus \cup_{j=1}^d\{\xi_j = 0\} \tag{3}$$

for every multi-index $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d) \in \mathbf{N}_0^d$ such that $|\boldsymbol{\alpha}| = \alpha_1+\alpha_2+\dots+\alpha_d \le d$. Then, the function ψ is an L^p-multiplier for $p\in\langle 1,\infty\rangle$, and the operator norm of $\mathcal{A}_\psi$ equals to $C_{d,p}\cdot C$, where $C_{d,p}$ depends only on p and d. ∎

Lemma. *If ψ is a symbol of a multiplier bounded on $\mathrm{L}^p(\mathbf{R}^d)$, then the functions defined by $\psi(\,\cdot + \mathbf{y}_0)$, $\mathbf{y}_0\in\mathbf{R}^d$, $\psi(\lambda\,\cdot\,)$, $\lambda > 0$, and $\psi(Q\,\cdot\,)$, Q orthogonal matrix, are symbols of multipliers bounded on L^p with the same operator norm as $\mathcal{A}_\psi$.* ∎

Fourier multipliers I

Let $a : \mathbf{R} \to M^{d\times d}$ be a non-negative definite matrix. Let

$$\pi_P(\tau,\boldsymbol{\xi},\lambda) = \frac{(\tau,\boldsymbol{\xi})}{|(\tau,\boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}.$$

By Π we will denote the closure of the set $\pi_P(\mathbf{R}\times\mathbf{R}^d\times\mathbf{R})$. Our next step is to show that for $\psi\in\mathrm{C}^{d+1}(\Pi)$ the composition $\psi(\pi_P)$ is a symbol of an $\mathrm{L}^p(\mathbf{R}^{d+1})$ multiplier (here, we consider λ to be fixed).

Lemma. *Under conditions stated above, $\psi(\pi_P)$ is an L^p multiplier.* ∎

We will show that a Fourier multiplier with a symbol $\partial_j^{1/2}\circ\partial_\lambda\left(\frac{1}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\right)$ satisfies conditions of Marcinkiewicz's multiplier theorem.
The symbol of $\partial_\lambda\left(\mathcal{A}_{\frac{1}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}}\right)$ is:

$$\partial_\lambda\left(\frac{1}{|(\tau,\boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\right) = \frac{-\langle a'(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{(|(\tau,\boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle)^2}.$$

Fourier multipliers II

Using a representation $a(\lambda) = \sigma(\lambda)^T\sigma(\lambda)$ and the change of variables $\boldsymbol{\eta} = M\boldsymbol{\xi}$, we have

$$\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle = |\tilde{\boldsymbol{\eta}}|^2, \quad \partial_\lambda(\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle) = 2\langle\sigma'(\lambda)M^{-1}\boldsymbol{\eta},\tilde{\boldsymbol{\eta}}\rangle.$$

In the new coordinates, the symbol has the form:

$$\partial_\lambda\left(\frac{1}{|(\tau,\boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\right) = \frac{-2\,\langle\sigma'(\lambda)M^{-1}\boldsymbol{\eta},\tilde{\boldsymbol{\eta}}\rangle}{(\,|(\tau,M^{-1}\boldsymbol{\eta})| + |\tilde{\boldsymbol{\eta}}|^2\,)^2}.$$

The symbol of $\partial_j^{1/2}\circ\partial_\lambda\left(\frac{1}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\right)$ is:

$$\frac{-2(2\pi i\eta_j)^{1/2}\,\langle\sigma'(\lambda)M^{-1}\boldsymbol{\eta},\tilde{\boldsymbol{\eta}}\rangle}{(\,|(\tau,M^{-1}\boldsymbol{\eta})| + |\tilde{\boldsymbol{\eta}}|^2\,)^2},$$

but, for reasons of simplicity, we will show the result for the following symbol (which will yield the same result):

$$\Psi(\tau,\boldsymbol{\eta},\lambda) = \frac{-2(1+|\boldsymbol{\eta}|^2)^{1/4}\,\langle\sigma'(\lambda)M^{-1}\boldsymbol{\eta},\tilde{\boldsymbol{\eta}}\rangle}{(\,|(\tau,M^{-1}\boldsymbol{\eta})| + |\tilde{\boldsymbol{\eta}}|^2\,)^2}. \tag{4}$$

Fourier multipliers III

We will assume the following uniform bounds:

$$0 < c \le \|M^{-1}\|_2 \le \widehat{C} < \infty\,, \qquad \|M\|_2 \le \widetilde{C}\,, \qquad \|\sigma'\|_2 \le \bar{C}\,, \tag{5}$$

where $c, \widehat{C}, \widetilde{C}$ and $\bar{C}$ are positive numbers. We already have $\widetilde{C} = \max\{1,\|a\|_2\} + \|a\|_2$ and $c = 1/\widetilde{C}$. For $\widehat{C}$ we do not have a uniform bound, so this together with assumption on $\bar{C}$ are the only new assumptions here.

Lemma. *Under the conditions (5), Ψ given in (4) is an L^p multiplier.* ∎

Corollary.

- *Let $p\in\langle 1,\infty\rangle$. Then $\partial_\lambda\left(\mathcal{A}_{\frac{1}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}}\right)$ continuously maps $\mathrm{L}^p(\mathbf{R}\times\mathbf{R}^d)$ to $\mathrm{W}^{1/2,p}(\mathbf{R}\times\mathbf{R}^d)$.*
- *Let $r > 2(d+1)$. Then $\partial_\lambda\left(\mathcal{A}_{\frac{1}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}}\right)$ continuously maps $\mathrm{L}^r(\mathbf{R}\times\mathbf{R}^d)$ to $\mathrm{C}^0(\mathbf{R}\times\mathbf{R}^d)$.*

∎

H-measures, H-distributions, and velocity averaging

H-distributions

Theorem. *Let $(u_n(t,\mathbf{x},\lambda))$ be an uniformly compactly supported sequence weakly converging to zero in $\mathrm{L}^p(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})$, $p > 1$. Let $(v_n(t,\mathbf{x}))$ be an uniformly compactly supported sequence bounded in $\mathrm{L}^\infty(\mathbf{R}^+\times\mathbf{R}^d)$. Then for every $\varepsilon > 0$ there exists a subsequence (not relabelled) and a continuous bilinear functional B on $\mathrm{L}^{p'+\varepsilon}(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})\otimes\mathrm{C}^{d+1}(\Pi)$ such that for every $\varphi\in\mathrm{L}^{p'+\varepsilon}(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})$ and $\psi\in\mathrm{C}^{d+1}(\Pi)$ it holds*

$$B(\varphi,\psi) = \lim_{n\to\infty}\int_{\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}} \varphi(t,\mathbf{x},\lambda)u_n(t,\mathbf{x},\lambda)\overline{\mathcal{A}_{\psi(\pi_P(\tau,\boldsymbol{\xi},\lambda))}(v_n)(t,\mathbf{x})}\,dtd\mathbf{x}d\lambda.$$

Furthermore, the bound of functional B is $C_uC_{v,s}C_{d,s}\sqrt[s]{C_\lambda}$, where C_u is the L^p-bound of sequence (u_n); $C_{v,s}$ is the L^s-bound of the sequence (v_n), where $1/p+1/(p'+\varepsilon)+1/s = 1$; and $C_{d,s}$ is a constant from the Marcinkiewicz multiplier theorem. ∎

Theorem. *The bilinear functional B from the previous Theorem can be extended by continuity to a functional on $\mathrm{L}^{p'+\varepsilon}(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R};\mathrm{C}^{d+1}(\Pi))$. The bound of the extension is equal to $2C_uC_{v,s}C_{d,s}C_\lambda$.* ∎

H-measures

Corollary. *If the sequence $(u_n(t,\mathbf{x},\lambda))$ is bounded in $\mathrm{L}^p(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})$, $p>2$, then $\mu\in\mathrm{L}^{(p'+\varepsilon)'}_{w*}(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R};\mathcal{M}(\Pi))$, where $\mathcal{M}(\Pi)$ is the space of Radon measures.* ∎

Lemma. *Let $\mu\in\mathrm{L}^{(p'+\varepsilon)'}_{w*}(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R};\mathcal{M}(\Pi))$ be the functional defined in the previous Corollary. Let $K_\lambda\subset\mathbf{R}$ be a fixed arbitrary compact set.*
If the function $F\in\mathrm{C}_0(\mathbf{R}^+\times\mathbf{R}^{d+1}\times\Pi)$ is such that for some $\alpha>0$ exists $N>0$ such that

$$\operatorname{esssup}_{(t,\mathbf{x})\in\mathbf{R}^+\times\mathbf{R}^d}\sup_{|(\tau,\boldsymbol{\xi})|>N}\operatorname{meas}\{\lambda\in K_\lambda:\ |F(t,\mathbf{x},\lambda,\pi_P(\tau,\boldsymbol{\xi},\lambda))|\le\sigma\}\ \le\ \sigma^\alpha \tag{6}$$

and for a.e. $(t,\mathbf{x},\lambda)\in\mathbf{R}^+\times\mathbf{R}^{d+1}$ it holds (in the sense of the dual pairing between $\mathcal{M}(\mathbf{R}^{d+1})$ and $\mathrm{C}_0(\mathbf{R}^{d+1})$, where $\mu\circ\pi_P$ is push-forward of measure μ by projection π_P; for simplicity reasons we use notation $\mu\circ\pi_P$ instead of $(\pi_P)_*\mu$):

$$\langle\,(\mu\circ\pi_P(\cdot,\cdot,\lambda))\,(t,\mathbf{x},\lambda),\ F(t,\mathbf{x},\lambda,\pi_P(\cdot,\cdot,\lambda))\,\rangle\equiv 0\,, \tag{7}$$

then

$$\mu\equiv 0.$$

∎

Velocity averaging

$$\begin{aligned}&\partial_t u_n(t,\mathbf{x},\lambda) + \operatorname{div}(f(t,\mathbf{x},\lambda)u_n(t,\mathbf{x},\lambda)) \\ &= \operatorname{div}(\operatorname{div}(a(\lambda)u_n(t,\mathbf{x},\lambda))) + \partial_\lambda G_n(t,\mathbf{x},\lambda) + \operatorname{div}P_n(t,\mathbf{x},\lambda),\end{aligned} \tag{8}$$

where

a) (u_n) weakly converges to zero in $\mathrm{L}^q(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})$, $q\ge 2$;

b) $a\in\mathrm{C}^{0,1}(\mathbf{R};\mathbf{R}^{d\times d})$;

c) $f\in\mathrm{L}^p(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R};\mathbf{R}^d)$, $p>1$;

d) $G_n\to 0$ strongly in $\mathrm{L}^{r_0}(\mathbf{R};\mathrm{W}^{-1/2,r_0}(\mathbf{R}^+\times\mathbf{R}^d))$ for some $r_0\in\langle 1,\infty\rangle$;

e) $P_n\to 0$ strongly in $\mathrm{L}^{p_0}(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R};\mathbf{R}^d)$ for some $p_0\in\langle 1,\infty\rangle$.

From assumptions on a it follows that $\sigma\in\mathrm{C}^{0,1}(\mathbf{R};\mathbf{R}^{d\times d})$.

Theorem. *Assume that the function*

$$F(t,\mathbf{x},\lambda,\pi_P(\tau,\boldsymbol{\xi},\lambda)) = i\frac{\tau+\langle\boldsymbol{\xi},f(t,\mathbf{x},\lambda)\rangle}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle} + \frac{\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}$$

satisfies non-degeneracy condition (6). Then, for any $\rho\in\mathrm{C}^1_c(\mathbf{R})$, the sequence $(\int_{\mathbf{R}}\rho(\lambda)u_n(t,\mathbf{x},\lambda)d\lambda)$ is strongly precompact in $\mathrm{L}^1_{loc}(\mathbf{R}^+\times\mathbf{R}^d)$. ∎

Degenerate parabolic equation

Cauchy problem

$$\partial_t u + \operatorname{div}_{\mathbf{x}}\mathfrak{f}(t,\mathbf{x},u) = D^2\cdot A(u) \tag{9}$$

$$u|_{t=0} = u_0(\mathbf{x}) \in \mathrm{L}^1(\mathbf{R}^d)\cap\mathrm{L}^\infty(\mathbf{R}^d). \tag{10}$$

The equation describes a flow governed by

- the convection effects (bulk motion of particles) which are represented by the first order terms;
- diffusion effects which are represented by the second order term and the matrix $A(\lambda) = [A_{ij}(\lambda)]_{i,j=1,\dots,d}$ (more precisely its derivative with respect to λ) describes direction and intensity of the diffusion;

The equation is degenerate in the sense that the derivative of the diffusion matrix $A' = a$ can be equal to zero in some direction. Roughly speaking, if this is the case (i.e. if for some vector $\boldsymbol{\xi}\in\mathbf{R}^d$ we have $\langle A'(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle = 0$), then diffusion effects do not exist at the point $\mathbf{x}$ for the state λ in the direction $\boldsymbol{\xi}$.

Example. $A(u) = \begin{bmatrix} u & -\frac{u^2}{2} \\ -\frac{u^2}{2} & \frac{u^3}{3}\end{bmatrix}$, $\quad a(\lambda) = \begin{bmatrix} 1 & -\lambda \\ -\lambda & \lambda^2\end{bmatrix}$, $\quad M(\lambda) = \begin{bmatrix} 1 & -\lambda \\ 0 & 1\end{bmatrix}$.

Assumptions

- The initial data are bounded between a and b and the flux function annuls at $\lambda = \tilde{a}$ and $\lambda = \tilde{b}$:
$$\tilde{a}\le u_0(\mathbf{x})\le\tilde{b}\quad\text{and}\quad\mathfrak{f}(t,\mathbf{x},\tilde{a}) = \mathfrak{f}(t,\mathbf{x},\tilde{b}) = 0\ \text{ a.e. }(t,\mathbf{x})\in\mathbf{R}^+\times\mathbf{R}^d. \tag{11}$$
- The convective term $\mathfrak{f}(t,\mathbf{x},\lambda)$ is continuously differentiable with respect to $\lambda\in\mathbf{R}$, and it belongs to $\mathrm{L}^r(\mathbf{R}^+\times\mathbf{R}^d\times[\tilde{a},\tilde{b}])$, $r>1$
We also assume:
$$\operatorname{div}_{\mathbf{x}}\mathfrak{f}(t,\mathbf{x},\lambda)\in\mathcal{M}(\mathbf{R}^+\times\mathbf{R}^d\times[\tilde{a},\tilde{b}]). \tag{12}$$
- The matrix $A(\lambda) = (A_{ij}(\lambda))_{i,j=1,\dots,d}\in\mathrm{C}^{1,1}(\mathbf{R};\mathbf{R}^{d\times d})$, is non-decreasing with respect to $\lambda\in\mathbf{R}$, i.e. the (diffusion) matrix $a(\lambda) = A'(\lambda)$ satisfies
$$\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle\ge 0.$$

Quasi-solution and kinetic formulation

Definition. A measurable function u defined on $\mathbf{R}^+\times\mathbf{R}$ is called a quasi-solution to (9) if $\mathfrak{f}_k(t,\mathbf{x},u)$, $A_{kj}(u)\in\mathrm{L}^1_{loc}(\mathbf{R}^+\times\mathbf{R}^d)$, $k,j=1,\dots,d$, and for a.e. $\lambda\in\mathbf{R}$ the Kruzhkov type entropy equality holds

$$\begin{aligned}&\partial_t|u-\lambda| + \operatorname{div}[\operatorname{sgn}(u-\lambda)(\mathfrak{f}(t,\mathbf{x},u)-\mathfrak{f}(t,\mathbf{x},\lambda))] \\ &- D^2\cdot[\operatorname{sgn}(u-\lambda)(A(u)-A(\lambda))] = -\zeta(t,\mathbf{x},\lambda),\end{aligned} \tag{13}$$

where $\zeta\in\mathrm{C}(\mathbf{R}_\lambda; w\star-\mathcal{M}(\mathbf{R}^+\times\mathbf{R}^d))$ we call the quasi-entropy defect measure.

Remark that for a regular flux $\mathfrak{f}$, the measure $\zeta(t,\mathbf{x},\lambda)$ can be rewritten in the form $\zeta(t,\mathbf{x},\lambda) = \bar{\zeta}(t,\mathbf{x},\lambda) + \operatorname{sgn}(u-\lambda)\operatorname{div}_{\mathbf{x}}\mathfrak{f}(t,\mathbf{x},\lambda)$, for a measure $\bar{\zeta}$. If $\bar{\zeta}$ is non-negative, then the quasi-solution u is an entropy solution to (9). For the uniqueness of such entropy solution, we additionally need the chain rule.

Theorem. *If function u is a quasi-solution to (9), then the function*

$$h(t,\mathbf{x},\lambda) = \operatorname{sgn}(u(t,\mathbf{x})-\lambda) = -\partial_\lambda|u(t,\mathbf{x})-\lambda| \tag{14}$$

is a weak solution to the following linear equation:

$$\partial_t h + \operatorname{div}(\mathfrak{F}(t,\mathbf{x},\lambda)h) - D^2\cdot[a(\lambda)h] = \partial_\lambda\zeta(t,\mathbf{x},\lambda)\,, \tag{15}$$

where $\mathfrak{F} = \mathfrak{f}'_\lambda$ and $a = A'_\lambda$. ∎

Theorem. *Assume that $\mathfrak{F} = \mathfrak{f}'_\lambda$ and $a = A'_\lambda$ are such that the function*

$$F(t,\mathbf{x},\pi_P(\tau,\boldsymbol{\xi},\lambda)) = i\frac{\tau+\langle\boldsymbol{\xi},\mathfrak{F}(t,\mathbf{x},\lambda)\rangle}{|(\tau,\boldsymbol{\xi})|+\langle A(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle} + \frac{\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{|(\tau,\boldsymbol{\xi})|+\langle A(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}$$

satisfies (6).
Then, there exists a solution to (9) augmented with the initial conditions $u|_{t=0} = u_0(\mathbf{x})$, $\tilde{a}\le u_0\le\tilde{b}$. ∎

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