Applications of a version of the Schwartz kernel theorem for anisotropic distributions

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H-measures

Theorem. *If $(u_n)_{n\in\mathbf{N}}$ is a sequence in $\mathrm{L}^2_{loc}(\Omega;\mathbf{R}^r)$, $\Omega\subset\mathbf{R}^{d+1}$, such that $u_n \rightharpoonup 0$ in $\mathrm{L}^2_{loc}(\Omega)$, then there exists subsequence $(u_{n'})_{n'}\subset(u_n)_n$ and positive complex bounded measure $\mu=\{\mu^{jk}\}_{j,k=1,\dots,r}$ on $\mathbf{R}^{d+1}\times\mathrm{S}^d$ such that for all $\varphi_1,\varphi_2\in\mathrm{C}_0(\Omega)$ and $\psi\in\mathrm{C}(\mathrm{S}^d)$,*

$$\lim_{n'\to\infty}\int_\Omega(\varphi_1 u^j_{n'})(\boldsymbol{\xi})\overline{\mathcal{A}_{\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}(\varphi_2 u^k_{n'})(\boldsymbol{\xi})}d\mathbf{x}=\langle\mu^{jk},\varphi_1\bar\varphi_2\psi\rangle$$
$$=\int_{\mathbf{R}^{d+1}\times\mathrm{S}^d}\varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})}\psi(\boldsymbol{\xi})d\mu^{jk}(\mathbf{x},\boldsymbol{\xi})$$

where $\mathcal{A}_{\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}$ is the multiplier operator with the symbol $\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)$. ∎

H-distributions

Theorem. *If $u_n \longrightarrow 0$ in $\mathrm{L}^p_{\rm loc}(\mathbf{R}^d)$ and $v_n \overset{*}{\rightharpoonup} v$ in $\mathrm{L}^q_{\rm loc}(\mathbf{R}^d)$ for some $p\in\langle 1,\infty\rangle$ and $q\geq p'$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a complex valued distribution $\mu\in\mathcal{D}'(\mathbf{R}^d\times\mathrm{S}^{d-1})$, such that, for every $\varphi_1,\varphi_2\in\mathrm{C}^\infty_c(\mathbf{R}^d)$ and $\psi\in\mathrm{C}^\kappa(\mathrm{S}^{d-1})$, for $\kappa=[d/2]+1$, one has:*

$$\lim_{n'\to\infty}\int_{\mathbf{R}^d}\mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}d\mathbf{x}=\lim_{n'\to\infty}\int_{\mathbf{R}^d}(\varphi_1 u_{n'})(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})}d\mathbf{x}$$
$$=\langle\mu,\varphi_1\overline{\varphi}_2\boxtimes\psi\rangle,$$

where $\mathcal{A}_\psi:\mathrm{L}^p(\mathbf{R}^d)\longrightarrow\mathrm{L}^p(\mathbf{R}^d)$ is the Fourier multiplier operator with symbol $\psi\in\mathrm{C}^\kappa(\mathrm{S}^{d-1})$. ∎

Hörmander-Mihlin theorem

Theorem. *Let $\psi\in\mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa=[d/2]+1$. If for some $k>0$*

$$(\forall r>0)(\forall\boldsymbol{\alpha}\in\mathbf{N}^d_0)\quad|\boldsymbol{\alpha}|\leq\kappa\Longrightarrow\int_{r/2\leq|\boldsymbol{\xi}|\leq r}|\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})|^2d\boldsymbol{\xi}\leq k^2r^{d-2|\boldsymbol{\alpha}|},$$

then for any $p\in\langle 1,\infty\rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a constant C_d such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p\to\mathrm{L}^p}\leq C_d\max\{p,1/(p-1)\}(k+\|\psi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}).$$

∎

For $\psi\in\mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d\setminus\{0\}$, we can take $k=\|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}$.

Anisotropic distributions

Let X and Y be open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$ (or C^∞ manifolds of dimenions d and r) and $\Omega\subseteq X\times Y$ an open set. By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha}\in\mathbf{N}^d_0$ and $\boldsymbol{\beta}\in\mathbf{N}^r_0$, if $|\boldsymbol{\alpha}|\leq l$ and $|\boldsymbol{\beta}|\leq m$, $\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}f=\partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\partial^{\boldsymbol{\beta}}_{\mathbf{y}}f\in\mathrm{C}(\Omega)$.

$\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$p^{l,m}_{K_n}(f):=\max_{|\boldsymbol{\alpha}|\leq l,|\boldsymbol{\beta}|\leq m}\|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\|_{\mathrm{L}^\infty(K_n)}\,,$$

where $K_n\subseteq\Omega$ are compacts, such that $\Omega=\cup_{n\in\mathbf{N}}K_n$ and $K_n\subseteq\mathrm{Int}K_{n+1}$,
Consider the space

$$\mathrm{C}^{l,m}_c(\Omega):=\bigcup_{n\in\mathbf{N}}\mathrm{C}^{l,m}_{K_n}(\Omega)\,,$$

and equip it by the topology of *strict inductive limit*.

Anisotropic distributions II

Definition. *A distribution of order l in $\mathbf{x}$ and order m in $\mathbf{y}$ is any linear functional on* $\mathrm{C}^{l,m}_c(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

Conjecture. *Let X,Y be C^∞ manifolds and let u be a linear functional on $\mathrm{C}^{l,m}_c(X\times Y)$. If $u\in\mathcal{D}'(X\times Y)$ and satisfies*
$(\forall K\in\mathcal{K}(X))(\forall L\in\mathcal{K}(Y))(\exists C>0)(\forall\varphi\in\mathrm{C}^\infty_K(X))(\forall\psi\in\mathrm{C}^\infty_L(Y))$

$$|\langle u,\varphi\boxtimes\psi\rangle|\leq Cp^l_K(\varphi)p^m_L(\psi),$$

then u can be uniquely extended to a continuous functional on $\mathrm{C}^{l,m}_c(X\times Y)$ (i.e. it can be considered as an element of $\mathcal{D}'_{l,m}(X\times Y)$). ∎

Anisotropic distributions III

From the proof of the existence of H-distributions, we already have $\mu\in\mathcal{D}'(\mathbf{R}^d\times\mathrm{S}^{d-1})$ and the following bound with $\varphi:=\varphi_1\overline{\varphi_2}$:

$$|\langle\mu,\varphi\boxtimes\psi\rangle|\leq C\|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}\|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)}\,,$$

where C does not depend on φ and ψ.

If the conjecture were true, then the H-distribution μ belongs to the space $\mathcal{D}'_{0,\kappa}(\mathbf{R}^d\times\mathrm{S}^{d-1})$, i.e. it is a distribution of order 0 in $\mathbf{x}$ and of order not more than κ in $\boldsymbol{\xi}$.

The Schwartz kernel theorem

Let X and Y be two C^∞ manifolds. Then the following statements hold:

Theorem. a) *Let $K\in\mathcal{D}'(X\times Y)$. Then for every $\varphi\in\mathcal{D}(X)$, the linear form K_φ defined as $\psi\mapsto\langle K,\varphi\boxtimes\psi\rangle$ is a distribution on Y. Furthermore, the mapping $\varphi\mapsto K_\varphi$, taking $\mathcal{D}(X)$ to $\mathcal{D}'(Y)$ is linear and continuous.*

b) *Let $A:\mathcal{D}(X)\to\mathcal{D}'(Y)$ be a continous linear operator. Then there exists unique distribution $K\in\mathcal{D}'(X\times Y)$ such that for any $\varphi\in\mathcal{D}(X)$ and $\psi\in\mathcal{D}(Y)$*

$$\langle K,\varphi\boxtimes\psi\rangle=\langle K_\varphi,\psi\rangle=\langle A\varphi,\psi\rangle.$$

∎

The Schwartz kernel theorem for anisotropic distributions

Let X and Y be two C^∞ manifolds of dimensions d and r, respectively. Then the following statements hold:

Theorem. a) *Let $K\in\mathcal{D}'_{l,m}(X\times Y)$. Then for every $\varphi\in\mathrm{C}^l_c(X)$, the linear form K_φ defined as $\psi\mapsto\langle K,\varphi\boxtimes\psi\rangle$ is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi\mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.*

b) *Let $A:\mathrm{C}^l_c(X)\to\mathcal{D}'_m(Y)$ be a continous linear operator. Then there exists unique distribution $K\in\mathcal{D}'(X\times Y)$ such that for any $\varphi\in\mathcal{D}(X)$ and $\psi\in\mathcal{D}(Y)$*

$$\langle K,\varphi\boxtimes\psi\rangle=\langle K_\varphi,\psi\rangle=\langle A\varphi,\psi\rangle.$$

Furthermore, $K\in\mathcal{D}'_{l,r(m+2)}(X\times Y)$. ∎

How to prove it?

Standard attempts:
- regularisation? (Schwartz)
- constructive proof? (Simanca, Gask, Ehrenpreis)
- nuclear spaces? (Treves)

Use the structure theorem of distributions (Dieudonne). There are two steps:
Step I: assume the range of A is $\mathrm{C}(Y)$
Step II: use structure theorem and go back to Step I

Consequence: H-distributions are of order 0 in $\mathbf{x}$ and of finite order not greater than $d(\kappa+2)$ with respect to $\boldsymbol{\xi}$.

Applications: localisation principle for H-distributions and Peetre's theorem

Localisation principle

Theorem. *Assume that $u_n\longrightarrow 0$ in $\mathrm{L}^p_{\rm loc}(\mathbf{R}^d)$ and $f_n\longrightarrow 0$ in $\mathrm{W}^{-1,q}_{loc}(\mathbf{R}^d)$ for some $p\in\langle 1,\infty\rangle$ and $q\in\langle 1,d\rangle$, such that they satisfy*

$$\sum_{i=1}^d\partial_i(a_i(\mathbf{x})u_n(\mathbf{x}))=f_n(\mathbf{x})\,,$$

where $a_i\in\mathrm{C}_c(\mathbf{R}^d)$. Take an arbitrary sequence (v_n) bounded in $\mathrm{L}^\infty_{loc}(\mathbf{R}^d)$, and by μ denote the H-distribution corresponding to some subsequences of sequences (u_n) and (v_n). Then,

$$\sum_{i=1}^d a_i(\mathbf{x})\xi_i\mu(\mathbf{x},\boldsymbol{\xi})=0$$

in the sense of distributions on $\mathbf{R}^d\times\mathrm{S}^{d-1}$. ∎

We can also obtain a corresponding variant of compactness by compensation theory.

Classical Peetre's result and notation

Theorem. *Let $A:\mathrm{C}^\infty_c(\Omega)\to\mathrm{C}^\infty_c(\Omega)$ be a linear mapping such that the following holds:*

$$\mathrm{supp}(Au)\subset\mathrm{supp}(u)\,,\qquad u\in\mathrm{C}^\infty_c(\Omega).$$

Then A is a differential operator on Ω with C^∞ coefficients. ∎

Let $\Omega\subset\mathbf{R}^d$ be an open set and by $U\subset\Omega$ let us denote its arbitrary open and relatively compact subset. For $k\in\mathbf{N}$, $f\in\mathrm{C}^\infty_c(\Omega)$ and $g\in\mathcal{D}'(\Omega)$, let us introduce the following seminorms and operator norms:

$$\|f\|_k:=\sup_{\mathbf{x}\in\Omega,\boldsymbol{\alpha}\in\mathbf{N}^d_0,|\boldsymbol{\alpha}|\leq k}|D^{\boldsymbol{\alpha}}f(\mathbf{x})|\,,$$

$$\|g\|_{-k}=\|g,U\|_{-k}:=\sup_{h\in\mathrm{C}^\infty_c(U)}\frac{|\langle g,h\rangle|}{\|h\|_k}\,.$$

Let us remark that $\|g,U\|_{-k}<\infty$ for k large enough (this follows from the properties of distributions with compact support).
Assume we are given $A:\mathrm{C}^\infty_c(\Omega)\to\mathcal{D}'(\Omega)$, a linear (not neccessary continuous) operator such that:

$$\mathrm{supp}\,(Af)\subset\mathrm{supp}\,f\,,\qquad f\in\mathrm{C}^\infty_c(\Omega)\,.$$

For given $k\in\mathbf{N}$ and $\mathbf{x}\in\Omega$, let us define $j=j(k,\mathbf{x})\in\mathbf{N}$ in the following way:

$$j(k,\mathbf{x}):=\inf\left\{j\in\mathbf{N}:\ \exists U\ni\mathbf{x}\,\text{neighbourhood},\ \sup_{h\in\mathrm{C}^\infty_c(U)}\frac{\|Ah,U\|_{-k}}{\|h\|_j}<\infty\right\}\,,$$

if it exists, otherwise we set $j(k,\mathbf{x}):=\infty$. Neighbourhoods U of $\mathbf{x}$ in the definition of $j(k,\mathbf{x})$ are assumed to be open and relatively compact.

A variant by application of the Bogdanowicz result

We can reformulate the main result of Bogdanowicz's article to our setting:

Theorem. *For every bilinear functional B on the space $\mathrm{C}^\infty_c(X_1)\times\mathrm{C}^l_c(X_2)$ which is continuous with respect to each variable separately, there exists a unique anisotropic distribution $T\in\mathcal{D}'_{\infty,l}(X_1\times X_2)$ such that*

$$B(\varphi,\phi)=\langle T,\varphi\otimes\phi\rangle\,,\qquad\varphi\in\mathrm{C}^\infty_c(X_1),\phi\in\mathrm{C}^l_c(X_2)\,.$$

∎

It is worth noting that Bogdanowicz's result also holds when X_2 is a smooth manifold and that only elementary properties of Frechet and (LF)-spaces were used to prove it.

The same result can be obtained using the adjoint of operator A.

Peetre's result with distributions

Definition. We say that $\mathbf{x}\in\Omega$ is a point of continuity of A if there exists an open and relatively compact neighbourhood U of $\mathbf{x}$ such that the restriction $A|_{\mathrm{C}^\infty_c(U)}:\mathrm{C}^\infty_c(U)\to\mathcal{D}'(U)$ is continuous.
Otherwise, we say that it is a point of discontinuity and the set of all points of discontinuity we denote by Λ.

From the definitions of Λ and $j(k,\mathbf{x})$, we conclude:
- if $\mathbf{x}\in\Lambda$, then $j(k,\mathbf{x})=\infty$ for every k;
- if $\mathbf{x}\notin\Lambda$, then there exists $k\in\mathbf{N}$ such that $j(k,\mathbf{x})<\infty$.

Lemma. *The set Λ is locally finite (i.e. discrete). For every $U\subset\Omega$ open and relatively compact set, the function $j(k,\cdot)$ is bounded on $U\setminus\Lambda$ for k large enough.* ∎

Theorem. *Let $A:\mathrm{C}^\infty_c(\Omega)\to\mathcal{D}'(\Omega)$ be linear operator such that* $\mathrm{supp}\,(Af)\subset\mathrm{supp}\,f$ *for $f\in\mathrm{C}^\infty_c(\Omega)$. Then there exists locally finite family of distributions $(a_{\boldsymbol{\alpha}})\in\mathcal{D}'(\Omega)$, unique on $\Omega\setminus\Lambda$, such that it holds:*

$$\mathrm{supp}\left(Af-\sum_{\boldsymbol{\alpha}}a_{\boldsymbol{\alpha}}D^{\boldsymbol{\alpha}}f\right)\subset\Lambda\,,\qquad f\in\mathrm{C}^\infty_c(\Omega)\,.$$

∎

If the image of the operator A is contained in some $\mathcal{D}'_m(\Omega)$, then $\|Af,U\|_{-m}<\infty$, and we write $j(\mathbf{x})$ for $j(m,\mathbf{x})$. The definition of point of continuity remains unchanged and we have that $j(\cdot)$ is locally bounded on $\Omega\setminus\Lambda$.

Theorem. *Let $A:\mathrm{C}^\infty_c(\Omega)\to\mathcal{D}'_m(\Omega)$ be linear operator such that*

$$\mathrm{supp}\,(Af)\subset\mathrm{supp}\,f\,,\qquad f\in\mathrm{C}^\infty_c(\Omega)\,.\qquad(1)$$

Then there exists locally finite family of distributions $(a_{\boldsymbol{\alpha}})\in\mathcal{D}'_m(\Omega)$, unique on $\Omega\setminus\Lambda$, such that it holds:

$$\mathrm{supp}\left(Af-\sum_{\boldsymbol{\alpha}}a_{\boldsymbol{\alpha}}D^{\boldsymbol{\alpha}}f\right)\subset\Lambda\,,\qquad f\in\mathrm{C}^\infty_c(\Omega)\,.$$

∎

Additional results

Lemma. *If $u\in\mathcal{D}'_{l,m}(X_1\times X_2)$ is of compact support such that $\mathrm{supp}\,u\subset\{0\}\times X_2$, then for any $\Phi\in\mathrm{C}^\infty_c(X_1\times X_2)$ it holds:*

$$u=\sum_{\boldsymbol{\alpha}\in\mathbf{N}^d_0,|\boldsymbol{\alpha}|\leq l}\langle u_{\boldsymbol{\alpha}},\Phi_{\boldsymbol{\alpha}}\rangle\,,$$

where $u_{\boldsymbol{\alpha}}\in\mathcal{D}'_m(X_2)$ and $\Phi_{\boldsymbol{\alpha}}(\mathbf{y})=D^{\boldsymbol{\alpha}}_{\mathbf{x}}\Phi(0,\mathbf{y})$. ∎

Corollary. *If $u\in\mathcal{D}'_{l,m}(X_1\times X_2)$ has compact support such that $\mathrm{supp}\,u\subset\{\mathbf{x}_0\}\times X_2$, for some $\mathbf{x}_0\in X_1$, then*

$$u=\sum_{\boldsymbol{\alpha}\in\mathbf{N}^d_0,|\boldsymbol{\alpha}|\leq l}D^{\boldsymbol{\alpha}}_{\mathbf{x}}\delta_{\mathbf{x}_0}\otimes u_{\boldsymbol{\alpha}}\,,$$

where $u_{\boldsymbol{\alpha}}\in\mathcal{D}'_m(X_2)$. ∎

Theorem. *Let $A:\mathrm{C}^\infty_c(X)\to\mathcal{D}'_m(X)$ be a continuous map. Its kernel is supported by the diagonal $\{(\mathbf{x},\mathbf{x}):\mathbf{x}\in X\}$ if and only if for any $\varphi\in\mathrm{C}^\infty_c(X)$:*

$$A\varphi=\sum_{\boldsymbol{\alpha}\in\mathbf{N}^d_0}a_{\boldsymbol{\alpha}}\otimes D^{\boldsymbol{\alpha}}\varphi,$$

where $a_{\boldsymbol{\alpha}}\in\mathcal{D}'_m(X)$ and the above sum is locally finite. Moreover, this representation is unique. ∎

Proof of the Peetre's theorem

Proof. Let $U\subset\Omega$ be an open and relatively compact set. Then there exists $j=j(U)\in\mathbf{N}$ such that for any $\mathbf{x}_0\in U\setminus\Lambda$, there is a neighbourhood V of $\mathbf{x}_0$ such that

$$|\langle Af,g\rangle|\leq C\|f\|_j\|g\|_m\,,\qquad f\in\mathrm{C}^\infty_c(V),g\in\mathrm{C}^m_c(V)\,.$$

Schwartz kernel theorem gives existence of $K\in\mathcal{D}'_{\infty,m}(V\times V)$ such that $\langle Af,g\rangle=\langle K,f\otimes g\rangle$. The locality assumption (1) implies that the distribution K is supported on a diagonal of a set V:

$$\mathrm{supp}\,K\subset\{(\mathbf{x},\mathbf{x}):\ \mathbf{x}\in V\}\,,$$

which puts us within the conditions of the theorem on diagonal support and we can write:

$$A\varphi=\sum_{\boldsymbol{\alpha}\in\mathbf{N}^d_0}a_{\boldsymbol{\alpha}}\otimes D^{\boldsymbol{\alpha}}\varphi\,,\qquad\varphi\in\mathrm{C}^\infty_c(V)\,,$$

where family $(a_{\boldsymbol{\alpha}})\subset\mathcal{D}'_m(V)$ is locally finite (remark that in Peetre's article, who used results of Ehrenpreis' on Schwartz kernel theorem, it is obtained that $\boldsymbol{\alpha}$ in the above representation formula are at most of order $j+m+2d+1$, with j dependent on V, that is U).
Taking $\varphi\in\mathrm{C}^\infty_c(\Omega)$ equal to one on V, we obtain $A\varphi=a_0$ in V. Therefore, a_0 in uniquely determined by the operator A, and we can extend it to a distribution on the whole U. By using monomials $\mathbf{x}^{\boldsymbol{\alpha}}$, we can obtain the same conclusion for other $a_{\boldsymbol{\alpha}}$.
Now, we conclude:

$$\mathrm{supp}\left(Af-\sum_{\boldsymbol{\alpha}}a_{\boldsymbol{\alpha}}D^{\boldsymbol{\alpha}}f\right)\subset\Lambda\,,\qquad f\in\mathrm{C}^\infty_c(U)\,.$$

Since $U\subset\Omega$ was arbitrary, the claim of the theorem follows.

Counterexample

As already noticed by Peetre in the standard case, the result in the statement of the preceeding theorem is the best possible. Namely, it can happen $A-\sum_{\boldsymbol{\alpha}}a_{\boldsymbol{\alpha}}D^{\boldsymbol{\alpha}}\neq 0$, as we can easily see from the following example:

for $\mathbf{x}_0\in\Omega$, take a linear form F defined for sequence $(c_{\boldsymbol{\alpha}})$ such that it can not be written in the form $F=\sum_{\boldsymbol{\alpha}}b^{\boldsymbol{\alpha}}c_{\boldsymbol{\alpha}}$, for any finite collection of $b^{\boldsymbol{\alpha}}$. Then

$$(Af)(\mathbf{x})=F(D^{\boldsymbol{\alpha}}f(\mathbf{x}_0))\delta_0(\mathbf{x}-\mathbf{x}_0)$$

has desired properties without being continuous: we have $\mathrm{supp}\,(Af)\subset\{\mathbf{x}_0\}$ and A is continuous everywhere except at the point $\mathbf{x}_0$.

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