Generalisation of compactness by compensation

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WeConMApp

Panov's result

The most general version of the classical L^2 results has recently been proved by E. Yu. Panov (2011):
Assume that the sequence $(\mathbf{u}_n)$ is bounded in $L^p(\mathbf{R}^d;\mathbf{R}^r)$, $2 \le p < \infty$, and converges weakly in $\mathcal{D}'(\mathbf{R}^d)$ to a vector function $\mathbf{u}$.
Let $q = p'$ if $p < \infty$, and $q > 1$ if $p = \infty$. Assume that the sequence

$$\sum_{k=1}^{\nu} \partial_k(\mathbf{A}^k \mathbf{u}_n) + \sum_{k,l=\nu+1}^{d} \partial_{kl}(\mathbf{B}^{kl}\mathbf{u}_n)$$

is precompact in the anisotropic Sobolev space $W^{-1,-2;q}_{loc}(\mathbf{R}^d;\mathbf{R}^m)$, where $m \times r$ matrices $\mathbf{A}^k$ and $\mathbf{B}^{kl}$ have variable coefficients belonging to $L^{2\bar q}(\mathbf{R}^d)$, $\bar q = \frac{p}{p-2}$ if $p > 2$, and to the space $C(\mathbf{R}^d)$ if $p = 2$.

We introduce the set $\Lambda(\mathbf{x})$

$$\Lambda(\mathbf{x}) = \left\{ \boldsymbol{\lambda} \in \mathbf{C}^r \,\middle|\, (\exists \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}) : \left(i \sum_{k=1}^{\nu} \xi_k \mathbf{A}^k(\mathbf{x}) - 2\pi \sum_{k,l=\nu+1}^{d} \xi_k \xi_l \mathbf{B}^{kl}(\mathbf{x}) \right) \boldsymbol{\lambda} = \mathbf{0}_m \right\},$$

and consider the bilinear form on $\mathbf{C}^r$

$$q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\eta}) = \mathbf{Q}(\mathbf{x})\boldsymbol{\lambda} \cdot \boldsymbol{\eta}, \tag{1}$$

where $\mathbf{Q} \in L^{\bar q}_{loc}(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p > 2$ and $\mathbf{Q} \in C(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p = 2$.
Finally, let $q(\mathbf{x}, \mathbf{u}_n, \mathbf{u}_n) \rightharpoonup \omega$ weakly in the space of distributions.

The following theorem holds

Theorem. *Assume that* $(\forall \boldsymbol{\lambda} \in \Lambda(\mathbf{x}))\ q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\lambda}) \ge 0$ *(a.e.* $\mathbf{x} \in \mathbf{R}^d$*) and* $\mathbf{u}_n \rightharpoonup \mathbf{u}$*, then* $q(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})) \le \omega$. ∎

The connection between q and Λ given in the previous theorem, we shall call *the consistency condition*.

Goal: to formulate and extend the results from the preceeding theorem to the $L^p - L^q$ framework for appropriate (greater than one) indices p and q where $p < 2$.

H-distributions

H-distributions were introduced by N. Antonić and D. Mitrović (2011) as an extension of H-measures to the $L^p - L^q$ context.
M. Lazar and D. Mitrović (2012) extended and applied them on a velocity averaging problem.

We need multiplier operators with symbols defined on a manifold P determined by an d-tuple $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbf{R}^d_+$ where $\alpha_k \in \mathbf{N}$ or $\alpha_k \ge d$:

$$\mathrm{P} = \left\{ \boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{2\alpha_k} = 1 \right\},$$

We shall use the following variant of H-distributions.

Theorem. *Let* (u_n) *be a bounded sequence in* $L^p(\mathbf{R}^d)$*,* $p > 1$*, and let* (v_n) *be a bounded sequence of uniformly compactly supported functions in* $L^q(\mathbf{R}^d)$*,* $1/q + 1/p < 1$*, weakly converging to 0 in the sense of distributions. Then, after passing to a subsequence (not relabelled), for any* $\bar s \in (1, \frac{pq}{p+q})$ *there exists a continuous bilinear functional* B *on* $L^{\bar s'}(\mathbf{R}^d) \otimes C^d(\mathrm{P})$ *such that for every* $\varphi \in L^{\bar s'}(\mathbf{R}^d)$ *and* $\psi \in C^d(\mathrm{P})$*, it holds*

$$B(\varphi, \psi) = \lim_{n \to \infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_n(\mathbf{x}) \big(\mathcal{A}_{\psi_{\mathrm{P}}} v_n\big)(\mathbf{x}) d\mathbf{x},$$

where $\mathcal{A}_{\psi_{\mathrm{P}}}$ *is the Fourier multiplier operator on* $\mathbf{R}^d$ *associated to* $\psi \circ \pi_{\mathrm{P}}$*.*
The bilinear functional B *can be continuously extended as a linear functional on* $L^{\bar s'}(\mathbf{R}^d; C^d(\mathrm{P}))$*.* ∎

Localisation principle

For $\alpha \in \mathbf{R}^+$, we define $\partial^{\alpha}_{x_k}$ to be a pseudodifferential operator with a polyhomogeneous symbol $(2\pi i \xi_k)^{\alpha}$, i.e.

$$\partial^{\alpha}_{x_k} u = \big((2\pi i \xi_k)^{\alpha} \hat u(\boldsymbol{\xi})\big)^{\check{}}.$$

In the sequel, we shall assume that sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ are uniformly compactly supported. This assumption can be removed if the orders of derivatives $(\alpha_1, \dots, \alpha_d)$ are natural numbers.

Lemma. *Assume that sequences* $(\mathbf{u}_n)$ *and* $(\mathbf{v}_n)$ *are bounded in* $L^p(\mathbf{R}^d;\mathbf{R}^r)$ *and* $L^q(\mathbf{R}^d;\mathbf{R}^r)$*, respectively, and converge toward* $\mathbf{0}$ *and* $\mathbf{v}$ *in the sense of distributions.*
Furthermore, assume that sequence $(\mathbf{u}_n)$ *satisfies:*

$$\mathbf{G}_n := \sum_{k=1}^{d} \partial^{\alpha_k}_k (\mathbf{A}^k \mathbf{u}_n) \ \to \mathbf{0} \text{ in } W^{-\alpha_1, \dots, -\alpha_d; p}(\Omega; \mathbf{R}^m), \tag{2}$$

where either $\alpha_k \in \mathbf{N}$*,* $k = 1, \dots, d$ *or* $\alpha_k > d$*,* $k = 1, \dots, d$*, and elements of matrices* $\mathbf{A}^k$ *belong to* $L^{\bar s'}(\mathbf{R}^d)$*,* $\bar s \in (1, \frac{pq}{p+q})$*.*
Finally, by $\boldsymbol{\mu}$ *denote a matrix* H*-distribution corresponding to subsequences of* $(\mathbf{u}_n)$ *and* $(\mathbf{v}_n - \mathbf{v})$*. Then the following relation holds*

$$\Big(\sum_{k=1}^{d} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k \Big) \boldsymbol{\mu} = \mathbf{0}.$$

∎

Compactness by compensation result

Introduce the set

$$\Lambda_{\mathcal{D}} = \Big\{ \boldsymbol{\mu} \in L^{\bar s}(\mathbf{R}^d; (C^d(\mathrm{P}))')^r : \Big(\sum_{k=1}^{n} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k \Big) \boldsymbol{\mu} = \mathbf{0}_m \Big\},$$

where the given equality is understood in the sense of $L^{\bar s}(\mathbf{R}^d; (C^d(\mathrm{P}))')^m$.
Let us assume that coefficients of the bilinear form q from (1) belong to space $L^t_{loc}(\mathbf{R}^d)$, where $1/t + 1/p + 1/q < 1$.

Definition. We say that set $\Lambda_{\mathcal{D}}$, bilinear form q from (1) and matrix $\boldsymbol{\mu} = [\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r]$, $\boldsymbol{\mu}_j \in L^{\bar s}(\mathbf{R}^d; (C^d(\mathrm{P}))')^r$ satisfy the strong consistency condition if $(\forall j \in \{1, \dots, r\})\ \boldsymbol{\mu}_j \in \Lambda_{\mathcal{D}}$, and it holds

$$\langle \phi \mathbf{Q} \otimes 1, \boldsymbol{\mu} \rangle \ge \mathbf{0}, \quad \phi \in L^{\bar s}(\mathbf{R}^d; \mathbf{R}^+_0).$$

Theorem. *Assume that sequences* $(\mathbf{u}_n)$ *and* $(\mathbf{v}_n)$ *are bounded in* $L^p(\mathbf{R}^d;\mathbf{R}^r)$ *and* $L^q(\mathbf{R}^d;\mathbf{R}^r)$*, respectively, and converge toward* $\mathbf{u}$ *and* $\mathbf{v}$ *in the sense of distributions.*
Assume that (2) holds and that

$$q(\mathbf{x}; \mathbf{u}_n, \mathbf{v}_n) \rightharpoonup \omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d).$$

If the set $\Lambda_{\mathcal{D}}$*, the bilinear form (1), and matrix* H*-distribution* $\boldsymbol{\mu}$*, corresponding to subsequences of* $(\mathbf{u}_n - \mathbf{u})$ *and* $(\mathbf{v}_n - \mathbf{v})$*, satisfy the strong consistency condition, then*

$$q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \le \omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d).$$

∎

Case $L^p - L^{p'}$, $p > 1$

In the case $1/p + 1/q = 1$, applying the same proof gives us continuous bilinear functional on $C(\mathbf{R}^d) \otimes C^d(\mathrm{P})$. Using Schwartz's kernel theorem, we can only extend it to a distribution from $\mathcal{D}'(\mathbf{R}^d \times \mathrm{P})$.
Introduce the truncation operator $T_l(v) = v$ if $|v| \le l$ and $T_l(v) = 0$ if $|v| \ge l$, for $l \in \mathbf{N}$.

Theorem. *Assume that*

- *sequences* $(\mathbf{u}_r)$ *and* $(\mathbf{v}_r)$ *are bounded in* $L^p(\mathbf{R}^d;\mathbf{R}^N)$ *and* $L^{p'}(\mathbf{R}^d;\mathbf{R}^N)$*, where* $1/p + 1/p' = 1$*, and converge toward* $\mathbf{u}$ *and* $\mathbf{v}$ *in the sense of distributions;*
- *for every* $l \in \mathbf{N}$*, the sequences* $(T_l(\mathbf{v}_r))$ *converge weakly in* $L^{p'}(\mathbf{R}^d;\mathbf{R}^N)$ *toward* $\mathbf{h}^l$*, where the truncation operator* T_l *is understood coordinatewise;*
- *there exists a vector valued function* $\mathbf{V} \in L^{p'}(\mathbf{R}^d;\mathbf{R}^N)$ *such that* $\mathbf{v}_r \le \mathbf{V}$ *holds coordinatewise for every* $r \in \mathbf{N}$*;*
- *(2) holds with* $a_{skl} \in C_0(\mathbf{R}^d)$ *and that* $q_{jm} \in C(\mathbf{R}^d)$*.*

Assume that

$$q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) \rightharpoonup \omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d).$$

If for every $l \in \mathbf{N}$*, the set* $\Lambda_{\mathcal{D}}$*, the bilinear form (1), and the (matrix of)* H*-distributions* $\boldsymbol{\mu}_l$ *corresponding to the sequences* $(\mathbf{u}_r - \mathbf{u})$ *and* $(T_l(\mathbf{v}_r) - \mathbf{h}^l)_r$ *satisfy the strong consistency condition, then it holds*

$$q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \le \omega \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d).$$

∎

Application

Let us consider the non-linear parabolic type equation

$$L(u) = \partial_t u - \operatorname{div}\operatorname{div}\big(g(t, \mathbf{x}, u)\mathbf{A}(t, \mathbf{x})\big),$$

on $(0, \infty) \times \Omega$, where Ω is an open subset of $\mathbf{R}^d$. For p, q and s such that $1/p + 1/q + 1/s < 1$, assume

$$u \in L^p((0,\infty)\times\Omega), \ \ g(t, \mathbf{x}, u(t, \mathbf{x})) \in L^q((0,\infty)\times\Omega),$$

$$\mathbf{A} \in L^s_{loc}((0,\infty) \times \Omega)^{d \times d},$$

and that the matrix $\mathbf{A}$ is strictly positive definite, i.e.

$$\mathbf{A}\boldsymbol{\xi} \cdot \boldsymbol{\xi} > 0, \ \ \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{\mathbf{0}\}, \ \ (a.e. (t, \mathbf{x}) \in (0,\infty) \times \Omega).$$

Furthermore, assume that g is a Caratèodory function and non-decreasing with respect to the third variable.

Theorem. *Assume that sequences*

- (u_r) *and* $g(\cdot, u_r)$ *are such that* $u_r, g(u_r) \in L^2(\mathbf{R}^+ \times \mathbf{R}^d)$ *for every* $r \in \mathbf{N}$*;*
- *that they are bounded in* $L^p(\mathbf{R}^+ \times \mathbf{R}^d)$*,* $p \in (1, 2]$*, and* $L^q(\mathbf{R}^+ \times \mathbf{R}^d)$*,* $q > 2$*, respectively, where* $1/p + 1/q < 1$*;*
- $u_r \rightharpoonup u$ *and, for some,* $f \in W^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d)$*, the sequence*

$$L(u_r) = f_r \to f \quad \text{strongly in } W^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d).$$

Under the assumptions given above, it holds

$$L(u) = f \ \text{ in } \ \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d).$$

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References

[1] N. Antonić, D. Mitrović: *H-distributions: An Extension of H-Measures to an $L^p - L^q$ Setting*, Abstr. Appl. Anal. (2011), Article ID 901084, 12 pages.

[2] M. Lazar, D. Mitrović: *On an extension of a bilinear functional on $L^p(\mathbf{R}^n) \times E$ to Bôchner spaces with an application to velocity averaging*, C. R. Math. Acad. Sci. Paris 351 (2013) 261–264.

[3] E. Yu. Panov: *Ultraparabolic H-measures and compensated compactness*, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011) 47–62.

[4] L. Tartar: *H-measures, a new approach for studying homogenisation, oscillation and concentration effects in PDEs*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990) 193–230.