Classical optimal designs on annuli

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Introduction

Let $\Omega \subset \mathbb{R}^d$ be open and bounded set.
Two phases each with different isotropic conductivity: α, β $(0 < \alpha < \beta)$.
q_α is the prescribed volume of the first phase α $(0 < q_\alpha < |\Omega|)$.
$\chi \in L^\infty(\Omega, \{0,1\})$ a measurable characteristic function.

Conductivity can be expressed as

$$\mathbf{A}(\chi) := \chi\alpha\,\mathbf{I} + (1-\chi)\beta\,\mathbf{I},$$

where

$$\int_\Omega \chi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = q_\alpha.$$

Introduction

State functions $u_i \in \mathrm{H}^1_0(\Omega),\ i = 1, 2, ..., m$ are solutions of the following boundary value problems:

$$(1) \qquad \left\{ \begin{array}{cc} -\mathrm{div}(\mathbf{A}\,\nabla u_i) = f_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega, \end{array} \right. \qquad i = 1, 2, ..., m.$$

Notation: $\quad \boldsymbol{u} = (u_1, u_2, ..., u_m).$

Energy functional:

$$I(\chi) := \sum_{i=1}^{m} \mu_i \int\limits_{\Omega} f_i(\boldsymbol{x}) u_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

where $\mu_i > 0,\ i = 1, 2, ..., m.$

Statement of the problem

Optimal design problem:

$$(2) \qquad \begin{cases} I(\chi) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \,\mathrm{d}\boldsymbol{x} \to \max \\ \text{s.t.} \quad \chi \in L^\infty(\Omega, \{0,1\}), \quad \int_\Omega \chi \,\mathrm{d}\boldsymbol{x} = q_\alpha, \\ \boldsymbol{u} \text{ solves (1) with } \mathbf{A} = \chi\alpha\,\mathbf{I} + (1-\chi)\beta\,\mathbf{I}. \end{cases}$$

If solution χ exists for (2) we call it *classical solution*.

Important: For general optimal design problems the classical solutions usually does not exists.

Aim of this talk:

- to express design problem as max-min optimization.
- to present examples of classical solutions on annuli.

Relaxed design

For characteristic functions relaxation consists of:

$$\chi \in L^\infty(\Omega, \{0,1\}) \quad \rightsquigarrow \quad \theta \in L^\infty(\Omega, [0,1]), \tag{3}$$

with

$$\int_\Omega \theta \, \mathrm{d}\boldsymbol{x} := q_\alpha.$$

Notion of H-convergence is introduced for conductivity $\mathbf{A}$.

Effective conductivities:

$$\mathcal{K}(\theta) \subset M_d(\mathbb{R}) \text{ with local fraction } \theta \in [0,1].$$

Precisely, $A \in \mathcal{K}(\theta)$ iff there exists sequence of characteristic functions

$$\begin{cases} \chi_n \xrightharpoonup{L^\infty\star} \theta \\ \mathbf{A}^n = \chi_n \alpha \mathbf{I} + (1-\chi_n)\beta \mathbf{I} \xrightharpoonup{H} A. \end{cases}$$

Effective conductivities - set $\mathcal{K}(\theta)$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues

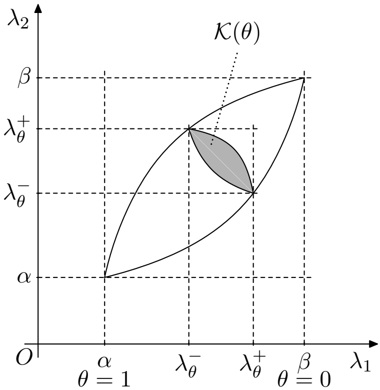
$$
\begin{aligned}
\lambda_\theta^- \leq & \quad \lambda_j \quad \leq \lambda_\theta^+ \quad j = 1, \ldots, d \\
\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} & \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha} \\
\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} & \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},
\end{aligned}
$$

where

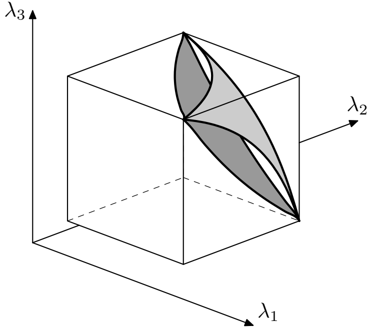
$$
\begin{aligned}
\lambda_\theta^+ &= \theta\alpha + (1-\theta)\beta \\
\frac{1}{\lambda_\theta^-} &= \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}.
\end{aligned}
$$

Visual representation of a set $\mathcal{K}(\theta)$

For dimension $d = 2$:



For dimension $d = 3$:



Relaxed problem A

Relaxed design:

$$\mathcal{A} = \left\{ (\theta, \mathbf{A}) \in L^\infty(\Omega, [0,1] \times \mathrm{Sym}_d) \;\middle|\; \begin{array}{c} \int_\Omega \theta \,\mathrm{d}\boldsymbol{x} = q_\alpha, \\ \mathbf{A}(\boldsymbol{x}) \in \mathcal{K}(\theta(\boldsymbol{x})), \;\; \text{a.e. } \boldsymbol{x} \end{array} \right\}$$

Relaxed problem can be written as:

(A)
$$\max_{(\theta,\mathbf{A})\in\mathcal{A}} J(\theta, \mathbf{A}) = \max_{(\theta,\mathbf{A})\in\mathcal{A}} \sum_{i=1}^{m} \mu_i \int_\Omega f_i u_i \,\mathrm{d}\boldsymbol{x}$$

Unfortunately, $\mathcal{A}$ is not a convex set.

Generalized (convex) problem B

To achieve convexity, an enlarged set is introduced:

$$\mathcal{B} = \left\{ (\theta, \mathbf{A}) \in L^\infty(\Omega, [0,1] \times \mathrm{Sym}_d) \;\middle|\; \begin{array}{c} \int_\Omega \theta \, \mathrm{d}\boldsymbol{x} = q_\alpha, \\ \lambda^-_{\theta(\boldsymbol{x})} \mathbf{I} \le \mathbf{A}(\boldsymbol{x}) \le \lambda^+_{\theta(\boldsymbol{x})} \mathbf{I}, \text{ a.e. } \boldsymbol{x} \end{array} \right\}$$

and with it

$$\text{(B)} \qquad \max_{(\theta,\mathbf{A})\in\mathcal{B}} J(\theta, \mathbf{A}) = \max_{(\theta,\mathbf{A})\in\mathcal{B}} \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, \mathrm{d}\boldsymbol{x}$$

Set $\mathcal{B}$ is convex and closed.
Observe that

$$\max_{(\theta,\mathbf{A})\in\mathcal{A}} J(\theta, \mathbf{A}) \le \max_{(\theta,\mathbf{A})\in\mathcal{B}} J(\theta, \mathbf{A}).$$

Rewrite B as a max-min problem

Define $\mathcal{S} := \left\{ \boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, ..., \boldsymbol{\sigma}_m) \;\middle|\; \boldsymbol{\sigma}_i \in L^2(\Omega, \mathbb{R}^d),\; -\operatorname{div}(\boldsymbol{\sigma}_i) = f_i \right\}$

One can rewrite functional J in terms of fluxes:

$$J(\theta, \mathbf{A}) = \min_{\boldsymbol{\sigma} \in \mathcal{S}} \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{A}^{-1} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i$$

With notation $\mathcal{C} = \left\{ (\theta, \mathbf{A}) \,|\, (\theta, \mathbf{A}^{-1}) \in \mathcal{B} \right\}$

$$\begin{aligned} \max_{(\theta,\mathbf{A}) \in \mathcal{B}} J(\theta, \mathbf{A}) &= \max_{(\theta,\mathbf{A}) \in \mathcal{B}} \min_{\boldsymbol{\sigma} \in \mathcal{S}} \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{A}^{-1} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i \\ &= \max_{(\theta,\mathbf{B}) \in \mathcal{C}} \min_{\boldsymbol{\sigma} \in \mathcal{S}} \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{B} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i \end{aligned}$$

Observe that

$$L(\boldsymbol{\sigma}, (\theta, \mathbf{B})) = \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{B}\,\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i$$

$$\boldsymbol{\sigma} \mapsto L(\boldsymbol{\sigma}, (\theta, \mathbf{B}))$$

- quadratic (strictly convex)
- continuous in $L^2(\Omega)$ (l.s.c.)
- $(\exists(\theta, \mathbf{B})) \quad \boldsymbol{\sigma} \mapsto L(\boldsymbol{\sigma}, (\theta, \mathbf{B}))$ $\lim_{\|\boldsymbol{\sigma}\| \to +\infty} L(\boldsymbol{\sigma}, (\theta, \mathbf{B}))) = +\infty$

$$(\theta, \mathbf{B}) \mapsto L(\boldsymbol{\sigma}, (\theta, \mathbf{B}))$$

- linear (concave)
- continuous in $L^\infty\star$ (u.s.c.)
- set $\mathcal{C}$ is compact (in $L^\infty\star$).

Min-max theory

Previous conclusions for the Lagrange functional L implies:

- set of saddle points $\mathcal{S}_0 \times \mathcal{C}_0 \subset \mathcal{S} \times \mathcal{C}$ is not empty
- min and max are interchangeable
- $\boldsymbol{\sigma} \mapsto L(\boldsymbol{\sigma}, (\theta, \mathbf{B}))$ is strictly convex $\quad \Rightarrow \quad \mathcal{S}_0 = \{\boldsymbol{\sigma}^*\}$.

This means that there exists **unique** $\boldsymbol{\sigma}^*$ such that this holds

$$
\begin{aligned}
\max_{(\theta,\mathbf{A})\in\mathcal{B}} J(\theta, \mathbf{A}) &= \max_{(\theta,\mathbf{B})\in\mathcal{C}} \min_{\boldsymbol{\sigma}\in\mathcal{S}} L(\boldsymbol{\sigma}, (\theta, \mathbf{B})) \\
&= \max_{(\theta,\mathbf{B})\in\mathcal{C}} L(\boldsymbol{\sigma}^*, (\theta, \mathbf{B})) \\
&= \max_{(\theta,\mathbf{B})\in\mathcal{C}} \sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{B}\,\boldsymbol{\sigma}_i^* \cdot \boldsymbol{\sigma}_i^*
\end{aligned}
$$

$$\sum_{i=1}^{m} \mu_i \int_\Omega \mathbf{B}\,\boldsymbol{\sigma}_i^* \cdot \boldsymbol{\sigma}_i^* \text{ achieves max in } (\theta^*, \mathbf{B}^*) \in \mathcal{C}$$

$$\iff \qquad \mathbf{B}^*\,\boldsymbol{\sigma}_i^* = \frac{1}{\lambda_{\theta^*}^-}\boldsymbol{\sigma}_i^* \quad i = 1, 2, ..., m$$

This means problem (B) achieves max in $(\theta^*, \mathbf{A}^*) \in \mathcal{B}$

$$\iff \qquad \mathbf{A}^*\,\nabla u_i^* = \lambda_{\theta^*}^- \nabla u_i^* \quad i = 1, 2, ..., m$$

Instead of solving convex problem B, one can solve the following optimization problem:

$$(\mathrm{I}) \qquad \begin{cases} I(\theta) = \sum_{i=1}^{m} \mu_i \int_\Omega f_i u_i \,\mathrm{d}x \to \max \\ s.t. \quad \theta \in L^\infty(\Omega, [0,1]), \ \int_\Omega \theta = q_\alpha, \ \text{where } \boldsymbol{u} \text{ satisfies} \\ -\operatorname{div}(\lambda_\theta^- \nabla u_i) = f_i, \quad u_i \in \mathrm{H}_0^1(\Omega), \quad i = 1, ..., m, \end{cases}$$

The necessary and sufficient condition of optimality

Define

$$\psi := \sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2.$$

Lemma

The necessary and sufficient condition of optimality for solution θ^* of optimal design problem (I) *simplifies to the existence of a Lagrange multiplier $c \geq 0$ such that*

$$\begin{aligned} \psi = \sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 > c \quad &\Rightarrow \quad \theta^* = 1\,, \\ \psi = \sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i^*|^2 < c \quad &\Rightarrow \quad \theta^* = 0\,. \end{aligned} \tag{4}$$

Spherical symmetry

For spherically symmetric problem such that:

- $\Omega = R(\Omega)$ for any rotation R
- f_i are radial functions

it can be proved that there exists radial solution θ^*_R of (I).

In particular, it can be shown that

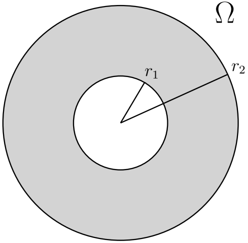
$$\max_{(\theta,\mathbf{A})\in\mathcal{A}} J(\theta,\mathbf{A}) = I(\theta^*_R).$$

Examples

Design problems with spherical symmetry were studied for ball:

- **Single state equations**
  Murat & Tartar (1985) Calculus of Variations and Homogenization - there exists relaxed solution $(\theta^*, \mathbf{A}^*)$ among simple laminates.
- **Multiple state equations**
  Vrdoljak, M. (2016) Classical Optimal Design in Two-Phase Conductivity Problems. SIAM Journal on Control and Optimization: 2020-2035

Single state optimal design problem



$\Omega = \overline{K}(0, r_2) \setminus K(0, r_1)$

Single state equation:

$$(5) \qquad \begin{cases} -\operatorname{div}(\lambda_\theta^-(x)\nabla u) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda_\theta^-(x) = \left(\frac{\theta(x)}{\alpha} + \frac{1-\theta(x)}{\beta}\right)^{-1}$.

Optimization problem:

For

$\theta \in \mathcal{T} := \left\{\theta \in L^\infty(\Omega, [0,1]) : \int_\Omega \theta \, \mathrm{d}\boldsymbol{x} = q_\alpha\right\}$

$$I(\theta) = \int_\Omega u \, \mathrm{d}x \to \max$$

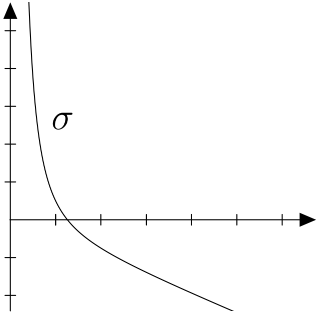
(5) in polar coordinates :

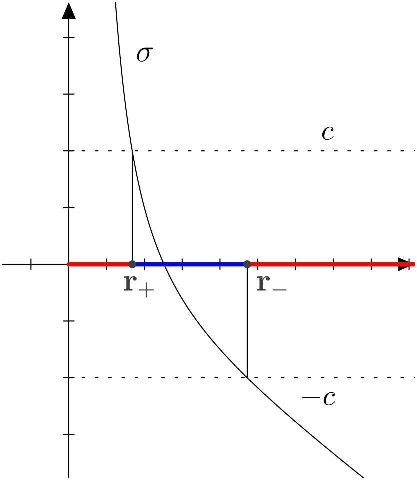
$$\begin{cases} -\frac{1}{r^{d-1}}(r^{d-1}\underbrace{\lambda_\theta^- u'(r)}_{\sigma})' = 1 \quad \text{in } \langle r_1, r_2 \rangle \\ \\ u(r_1) = u(r_2) = 0 \end{cases}$$

Observe that σ satisfies

$$\sigma = -\frac{r}{d} + \frac{\gamma}{r^{d-1}}, \quad \gamma > 0$$

$\sigma(r) : \langle 0, \infty \rangle \to \mathbb{R}$ is a strictly decreasing function.





The necessary and sufficient condition of optimality for θ^* states

$$
\begin{aligned}
|\sigma^*| > c \quad &\Rightarrow \quad \theta^* = 1\,, \\
|\sigma^*| < c \quad &\Rightarrow \quad \theta^* = 0\,.
\end{aligned}
$$

There are only three possible candidates for optimal design:

1) $\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_-\rangle \\ 1, & r \in [r_-, r_2] \end{cases}$

2) $\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_2\rangle \end{cases}$

3) $\theta^*(r) = \begin{cases} 0, & r \in [r_1, r_-\rangle \\ 1, & r \in [r_-, r_2\rangle \end{cases}$

Simplification to a non-linear system

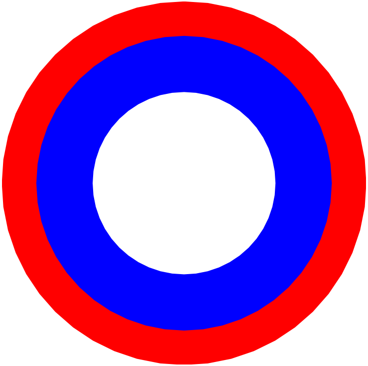
Necessary and sufficient condition of optimality can also be expressed as a non-linear system (unknowns $\gamma, c, r_+ r_-$):

$$
(6) \qquad \begin{cases}
S_d \displaystyle\int_{r_1}^{r_2} \theta(\rho)\rho^{d-1}\,\mathrm{d}\rho = q_\alpha \\[2ex]
u(r_2) = 0 \iff \gamma \displaystyle\int_{r_1}^{r_2} \left(\frac{1}{a(\rho)\rho^{d-1}}\right) \mathrm{d}\rho = \displaystyle\int_{r_1}^{r_2} \frac{\rho}{a(\rho)}\,\mathrm{d}\rho \\[2ex]
\sigma(r_+) = c, \quad \sigma(r_-) = -c, \qquad \text{where } c > 0
\end{cases}
$$

where

$$
\sigma(r) = \frac{\gamma}{r^{d-1}} - \frac{r}{d}, \quad \& \quad a(r) = \left(\frac{\theta(r)}{\alpha} + \frac{1-\theta(r)}{\beta}\right)^{-1}.
$$

3) case **beta**-**alpha**



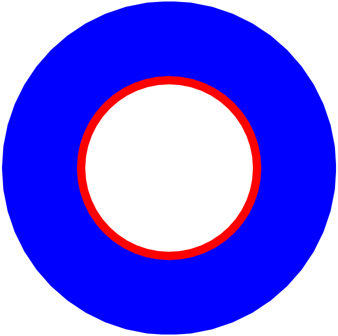
Non-linear system (6) does not admit a solution for design in the case beta-alpha.
We have proof for $d = 2$ and $d = 3$.

Therefore, cases: 1) and 2) should be considered as possible solutions.

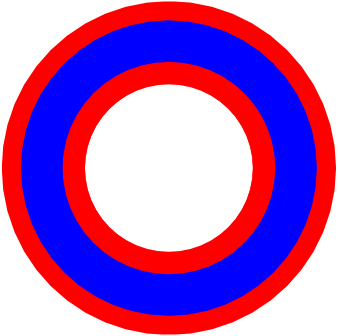
Results $d = 2$ or $d = 3$

Numerical results ($\alpha = 1, \beta = 2, r_1 = 1, r_2 = 2, d = 2$):

alpha-beta
($q_\alpha < 3.22\%$)



alpha-beta-alpha
($q_\alpha > 3.22\%$)



One can easily prove if q_α is very small, case alpha-beta is always solution (for arbitrary parameters α, β, r_1, r_2).

Multiple state optimal design problem, $d = 2$

Multiple state equations:

$$\begin{cases} -\operatorname{div}(\lambda_\theta^-(x)\nabla u_1) = 1 = f_1 & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} -\operatorname{div}(\lambda_\theta^-(x)\nabla u_2) = \frac{b}{r(b-r)^2} = f_2 & \text{in } \Omega \\ u_2 = 0 & \text{on } \partial\Omega \end{cases}$$

where $b > r_2$ and $\lambda_\theta^-(x) = \left(\frac{\theta(x)}{\alpha} + \frac{1-\theta(x)}{\beta}\right)$.

Optimization problem:

Find $\theta \in \mathcal{T}$ s.t.

$$I(\theta) = \mu_1 \int_\Omega f_1 u_1 \,\mathrm{d}x + \mu_2 \int_\Omega f_2 u_2 \,\mathrm{d}x \to \max$$

with $\mu_1, \mu_2 > 0$

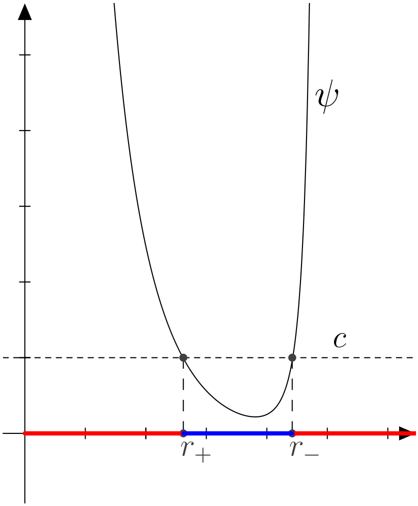
One can easily calculate:

$$\sigma_1 = -\frac{r}{2} + \frac{\gamma_1}{r},\ \gamma_1 > 0 \qquad \sigma_2 = \frac{1}{r-b} + \frac{\gamma_2}{r},\ \gamma_2 > 0$$

For any $\mu_1, \mu_2 > 0$ function $\psi : [r_1, r_2] \to \mathbb{R}$,

$$\psi := \mu_1 {\sigma_1}^2 + \mu_2 {\sigma_2}^2$$

is a strictly convex function.



The necessary and sufficient condition of optimality states that there exists $c > 0$ such that :

$$\psi^* > c \quad \Rightarrow \quad \theta^* = 1$$
$$\psi^* < c \quad \Rightarrow \quad \theta^* = 0$$

As before this implies that there can only be 3 cases:

1) $\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_-\rangle \\ 1, & r \in [r_-, r_2] \end{cases}$

2) $\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_2\rangle \end{cases}$

3) $\theta^*(r) = \begin{cases} 0, & r \in [r_1, r_-\rangle \\ 1, & r \in [r_-, r_2\rangle \end{cases}$

Results

Unlike previous one state example here all 3 cases (with **variations** of parameters: $\alpha, \beta, r_1, r_2, \mu_1, \mu_2$) can be possible solutions.

1) alpha-beta-alpha **2) alpha-beta** **3)beta-alpha**