Optimal design problem on an annulus for two-composite material maximizing the energy

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Introduction

Let $\Omega \subset \mathbb{R}^d$ be open and bounded set.
Two phases each with different isotropic conductivity: $\alpha, \beta$ ($0 < \alpha < \beta$).
$q_\alpha$ is prescribed amount (volume) of the first phase $\alpha$ ($0 < q_\alpha < |\Omega|$).
$\chi \in L^\infty(\Omega, \{0, 1\})$ a measurable characteristic function.

Conductivity can be expressed as

$$A(\chi) := \chi \alpha I + (1 - \chi) \beta I,$$

where

$$\int_\Omega \chi(\mathbf{x}) \, d\mathbf{x} = q_\alpha.$$
State functions $u_i \in H_0^1(\Omega), \ i = 1, 2, ..., m$ are given as a solution of the following boundary value problems:

\[
\begin{align*}
\begin{cases}
-\text{div}(A \nabla u_i) &= f_i \quad \text{in } \Omega \\
u_i &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

$i = 1, 2, ..., m$.

Notation: $\mathbf{u} = (u_1, u_2, ..., u_m)$.

Energy functional:

\[
I(\chi) := \sum_{i=1}^{m} \mu_i \int_{\Omega} f_i(\mathbf{x}) u_i(\mathbf{x}) \, d\mathbf{x}.
\]

where $\mu_i > 0, \ i = 1, 2, ..., m$. 
Optimal design problem:

\[
\begin{align*}
I(\chi) = \sum_{i=1}^{m} \mu_i &\int_{\Omega} f_i u_i \, d\mathbf{x} \to \text{max} \\
\text{s.t. } \chi &\in L^\infty(\Omega, \{0, 1\}), \quad \int_{\Omega} \chi \, d\mathbf{x} = q_{\alpha}, \\
\mathbf{u} &\text{ solves (1) with } \mathbf{A} = \chi \alpha \mathbf{I} + (1 - \chi) \beta \mathbf{I}.
\end{align*}
\]

(2)

If there exists solution $\chi$ for (2) we call it \textit{classical solution}.

\textbf{Important:} For general optimal design problems classical solutions doesn’t exist.

\textbf{Aim of this talk:}

- to present design problem as max-min optimization.
- to give examples on annulus with classical solutions
Relaxed design

For characteristic functions relaxation consists of:

\[(3) \quad \chi \in L^\infty(\Omega, \{0, 1\}) \leadsto \theta \in L^\infty(\Omega, [0, 1]),\]

with

\[\int_{\Omega} \theta \, d\mathbf{x} := \int_{\Omega} \chi \, d\mathbf{x} = q_\alpha.\]

Notion of H-convergence is introduced for conductivity \(A\).

**Effective conductivities:**

\[\mathcal{K}(\theta) \subset M_d(\mathbb{R}) \text{ with local fraction } \theta \in [0, 1].\]

Precisely, \(A \in \mathcal{K}(\theta)\) iff there exists sequence of characteristic functions

\[
\begin{cases}
\chi_n \xrightarrow{L^\infty} \theta \\
A^n = \chi_n \alpha \mathbf{I} + (1 - \chi_n) \beta \mathbf{I} \xrightarrow{H} A.
\end{cases}
\]
Effective conductivities - set $\mathcal{K}(\theta)$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues

$$
\lambda^-_\theta \leq \lambda_j \leq \lambda^+_\theta \quad j = 1, \ldots, d
$$

$$
\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda^-_\theta - \alpha} + \frac{d - 1}{\lambda^+_\theta - \alpha}
$$

$$
\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda^-_\theta} + \frac{d - 1}{\beta - \lambda^+_\theta},
$$

where

$$
\lambda^+_\theta = \theta \alpha + (1 - \theta) \beta
$$

$$
\frac{1}{\lambda^-_\theta} = \frac{\theta}{\alpha} + \frac{1 - \theta}{\beta}.
$$
Visual representation of a set $\mathcal{K}(\theta)$

For dimension $d = 2$:

For dimension $d = 3$: 
Relaxed problem A

Relaxed design:

\[ \mathcal{A} = \left\{ (\theta, A) \in L^\infty(\Omega, [0, 1] \times \text{Sym}_d) \mid \begin{align*} & \int_{\Omega} \theta \, dx = q_\alpha, \\ & A(x) \in K(\theta(x)), \text{ a.e. } x \end{align*} \right\} \]

Relaxed problem can be written as:

\[ (A) \max_{(\theta, A) \in \mathcal{A}} J(\theta, A) = \max_{(\theta, A) \in \mathcal{A}} \sum_{i=1}^{m} \mu_i \int_{\Omega} f_i u_i \, dx \]

Unfortunately, \( \mathcal{A} \) is not a convex set.
Generalized (convex) problem B

To achieve convexity, a new set is introduced:

\[ \mathcal{B} = \left\{ (\theta, A) \in L^\infty(\Omega, [0, 1] \times \text{Sym}_d) \ \bigg| \ \begin{array}{c} \int_{\Omega} \theta \, dx = q_\alpha, \\
\lambda^{-}_\theta(x) I \leq A(x) \leq \lambda^{+}_\theta(x) I, \ \text{a.e.} \ x \end{array} \right\} \]

and with it

\[ (B) \quad \max_{(\theta, A) \in \mathcal{B}} J(\theta, A) = \max_{(\theta, A) \in \mathcal{B}} \sum_{i=1}^{m} \mu_i \int_{\Omega} f_i u_i \, dx \]

Set \( \mathcal{B} \) is convex and closed.
Observe that

\[ \max_{(\theta, A) \in \mathcal{A}} J(\theta, A) \leq \max_{(\theta, A) \in \mathcal{B}} J(\theta, A). \]
Rewriting problem B as max-min problem

Define $S := \{ \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \mid \sigma_i \in L^2(\Omega, \mathbb{R}^d), \ -\text{div}(\sigma_i) = f_i \}$

One can rewrite functional $J$ in terms of fluxes:

$$J(\theta, A) = \min_{\sigma \in S} \sum_{i=1}^{m} \mu_i \int_{\Omega} A^{-1} \sigma_i \cdot \sigma_i$$

With notation $C = \{ (\theta, A) \mid (\theta, A^{-1}) \in B \}$

$$\max_{(\theta, A) \in B} \min_{\sigma \in S} \sum_{i=1}^{m} \mu_i \int_{\Omega} A^{-1} \sigma_i \cdot \sigma_i = \max_{(\theta, B) \in C} \min_{\sigma \in S} \sum_{i=1}^{m} \mu_i \int_{\Omega} B \sigma_i \cdot \sigma_i$$
Observe that

\[ L(\sigma, (\theta, B)) = \sum_{i=1}^{m} \mu_i \int_{\Omega} B \sigma_i \cdot \sigma_i \]

\[ \sigma \mapsto L(\sigma, (\theta, B)) \]

\[ (\theta, B) \mapsto L(\sigma, (\theta, B)) \]

- quadratic (strictly convex)
- continuous in \( L^2(\Omega) \) (l.s.c.)
- \((\exists (\theta, B)) \quad \sigma \mapsto L(\sigma, (\theta, B)) \)
- \( \lim_{\|\sigma\| \to +\infty} L(\sigma, (\theta, B)) = +\infty \)

- linear (concave)
- continuous in \( L^\infty \) (u.s.c.)
- set \( C \) is compact (in \( L^\infty \)).
Min-max theory

Previous conclusions for Lagrange function $L$ implies:
- set of saddle points $S_0 \times C_0 \subset S \times C$ is not empty
- min and max are interchangeable
- $\sigma \mapsto L(\sigma, (\theta, B))$ is strictly convex $\Rightarrow S_0 = \{\sigma^*\}$.

This means that $\sigma^*$ is unique.

$$
\max_{(\theta, A) \in B} J(\theta, A) = \max_{(\theta, B) \in C} \min_{\sigma \in S} L(\sigma, (\theta, B))
$$

$$
= \max_{(\theta, B) \in C} L(\sigma^*, (\theta, B))
$$

$$
= \max_{(\theta, B) \in C} \sum_{i=1}^{m} \mu_i \int_{\Omega} B \sigma_i^* \cdot \sigma_i^*
$$
\[
\sum_{i=1}^{m} \mu_i \int_{\Omega} \mathbf{B} \sigma_i^* \cdot \sigma_i^* \text{ achieves max. in } (\theta^*, \mathbf{B}^*) \in \mathcal{C}
\]

\[\iff\]

\[\mathbf{B}^* \sigma_i^* = \frac{1}{\lambda_{\theta_i^*}} \sigma_i^* \quad i = 1, 2, \ldots, m\]

This means problem (B) achieves max. in \((\theta^*, \mathbf{A}^*) \in \mathcal{B}\)

\[\iff\]

\[\mathbf{A}^* \nabla u_i^* = \lambda_{\theta_i^*} \nabla u_i^* \quad i = 1, 2, \ldots, m\]

Instead of solving convex problem B, one can solve following optimization problem:

\[
(I) \quad \left\{
\begin{array}{l}
I(\theta) = \sum_{i=1}^{m} \mu_i \int_{\Omega} f_i u_i \, dx \rightarrow \max \\
\text{s.t. } \theta \in L^\infty(\Omega, [0, 1]), \int_{\Omega} \theta = q_{\alpha}, \text{ where } \mathbf{u} \text{ satisfies }
\end{array}
\right.
\]

\[-\text{div}(\lambda_{\theta} \nabla u_i) = f_i, \quad u_i \in H^1_0(\Omega), \quad i = 1, \ldots, m,\]
Define $\psi := \sum_{i=1}^{m} \mu_i |\sigma_i^*|^2$.

**Lemma** (The necessary and sufficient condition of optimality)

The necessary and sufficient condition of optimality for solution $\theta^*$ of optimal design problem (I) simplifies to the existence of a Lagrange multiplier $c \geq 0$ such that

$$\psi = \sum_{i=1}^{m} \mu_i |\sigma_i^*|^2 > c \quad \Rightarrow \quad \theta^* = 1,$$

(4) $$\psi = \sum_{i=1}^{m} \mu_i |\sigma_i^*|^2 < c \quad \Rightarrow \quad \theta^* = 0.$$
Spherical symmetry

For spherically symmetric problem such that:
- \( \Omega = R(\Omega) \) for any rotation \( R \)
- \( f \) is radial function
it can be proved that there exists radial solution \( \theta^*_R \) of (I).

Specially, it can be shown that

\[
\max_{(\theta, \mathbf{A}) \in \mathcal{A}} J(\theta, \mathbf{A}) = I(\theta^*_R).
\]
Examples

Classical solutions where design problem has spherical symmetry were researched primarily for ball:

  **Single state equations** ... there exists relaxed solution \((\theta^*, A^*)\) among simple laminates.

  **Multiple state equations**
Single state optimal design problem

Single state equation:

\[
(5) \quad \begin{cases} 
- \text{div}(\lambda_\theta^{-}(x)\nabla u) = 1 & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega 
\end{cases}
\]

where \( \lambda_\theta^{-}(x) = \left( \frac{\theta(x)}{\alpha} + \frac{1-\theta(x)}{\beta} \right)^{-1} \).

Optimal problem:

For \( \theta \in \mathcal{T} := \{ \theta \in L^\infty(\Omega, [0, 1]) : \int_\Omega \theta \, dx = q_\alpha \} \)

\[
I(\theta) = \int_\Omega u \, dx \to \max
\]
Equations (5) in polar coordinates:

\[ \begin{aligned}
  \left\{ \begin{array}{l}
    -\frac{1}{r^{d-1}} \left( r^{d-1} \lambda \theta u'(r) \right)' = 1 \\
    u(r_1) = u(r_2) = 0
  \end{array} \right. \quad \text{in } \langle r_1, r_2 \rangle
\end{aligned} \]

Observe that \( \sigma \) satisfies

\[ \sigma = -\frac{r}{d} + \frac{\gamma}{r^{d-1}}, \quad \gamma > 0 \]

\( \sigma(r) : \langle 0, \infty \rangle \to \mathbb{R} \) is strictly decreasing function.
The necessary and sufficient condition of optimality for $\theta^*$ states

$$|\sigma^*| > c \implies \theta^* = 1,$$
$$|\sigma^*| < c \implies \theta^* = 0.$$ 

There are only three possible candidates for optimal design:

1) $\theta^*(r) = \begin{cases} 
1, & r \in [r_1, r_+) \\
0, & r \in [r_+, r_-) \\
1, & r \in [r_-, r_2] 
\end{cases}$

2) $\theta^*(r) = \begin{cases} 
1, & r \in [r_1, r_+) \\
0, & r \in [r_+, r_2] 
\end{cases}$

3) $\theta^*(r) = \begin{cases} 
0, & r \in [r_1, r_-] \\
1, & r \in [r_-, r_2] 
\end{cases}$
Simplification to a non-linear system

Necessary and sufficient condition of optimality can also be expressed as a non-linear system (unknowns $\gamma, c, r_+ r_-$):

\[
\begin{align*}
S_d \int_{r_1}^{r_2} \theta(\rho) \rho^{d-1} \, d\rho &= q_\alpha \\
u(r_2) &= 0 \iff \gamma \int_{r_1}^{r_2} \left( \frac{1}{a(\rho)\rho^{d-1}} \right) \, d\rho = \int_{r_1}^{r_2} \frac{\rho}{a(\rho)} \, d\rho \\
\sigma(r_+) &= c, \quad \sigma(r_-) = -c, \quad \text{where } c > 0
\end{align*}
\]

where

\[
\sigma(r) = \frac{\gamma}{r^{d-1}} - \frac{r}{d}, \quad \& \quad a(r) = \left( \frac{\theta(r)}{\alpha} + \frac{1 - \theta(r)}{\beta} \right)^{-1}
\]
Results $d = 2$ or $d = 3$

3) case beta-alpha

Non-linear system (6) does not admit solution for design in case beta-alpha.
Proof exists for $d = 2$ and $d = 3$.

Therefore 1) case and 2) case remains as possible solutions.
Results $d = 2$ or $d = 3$

Numerical results ($\alpha = 1, \beta = 2, r_1 = 1, r_2 = 2, d = 2$):

One can easily prove if $q_\alpha$ is scarce (very small), case alpha-beta is always solution (for whatever parameters $\alpha, \beta, r_1, r_2$).
Multiple state optimal design problem, $d = 2$

Multiple state equations:

\[
\begin{aligned}
- \text{div}(\lambda^{-}(x)\nabla u_1) &= 1 = f_1 \quad \text{in } \Omega \\
\quad u_1 &= 0 \quad \text{on } \partial \Omega \\
- \text{div}(\lambda^{-}(x)\nabla u_2) &= \frac{b}{r(b-r)^2} = f_2 \quad \text{in } \Omega \\
\quad u_2 &= 0 \quad \text{on } \partial \Omega 
\end{aligned}
\]

where $b > r_2$ and $\lambda^{-}(x) = \left(\frac{\theta(x)}{\alpha} + \frac{1-\theta(x)}{\beta}\right)$.

Optimal problem:
Finding $\theta \in \mathcal{T}$ s.t.

\[
I(\theta) = \mu_1 \int_{\Omega} f_1 u_1 \, dx + \mu_2 \int_{\Omega} f_2 u_2 \, dx \rightarrow \max
\]

with $\mu_1, \mu_2 > 0$
One can easily calculate:

$$\sigma_1 = -\frac{r}{2} + \frac{\gamma_1}{r}, \quad \gamma_1 > 0$$

$$\sigma_2 = \frac{1}{r - b} + \frac{\gamma_2}{r}, \quad \gamma_2 > 0$$

For $\mu_1, \mu_2 > 0$ we define $\psi : [r_1, r_2] \rightarrow \mathbb{R}$,

$$\psi := \mu_1 \sigma_1^2 + \mu_2 \sigma_2^2$$

is strictly convex function (for any $\mu_1, \mu_2 > 0$).
The necessary and sufficient condition of optimality states that there exists \( c > 0 \) such that:

\[
\psi^* > c \quad \Rightarrow \quad \theta^* = 1 \\
\psi^* < c \quad \Rightarrow \quad \theta^* = 0
\]

As before this implies that there can only be 3 cases:

1) \( \theta^*(r) = \begin{cases} 
1, & r \in [r_1, r_+), \\
0, & r \in [r_+, r_-), \\
1, & r \in (r_-, r_2]
\end{cases} \)

2) \( \theta^*(r) = \begin{cases} 
1, & r \in [r_1, r_+) \\
0, & r \in [r_+, r_2]
\end{cases} \)

3) \( \theta^*(r) = \begin{cases} 
0, & r \in [r_1, r_-) \\
1, & r \in (r_-, r_2]
\end{cases} \)
Unlike previous one state example here all 3 cases (with variations of parameters: $\alpha, \beta, r_1, r_2, \mu_1, \mu_2$) can be possible solutions.

1) alpha-beta-alpha  2) alpha-beta  3) beta-alpha
Thank you for your attention.