Compactness by compensation via H-distributions

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Panov's result

The most general version of the classical L^2 results has recently been proved by E. Yu. Panov (2011):
Assume that the sequence $(\mathbf{u}_n)$ is bounded in $L^p(\mathbf{R}^d; \mathbf{R}^r)$, $2 \leq p < \infty$, and converges weakly in $\mathcal{D}'(\mathbf{R}^d)$ to a vector function $\mathbf{u}$.
Let $q = p'$ if $p < \infty$, and $q > 1$ if $p = \infty$. Assume that the sequence

$$\sum_{k=1}^{\nu} \partial_k (\mathbf{A}^k \mathbf{u}_n) + \sum_{k,l=\nu+1}^{d} \partial_{kl} (\mathbf{B}^{kl} \mathbf{u}_n)$$

is precompact in the anisotropic Sobolev space $W_{loc}^{-1,-2;q}(\mathbf{R}^d; \mathbf{R}^m)$, where $m \times r$ matrices $\mathbf{A}^k$ and $\mathbf{B}^{kl}$ have variable coefficients belonging to $L^{2\bar{q}}(\mathbf{R}^d)$, $\bar{q} = \frac{p}{p-2}$ if $p > 2$, and to the space $C(\mathbf{R}^d)$ if $p = 2$.

We introduce the set $\Lambda(\mathbf{x})$

$$\Lambda(\mathbf{x}) = \left\{ \boldsymbol{\lambda} \in \mathbf{C}^r \,\middle|\, (\exists \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}) : \left(i \sum_{k=1}^{\nu} \xi_k \mathbf{A}^k(\mathbf{x}) - 2\pi \sum_{k,l=\nu+1}^{d} \xi_k \xi_l \mathbf{B}^{kl}(\mathbf{x}) \right) \boldsymbol{\lambda} = \mathbf{0}_m \right\},$$

and consider the bilinear form on $\mathbf{C}^r$

$$q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\eta}) = \mathbf{Q}(\mathbf{x}) \boldsymbol{\lambda} \cdot \boldsymbol{\eta}, \tag{1}$$

where $\mathbf{Q} \in L_{loc}^{\bar{q}}(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p > 2$ and $\mathbf{Q} \in C(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p = 2$.
Finally, let $q(\mathbf{x}, \mathbf{u}_n, \mathbf{u}_n) \rightharpoonup \omega$ weakly in the space of distributions.

The following theorem holds

Theorem. *Assume that $(\forall \boldsymbol{\lambda} \in \Lambda(\mathbf{x}))\ q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\lambda}) \geq 0$ (a.e. $\mathbf{x} \in \mathbf{R}^d$) and $\mathbf{u}_n \rightharpoonup \mathbf{u}$, then $q(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})) \leq \omega$.*

The connection between q and Λ given in the previous theorem, we shall call *the consistency condition*.

Goal: to formulate and extend the results from the preceeding theorem to the $L^p - L^q$ framework for appropriate (greater than one) indices p and q where $p < 2$.

Localisation principle

For $\alpha \in \mathbf{R}^+$, we define $\partial_{x_k}^{\alpha}$ to be a pseudodifferential operator with a polyhomogeneous symbol $(2\pi i \xi_k)^{\alpha}$, i.e.

$$\partial_{x_k}^{\alpha} u = ((2\pi i \xi_k)^{\alpha} \hat{u}(\boldsymbol{\xi}))\check{}.$$

In the sequel, we shall assume that sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ are uniformly compactly supported. This assumption can be removed if the orders of derivatives $(\alpha_1, \ldots, \alpha_d)$ are natural numbers.

Lemma. *Assume that sequences $(\mathbf{u}_n)$ and $(\mathbf{v}_n)$ are bounded in $L^p(\mathbf{R}^d; \mathbf{R}^r)$ and $L^q(\mathbf{R}^d; \mathbf{R}^r)$, respectively, and converge toward $\mathbf{0}$ and $\mathbf{v}$ in the sense of distributions. Furthermore, assume that sequence $(\mathbf{u}_n)$ satisfies:*

$$\mathbf{G}_n := \sum_{k=1}^{d} \partial_k^{\alpha_k} (\mathbf{A}^k \mathbf{u}_n) \to \mathbf{0} \text{ in } W^{-\alpha_1, \ldots, -\alpha_d; p}(\Omega; \mathbf{R}^m), \tag{2}$$

*where either $\alpha_k \in \mathbf{N}$, $k = 1, \ldots, d$ or $\alpha_k > d$, $k = 1, \ldots, d$, and elements of matrices $\mathbf{A}^k$ belong to $L^{\bar{s}'}(\mathbf{R}^d)$, $\bar{s} \in (1, \frac{pq}{p+q})$.
Finally, by $\boldsymbol{\mu}$ denote a matrix H-distribution corresponding to subsequences of $(\mathbf{u}_n)$ and $(\mathbf{v}_n - \mathbf{v})$. Then the following relation holds*

$$\left(\sum_{k=1}^{d} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k \right) \boldsymbol{\mu} = \mathbf{0}.$$

Compactness by compensation result

Introduce the set

$$\Lambda_{\mathcal{D}} = \left\{ \boldsymbol{\mu} \in L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}))')^r : \left(\sum_{k=1}^{n} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k \right) \boldsymbol{\mu} = \mathbf{0}_m \right\},$$

where the given equality is understood in the sense of $L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}))')^m$.
Let us assume that coefficients of the bilinear form q from (1) belong to space $L_{loc}^t(\mathbf{R}^d)$, where $1/t + 1/p + 1/q < 1$.

Definition. We say that set $\Lambda_{\mathcal{D}}$, bilinear form q from (1) and matrix $\boldsymbol{\mu} = [\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r]$, $\boldsymbol{\mu}_j \in L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}))')^r$ satisfy the strong consistency condition if $(\forall j \in \{1, \ldots, r\})\ \boldsymbol{\mu}_j \in \Lambda_{\mathcal{D}}$, and it holds

$$\langle \phi \mathbf{Q} \otimes 1, \boldsymbol{\mu} \rangle \geq 0, \quad \phi \in L^{\bar{s}}(\mathbf{R}^d; \mathbf{R}_0^+).$$

Theorem. *Assume that sequences $(\mathbf{u}_n)$ and $(\mathbf{v}_n)$ are bounded in $L^p(\mathbf{R}^d; \mathbf{R}^r)$ and $L^q(\mathbf{R}^d; \mathbf{R}^r)$, respectively, and converge toward $\mathbf{u}$ and $\mathbf{v}$ in the sense of distributions.
Assume that (2) holds and that*

$$q(\mathbf{x}; \mathbf{u}_n, \mathbf{v}_n) \rightharpoonup \omega \text{ in } \mathcal{D}'(\mathbf{R}^d).$$

If the set $\Lambda_{\mathcal{D}}$, the bilinear form (1), and matrix H-distribution $\boldsymbol{\mu}$, corresponding to subsequences of $(\mathbf{u}_n - \mathbf{u})$ and $(\mathbf{v}_n - \mathbf{v})$, satisfy the strong consistency condition, then

$$q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \leq \omega \text{ in } \mathcal{D}'(\mathbf{R}^d).$$

H-distributions

H-distributions were introduced by N. Antonić and D. Mitrović (2011) as an extension of H-measures to the $L^p - L^q$ context.
M. Lazar and D. Mitrović (2012) extended and applied them on a velocity averaging problem.

We need multiplier operators with symbols defined on a manifold P determined by an d-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}_+^d$ where $\alpha_k \in \mathbf{N}$ or $\alpha_k \geq d$:

$$\mathrm{P} = \left\{ \boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{2\alpha_k} = 1 \right\},$$

We shall use the following variant of H-distributions.

Theorem. *Let (u_n) be a bounded sequence in $L^p(\mathbf{R}^d)$, $p > 1$, and let (v_n) be a bounded sequence of uniformly compactly supported functions in $L^q(\mathbf{R}^d)$, $1/q + 1/p < 1$, weakly converging to 0 in the sense of distributions. Then, after passing to a subsequence (not relabelled), for any $\bar{s} \in (1, \frac{pq}{p+q})$ there exists a continuous bilinear functional B on $L^{\bar{s}'}(\mathbf{R}^d) \otimes C^d(\mathrm{P})$ such that for every $\varphi \in L^{\bar{s}'}(\mathbf{R}^d)$ and $\psi \in C^d(\mathrm{P})$, it holds*

$$B(\varphi, \psi) = \lim_{n \to \infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_n(\mathbf{x}) (\mathcal{A}_{\psi_{\mathrm{P}}} v_n)(\mathbf{x}) d\mathbf{x},$$

*where $\mathcal{A}_{\psi_{\mathrm{P}}}$ is the Fourier multiplier operator on $\mathbf{R}^d$ associated to $\psi \circ \pi_{\mathrm{P}}$.
The bilinear functional B can be continuously extended as a linear functional on $L^{\bar{s}'}(\mathbf{R}^d; C^d(\mathrm{P}))$.*

Case $L^p - L^{p'}$, $p > 1$

In the case $1/p + 1/q = 1$, applying the same proof gives us continuous bilinear functional on $C(\mathbf{R}^d) \otimes C^d(\mathrm{P})$. Using Schwartz's kernel theorem, we can only extend it to a distribution from $\mathcal{D}'(\mathbf{R}^d \times \mathrm{P})$.
Introduce the truncation operator $T_l(v) = v$ if $|v| \leq l$ and $T_l(v) = 0$ if $|v| \geq l$, for $l \in \mathbf{N}$.

Theorem. *Assume that*

- *sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ are bounded in $L^p(\mathbf{R}^d; \mathbf{R}^N)$ and $L^{p'}(\mathbf{R}^d; \mathbf{R}^N)$, where $1/p + 1/p' = 1$, and converge toward $\mathbf{u}$ and $\mathbf{v}$ in the sense of distributions;*

- *for every $l \in \mathbf{N}$, the sequences $(T_l(\mathbf{v}_r))$ converge weakly in $L^{p'}(\mathbf{R}^d; \mathbf{R}^N)$ toward $\mathbf{h}^l$, where the truncation operator T_l is understood coordinatewise;*

- *there exists a vector valued function $\mathbf{V} \in L^{p'}(\mathbf{R}^d; \mathbf{R}^N)$ such that $\mathbf{v}_r \leq \mathbf{V}$ holds coordinatewise for every $r \in \mathbf{N}$;*

- *(2) holds with $a_{skl} \in C_0(\mathbf{R}^d)$ and that $q_{jm} \in C(\mathbf{R}^d)$.*

Assume that

$$q(\mathbf{x}; \mathbf{u}_r, \mathbf{v}_r) \rightharpoonup \omega \text{ in } \mathcal{D}'(\mathbf{R}^d).$$

If for every $l \in \mathbf{N}$, the set $\Lambda_{\mathcal{D}}$, the bilinear form (1), and the (matrix of) H-distributions $\boldsymbol{\mu}_l$ corresponding to the sequences $(\mathbf{u}_r - \mathbf{u})$ and $(T_l(\mathbf{v}_r) - \mathbf{h}^l)_r$ satisfy the strong consistency condition, then it holds

$$q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \leq \omega \text{ in } \mathcal{D}'(\mathbf{R}^d).$$

Application

Let us consider the non-linear parabolic type equation

$$L(u) = \partial_t u - \mathrm{div\,div}\,(g(t, \mathbf{x}, u) \mathbf{A}(t, \mathbf{x})),$$

on $(0, \infty) \times \Omega$, where Ω is an open subset of $\mathbf{R}^d$. For p, q and s such that $1/p + 1/q + 1/s < 1$, assume

$$u \in L^p((0, \infty) \times \Omega), \quad g(t, \mathbf{x}, u(t, \mathbf{x})) \in L^q((0, \infty) \times \Omega),$$

$$\mathbf{A} \in L_{loc}^s((0, \infty) \times \Omega)^{d \times d},$$

and that the matrix $\mathbf{A}$ is strictly positive definite, i.e.

$$\mathbf{A} \boldsymbol{\xi} \cdot \boldsymbol{\xi} > 0, \quad \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}, \quad (a.e.(t, \mathbf{x}) \in (0, \infty) \times \Omega).$$

Furthermore, assume that g is a Carathèodory function and non-decreasing with respect to the third variable.

Theorem. *Assume that sequences*

- *(u_r) and $g(\cdot, u_r)$ are such that $u_r, g(u_r) \in L^2(\mathbf{R}^+ \times \mathbf{R}^d)$ for every $r \in \mathbf{N}$;*

- *that they are bounded in $L^p(\mathbf{R}^+ \times \mathbf{R}^d)$, $p \in (1, 2]$, and $L^q(\mathbf{R}^+ \times \mathbf{R}^d)$, $q > 2$, respectively, where $1/p + 1/q < 1$;*

- *$u_r \rightharpoonup u$ and, for some, $f \in W^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d)$, the sequence*

$$L(u_r) = f_r \to f \text{ strongly in } W^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d).$$

Under the assumptions given above, it holds

$$L(u) = f \text{ in } \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d).$$

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