Composite elastic plate via general homogenization theory

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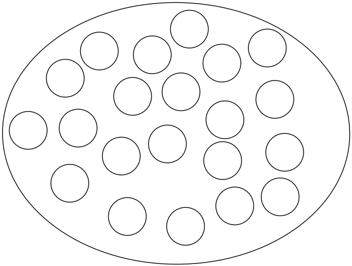
The physical idea of homogenization is to average a heterogeneous media in order to derive effective properties.

$$\begin{cases} Au = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition} \end{cases}$$



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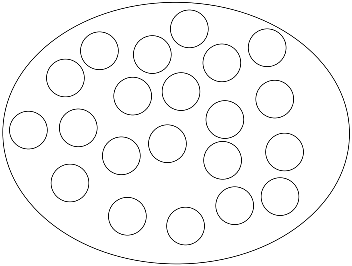

Sequence of similar problems

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Kirchhoff-Love plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega). \end{cases}$$

- $\Omega \subseteq \mathbb{R}^d$ bounded domain ($d = 2 \ldots$ plate)
- $f \in H^{-2}(\Omega)$ external load
- $u \in H_0^2(\Omega)$ vertical displacement of the plate
- $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{\mathbf{N} \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\, \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \alpha \mathbf{S} : \mathbf{S} \text{ and } \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \frac{1}{\beta}\mathbf{S} : \mathbf{S} \text{ a.e. } \mathbf{x}\}$ describes properties of material of the given plate

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Antonić, Balenović, 1999.

Definition

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha,\beta;\Omega)$ H-converges to $\mathbf{M}\in\mathfrak{M}_2(\alpha',\beta';\Omega)$ if for any $f\in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

$$\begin{cases} \mathrm{div}\,\mathrm{div}\,(\mathbf{M}^n\nabla\nabla u_n)=f \quad \text{in} \quad \Omega \\ u_n\in H^2_0(\Omega) \end{cases}$$

coverges weakly to a limit u in $H^2_0(\Omega)$, while the sequence $(\mathbf{M}^n\nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $L^2(\Omega;\mathrm{Sym})$.

Theorem

Let $(\mathbf{M}^n)$ be a sequence in $\mathfrak{M}_2(\alpha,\beta;\Omega)$. Then there is a subsequence $(\mathbf{M}^{n_k})$ and a tensor function $\mathbf{M}\in\mathfrak{M}_2(\alpha,\beta;\Omega)$ such that $(\mathbf{M}^{n_k})$ H-converges to $\mathbf{M}$.

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Properties

- Locality of the H-convergence
- Irrelevance of boundary conditions
- Energy convergence
- Ordering property
- Metrizability
- Corrector results

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Definition of correctors

Definition

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $\mathbf{M}$. Let $(w_n^{ij})_{1 \leq i,j \leq d}$ be a family of test functions satisfying

$$w_n^{ij} \rightharpoonup \frac{1}{2} x_i x_j \quad \text{in} \quad \mathrm{H}^2(\Omega)$$

$$\mathbf{M}^n \nabla\nabla w_n^{ij} \rightharpoonup \cdot \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$$

$$\operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla w_n^{ij}) \to \cdot \quad \text{in} \quad \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega).$$

The sequence of tensors $\mathbf{W}^n$ defined with $\mathbf{W}^n_{ijkm} = [\nabla\nabla w_n^{km}]_{ij}$ is called a sequence of correctors.

Uniqueness of correctors

Theorem

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor $\mathbf{M}$. A sequence of correctors $(\mathbf{W}^n)$ is unique in the sense that, if there exist two sequences of correctors $(\mathbf{W}^n)$ and $(\tilde{\mathbf{W}}^n)$, their difference $(\mathbf{W}^n - \tilde{\mathbf{W}}^n)$ converges strongly to zero in $L^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Corrector result

Theorem

Let $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *which H-converges to* $\mathbf{M}$*. For* $f \in H^{-2}(\Omega)$*, let* (u_n) *be the solution of*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega), \end{cases}$$

and let u *be the weak limit of* (u_n) *in* $H_0^2(\Omega)$*, i.e., the solution of the homogenized equation*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M} \nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega). \end{cases}$$

Then, $r_n := \nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u \to 0$ *strongly in* $L^1_{\mathrm{loc}}(\Omega; Sym)$*.*

Definition

Let $\chi^n \in \mathrm{L}^\infty(\Omega;[0,1])$ be a sequence of characteristic functions and $(\mathbf{M}^n)$ be a sequence of tensors defined by

$$\mathbf{M}^n(\mathbf{x}) = \chi^n(\mathbf{x})\mathbf{A} + (1-\chi^n(\mathbf{x}))\mathbf{B},$$

*where **A** and **B** are assumed to be symmetric, positive definite fourth order tensors. Assume that there exist $\theta \in \mathrm{L}^\infty(\Omega;[0,1])$ and $\mathbf{M}^* \in \mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$ such that*

$$\chi^n \xrightarrow{*} \theta \text{ in } \mathrm{L}^\infty(\Omega,[0,1]),$$

$$\mathbf{M}^n \xrightarrow{H} \mathbf{M}^* \text{ in } \mathfrak{M}_2(\alpha,\beta;\Omega).$$

*The H-limit $\mathbf{M}^*$ is said to be the homogenized tensor of a two-phase composite material obtained by mixing **A** and **B** in proportions θ and $(1-\theta)$, respectively, with a microstructure defined by the sequence (χ^n).*

Homogenization of laminated structures

Theorem

Let **A** *and* **B** *be two constant tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *and* $\chi_n(x \cdot e)$ *be a sequence of characteristic functions that converges to* $\theta(x \cdot e)$ *in* $\mathrm{L}^\infty(\Omega; [0,1])$ *weakly-*$*$*. Then, sequence* $(\mathbf{M}^n)$ *of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, defined as*

$$\mathbf{M}^n(x \cdot e) = \chi_n(x \cdot e)\mathbf{A} + (1 - \chi_n(x \cdot e))\mathbf{B}$$

H-converges to

$$\mathbf{M}^* = \theta\mathbf{A} + (1-\theta)\mathbf{B} - \frac{\theta(1-\theta)(\mathbf{A}-\mathbf{B})(e \otimes e) \otimes (\mathbf{A}-\mathbf{B})^T(e \otimes e)}{(1-\theta)\mathbf{A}(e \otimes e) : (e \otimes e) + \theta\mathbf{B}(e \otimes e) : (e \otimes e)}, \tag{3.1}$$

which also depends only on $x \cdot e$*.*

Corollary

If $(\mathbf{A}-\mathbf{B})$ is an invertible, symmetric, fourth order tensor, formula (3.1) is equivalent to

$$\theta(\mathbf{M}^*-\mathbf{B})^{-1} = (\mathbf{A}-\mathbf{B})^{-1} + \frac{1-\theta}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}). \quad (3.2)$$

If we repeat iterative process of lamination p times, in lamination directions $(\mathrm{e}_i)_{1\leq i\leq p}$ and proportions $(\theta_i)_{1\leq i\leq p}$, we obtain a rank-p sequential laminate with tensor **B** and core **A**, which is defined by the following formula:

$$\left(\prod_{j=1}^{p}\theta_j\right)(\mathbf{A}_p^*-\mathbf{B})^{-1} = (\mathbf{A}-\mathbf{B})^{-1} + \sum_{i=1}^{p}\left((1-\theta_i)\prod_{j=1}^{i-1}\theta_j\right)\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}.$$

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If $(\mathbf{A} - \mathbf{B})$ is an invertible, symmetric, fourth order tensor, formula (3.1) is equivalent to

$$\theta(\mathbf{M}^* - \mathbf{B})^{-1} = (\mathbf{A} - \mathbf{B})^{-1} + \frac{1-\theta}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}). \quad (3.2)$$

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Hashin-Shtrikman bounds

Definition

Let $\boldsymbol{\xi} \in \mathrm{Sym}$, $\theta \in [0, 1]$ the volume fraction of material **A** and $(1 - \theta)$ the volume fraction of material **B**. The function $f^-(\theta, \mathbf{A}, \mathbf{B}, \boldsymbol{\xi})$ (respectively, $f^+(\theta, \mathbf{A}, \mathbf{B}, \boldsymbol{\xi})$), which is real-valued, is said to be a lower bound (respectively, an upper bound) if for any $\mathbf{A}^* \in G_\theta$

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq f^-(\theta, \mathbf{A}, \mathbf{B}, \boldsymbol{\xi}) \quad (\text{respectively, } \mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq f^+(\theta, \mathbf{A}, \mathbf{B}, \boldsymbol{\xi})).$$

The lower bound $f^-(\theta, \mathbf{A}, \mathbf{B}, \boldsymbol{\xi})$ (respectively, the upper bound $f^+(\theta, \mathbf{A}, \mathbf{B}, \boldsymbol{\xi})$) is said to be optimal if for any $\boldsymbol{\xi} \in \mathrm{Sym}$ there exists $\mathbf{A}^* \in G_\theta$ such that

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} = f^-(\theta, \mathbf{A}, \mathbf{B}, \boldsymbol{\xi}) \quad (\text{respectively, } \mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} = f^+(\theta, \mathbf{A}, \mathbf{B}, \boldsymbol{\xi})).$$

Theorem

For any $\boldsymbol{\xi} \in \mathrm{Sym}$*, the effective energy of a composite material* $\mathbf{A}^* \in G_\theta$ *satisfies the following bounds:*

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \geq \mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} + (1-\theta) \max_{\boldsymbol{\eta}\in\mathrm{Sym}} [2\boldsymbol{\xi} : \boldsymbol{\eta} - (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - \theta g(\boldsymbol{\eta})], \quad (3.3)$$

where $g(\boldsymbol{\eta})$ *is defined by*

$$g(\boldsymbol{\eta}) = \sup_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} \frac{|(\mathbf{k}\otimes\mathbf{k}) : \eta|^2}{\mathbf{A}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})} \quad (3.4)$$

and

$$\mathbf{A}^*\boldsymbol{\xi} : \boldsymbol{\xi} \leq \mathbf{B}\boldsymbol{\xi} : \boldsymbol{\xi} + \theta \min_{\boldsymbol{\eta}\in\mathrm{Sym}} [2\boldsymbol{\xi} : \boldsymbol{\eta} + (\mathbf{B}-\mathbf{A})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta) h(\boldsymbol{\eta})], \quad (3.5)$$

where $h(\boldsymbol{\eta})$ *is defined by*

$$h(\boldsymbol{\eta}) = \inf_{\mathbf{k}\in\mathbf{Z}^d, \mathbf{k}\neq 0} \frac{|(\mathbf{k}\otimes\mathbf{k}) : \boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}. \quad (3.6)$$

Moreover, (3.3) and (3.5) are optimal in the sense of Definition 7 and optimality is achieved by a finite-rank sequential laminate.

Now what?

- G-closure problem
- Optimal design of plates
- Small-amplitude homogenization - non-periodic case

Thank you for your attention!

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